Review sheet 3: Modules of finite length

We review the definition of modules of finite length and their characterization over Noetherian rings.

Let R be a commutative ring. Recall that an R-module M is *simple* if $M \neq 0$ and for every submodule M' of M, we have either $M' = 0$ or $M' = M$. It is straightforward to see that a module M is simple if and only if it is isomorphic to $A/\mathfrak{m}$, for some maximal ideal $\mathfrak{m}$ of R.

Definition 0.1. An R-module M is of *finite length* if it has a *composition series*, that is, a sequence of submodules

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$$

such that M_i/M_{i-1} is a simple module for $1 \leq i \leq r$. It is a consequence of the Jordan-Hölder theorem that if M satisfies this property, then the quotients M_i/M_{i-1} are independent of the choice of composition series, up to reordering. In particular, the length r only depends on M; this is the *length* of M, denoted $\ell(M)$ (or $\ell_R(M)$ if the ring is not clear from the context).

Example 0.2. If R is a DVR with discrete valuation v, then for every $a \in R$, we have $\ell\big(R/(a)\big) = v(a)$.

We begin with some easy properties regarding finite length modules.

Proposition 0.3. *Given an exact sequence of R-modules*

$$0 \to M' \to M \to M'' \to 0,$$

the module M has finite length if and only if both M' and M'' have finite length, and in this case

$$\ell(M) = \ell(M') + \ell(M'').$$

Proof. It is clear that if M' and M'' have finite length, then we obtain a composition series for M by concatenating the composition series for M' and M''. This implies that $\ell(M) = \ell(M') + \ell(M'')$. The converse follows from the fact, easy to check, that given a composition series for M, by intersecting each submodule with M' (respectively, by taking the image of each submodule in M'') we obtain after removing repeated submodules a composition series for M' (respectively, M''). $\square$

Remark 0.4. Every R-module of finite length is Artinian: if

$$M = M_0 \supsetneq M_1 \supsetneq \ldots$$

is a strictly decreasing sequence of submodules, then it follows from the above proposition that we have a strictly decreasing sequence of non-negative integers

$$\ell(M_0) > \ell(M_1) > \ldots,$$

a contradiction.

Proposition 0.5. *If R is a Noetherian ring, then an R-module M has finite length if and only if M is finitely generated and* $\dim\big(R/\mathrm{Ann}_R(M)\big)=0$.

Proof. Suppose first that M has a composition series

$$0=M_0\subseteq M_1\subseteq\ldots\subseteq M_r=M,$$

with $M_i/M_{i-1}\simeq R/\mathfrak{m}_i$ for $1\leq i\leq r$, where each $\mathfrak{m}_i$ is a maximal ideal of R. Since each M_i/M_{i-1} is finitely generated, we conclude that M is finitely generated. Moreover, we have $\prod_{i=1}^r\mathfrak{m}_i\subseteq\mathrm{Ann}_R(M)$, hence the only primes containing $\mathrm{Ann}(R)$ are the $\mathfrak{m}_i$. This implies that $\dim\big(R/\mathrm{Ann}_R(M)\big)=0$.

Conversely, if M is finitely generated over a Noetherian ring, then it follows from Corollary 1.4 in Review Sheet 1 that we have submodules

$$0=M_0\subseteq M_1\subseteq\ldots\subseteq M_r=M,$$

such that $M_i/M_{i-1}\simeq A/\mathfrak{p}_i$ for $1\leq i\leq r$, where each $\mathfrak{p}_i$ is a prime ideal in R. If we have $\dim\big(R/\mathrm{Ann}_R(M)\big)=0$, then every prime ideal in $R/\mathrm{Ann}_R(M)$ is a maximal ideal. Since we clearly have $\mathrm{Ann}_R(M)\subseteq\mathfrak{p}_i$ for all i, we conclude that each quotient M_i/M_{i-1} is a simple module, hence M has finite length. $\square$

Example 0.6. If $(R,\mathfrak{m})$ is a Noetherian local ring, then an R-module M has finite length if and only if it is finitely generated and $\mathfrak{m}^r\cdot M=0$ for some $r\geq 1$.

Example 0.7. If k is a field and A is a finite k-algebra, then A is clearly Noetherian and $\dim(A)=0$ (if $\mathfrak{p}$ is a prime ideal in A, then $A/\mathfrak{p}$ is a domain which is a finite k-algebra, hence it is a field). In particular, we see that A has finite length as a module over itself.

We also note that in this case A is a product of local, finite k-algebras. Indeed, given a minimal primary decomposition

$$(0)=\mathfrak{q}_1\cap\ldots\cap\mathfrak{q}_r,$$

by the Chinese Remainder theorem we have

$$R\simeq\prod_{i=1}^r R/\mathfrak{q}_i$$

(note that the ideals $\mathrm{rad}(\mathfrak{q}_i)$ are mutually distinct maximal ideals, hence $\mathfrak{q}_i+\mathfrak{q}_j=R$ whenever $i\neq j$).