Review sheet 2: Completion

In these we review the basic results about completion of rings and modules. For a more general treatment, we refer to [Mat89, §8]. By way of motivation, let us recall the construction of the ring of p-adic integers, where p is a positive prime integer. One defines a topology on $\mathbb{Z}$ such that two integers are "close" if their difference is divisible by a large power of p; in other words, a basis of neighborhoods of $m \in \mathbb{Z}$ is given by $(m+p^n\mathbb{Z})_{n\geq 1}$. The topology comes from a metric space structure, but the choice of metric is not important. The ring of p-adic integers $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with respect to this topology. It can be described as the quotient of the set of Cauchy sequences in $\mathbb{Z}$ modulo a suitable equivalence relation; however, algebraically it is more convenient to describe it as

$$\mathbb{Z}_p = \varprojlim_{n\geq 1} \mathbb{Z}/p^n\mathbb{Z}.$$

In what follows we consider a similar construction for rather general rings and modules.

1. Completion with respect to an ideal

In what follows we fix a Noetherian ring A and let I be a fixed ideal in A. Note that for every $n \geq 1$ we have a canonical surjective homomorphism $A/I^{n+1} \to A/I^n$. By taking the inverse limit of these homomorphisms we obtain the *completion* of A with respect to I:

$$\widehat{A} := \varprojlim_{n\geq 1} A/I^n.$$

This is a ring and we have a canonical ring homomorphism $\psi_A\colon A \to \widehat{A}$ that maps $a \in A$ to $(a \bmod I^n)_{n\geq 1}$.

Suppose now that M is an A-module. For every $n \geq 1$, we have a surjective morphism of A-modules $M/I^{n+1}M \to M/I^nM$. The *completion* of M with respect to I is

$$\widehat{M} := \varprojlim_{n\geq 1} M/I^nM.$$

This is an A-module and we have a canonical morphism of A-modules $\psi_M\colon M \to \widehat{M}$ that maps $u \in M$ to $(u \bmod I^nM)_{n\geq 1}$. In fact, since each M/I^nM is an A/I^n-module, we have a natural $\widehat{A}$-module structure on $\widehat{M}$ that induces, by restriction of scalars via ψ_A, the original A-module structure on $\widehat{M}$.

If $\phi\colon M \to N$ is a morphism of A-modules, we obtain an induced morphism of $\widehat{A}$-modules $\widehat{M} \to \widehat{N}$. This gives a functor from A-modules to $\widehat{A}$-modules.

Example 1.1. If $A = R[x_1, \ldots, x_n]$ for some Noetherian ring R, and $I = (x_1, \ldots, x_n)$, then $\widehat{A}$ is isomorphic, as an A-algebra, to $R[\![x_1, \ldots, x_n]\!]$.

Remark 1.2. If we have a sequence of submodules $(M_n)_{n\geq 1}$ of M such that $M_{n+1} \subseteq M_n$ for every $n \geq 1$, then we have canonical morphisms $M/M_{n+1} \to M/M_n$ and we can consider $\varprojlim_{n\geq 1} M/M_n$. If $I^n M \subseteq M_n$ for every n, then we have an induced morphism

$$\widehat{M} \to \varprojlim_{n\geq 1} M/M_n. \tag{1}$$

If, in addition, for every n we can find ℓ such that $M_\ell \subseteq I^n M$, then (1) is an isomorphism (this follows easily using the fact that the inverse limit does not change if we pass to a final subset).

In particular, we see that the completion of M with respect to two ideals I and J are canonically isomorphic if $\mathrm{rad}(I) = \mathrm{rad}(J)$.

Remark 1.3. If there is n such that $I^n M = 0$, then it is clear that the morphism $M \to \widehat{M}$ is an isomorphism.

Remark 1.4. By definition, the kernel of the morphism $\psi_M \colon M \to \widehat{M}$ is equal to $\bigcap_{n\geq 1} I^n M = 0$. We thus see that if $(A, \mathfrak{m})$ is a local Noetherian ring, $I \subseteq \mathfrak{m}$, and M is a finitely generated A-module, then ψ_M is injective by Krull's Intersection theorem.

Remark 1.5. If $\phi \colon A \to B$ is a ring homomorphism and $I \subseteq A$ and $J \subseteq B$ are ideals such that $I \cdot B \subseteq J$, then we have a ring homomorphism $\widehat{\phi} \colon \widehat{A} \to \widehat{B}$ such that $\widehat{\phi} \circ \psi_A = \psi_B \circ \phi$ (where the completions of A and B are taken with respect to I and J, respectively). Indeed, for every n, we have an induced homomorphism $A/I^n \to B/J^n$, and by taking the inverse limit over n, we get the morphism $\widehat{\phi}$ that satisfies the required commutativity condition.

2. Basic properties of completion

We now derive some properties of $\widehat{A}$ and of the completion functor. We assume that A is a Noetherian ring and I is an ideal in A.

Proposition 2.1. *Given a short exact sequence of finitely generated A-modules*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0,$$

the induced sequence

$$0 \longrightarrow \widehat{M'} \longrightarrow \widehat{M} \longrightarrow \widehat{M''} \longrightarrow 0$$

is exact, too.

Proof. For every $n \geq 1$, we have an induced exact sequence of A/I^n-modules

$$0 \to M'/(I^n M \cap M') \to M/I^n M \to M''/I^n M'' \to 0.$$

A well-known (and easy to check) property of inverse limits implies that by passing to inverse limit we obtain an exact sequence

$$0 \to \varprojlim_{n\geq 1} M'/(I^n M \cap M') \to \widehat{M} \to \widehat{M''}.$$

Note first that we have a canonical morphism

$$\widehat{M'} = \varprojlim_{n\geq 1} M'/I^nM' \to \varprojlim_{n\geq 1} M'/(I^nM \cap M')$$

and we deduce from the Artin-Rees lemma and Remark 1.2 that this is an isomorphism.

In order to complete the proof it is thus enough to show that the morphism $\widehat{M} \to \widehat{M''}$ is surjective. Consider $u \in \widehat{M''}$ given by $(u_n \bmod I^nM)_{n\geq 1}$, where the elements $u_n \in M''$ are such that $u_n - u_{n+1} \in I^nM''$. We construct recursively elements $v_n \in M$ such that the following hold for all $n \geq 1$:

i) $u_n = \beta(v_n)$ and
ii) $v_n - v_{n+1} \in I^nM$.

We begin by choosing $v_1 \in M$ such that $\beta(v_1) = u_1$ (this is possible since β is surjective). Suppose now that $v_1, \ldots, v_r$ are chosen such that i) holds for $1 \leq n \leq r$ and ii) holds for $1 \leq n \leq r-1$. Since $u_r - u_{r+1} \in I^rM''$, we can write $u_r - u_{r+1} = \sum_{j=1}^{s} a_jw_j$, with $a_j \in I^r$ and $w_j \in M''$. We choose $\widetilde{w}_j \in M$ such that $\beta(\widetilde{w}_j) = w_j$ and put $v_{r+1} = v_r - \sum_{j=1}^{s} a_j\widetilde{w}_j$. It is then clear that i) holds also for $n = r+1$ and ii) holds also for $n = r$. By ii), we can thus consider $v = (v_n \bmod I^nM)_{n\geq 1} \in \widehat{M}$ and it follows from i) that v maps to $u \in \widehat{M''}$. This completes the proof of the proposition. □

Corollary 2.2. *For every finitely generated A-module M, the canonical morphism*

$$\widehat{A} \otimes_A M \to \widehat{M}$$

induced by ψ_M is an isomorphism. In particular, $\widehat{M}$ is a finitely generated $\widehat{A}$-module.

Proof. The assertion is clear if M is a finitely generated, free A-module. For the general case, consider an exact sequence of A-modules

$$F_1 \to F_0 \to M \to 0,$$

where F_1 and F_0 are finitely generated, free modules. We then obtain a commutative diagram

$$\begin{array}{ccccccc}
\widehat{A} \otimes_A F_1 & \longrightarrow & \widehat{A} \otimes_A F_0 & \longrightarrow & \widehat{A} \otimes_A M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\widehat{F_1} & \longrightarrow & \widehat{F_0} & \longrightarrow & \widehat{M} & \longrightarrow & 0.
\end{array}$$

The top row is exact by right-exactness of the tensor product, while the bottom row is exact by the proposition. Since the first and the second vertical maps are isomorphisms, it follows that the third one is an isomorphism as well. □

Corollary 2.3. *The A-algebra $\widehat{A}$ is flat.*

Proof. We need to show that for every injective morphism of A-modules $\phi\colon M' \hookrightarrow M$, the induced morphism

$$1_{\widehat{A}} \otimes \phi\colon \widehat{A} \otimes_A M' \to \widehat{A} \otimes_A M$$

is injective. We can write $M = \varinjlim_{i\in I} M_i$, where the M_i are the finitely generated submodules of M and similarly $M' = \varinjlim_{i\in I} \phi^{-1}(M_i)$. Since the tensor product commutes with direct limits and a filtered direct limit of injective morphisms is injective, we see that it is enough to consider the case when M (and thus also M') is finitely generated. In this case, the assertion follows by combining the proposition and the previous corollary. $\square$

Corollary 2.4. *For every $n \geq 1$ and every finitely generated A-module M, we have*

$$\widehat{I^n M} = I^n \widehat{M} = \widehat{I}^n \widehat{M}.$$

Moreover, the morphism $M \to \widehat{M}$ induces an isomorphism $M/I^n M \to \widehat{M}/I^n\widehat{M}$.

Proof. Since $\widehat{A}$ is flat over A, the canonical morphism $\widehat{A} \otimes_A I^n \to \widehat{A}$ is injective; its image is $I^n\widehat{A} = (I\widehat{A})^n$. Moreover, by Proposition 2.1 and Corollary 2.2, this is also the image of the morphism $\widehat{I^n} \to \widehat{A}$, which is injective. By taking $n = 1$, we see that $I\widehat{A} = \widehat{I}$, and thus

$$\widehat{I^n} = I^n\widehat{A} = \widehat{I}^n.$$

Given the finitely generated A-module M, by applying Proposition 2.1 to the exact sequence

$$0 \to I^n M \to M \to M/I^n M \to 0,$$

we obtain an exact sequence

$$0 \to \widehat{I^n M} \to \widehat{M} \xrightarrow{p} \widehat{M/I^n M} \to 0.$$

Note also that we have an isomorphism

$$M/I^n M \simeq \widehat{M/I^n M}$$

such that p gets identified to the canonical projection $\widehat{M} \to M/I^n M$ that comes from the definition of the projective limit (see Remark 1.3). On one hand, it follows from Corollary 2.2 that

$$\widehat{I^n M} = \mathrm{Im}(\widehat{A} \otimes_A I^n M \to \widehat{A} \otimes_A M = \widehat{M}) = I^n\widehat{M}.$$

On the other hand, it follows from what we have already proved that

$$I^n\widehat{M} = (I^n\widehat{A}) \cdot \widehat{M} = \widehat{I}^n\widehat{M}.$$

This completes the proof of the proposition. $\square$

Given a ring A, an ideal I in A, and an A-module M, we say that M is *complete* (with respect to I) if the canonical morphism $M \to \widehat{M}$ is an isomorphism. This applies, in particular, in the case $M = A$.

Example 2.5. Given a Noetherian ring A and a finitely generated A-module M, it follows from Corollary 2.4 that $\widehat{M}$ is complete as an A-module (with respect to I) and as an $\widehat{A}$-module (with respect to $\widehat{I} = I\widehat{A}$).

Remark 2.6. Let $\phi\colon A \to B$ be a ring homomorphism and $I \subseteq A$ and $J \subseteq B$ be ideals such that $I \cdot B \subseteq J$. If $\psi_A\colon A \to \widehat{A}$ is the morphism to the completion (with respect to I) and if B is complete (with respect to J), then there is a unique ring homomorphism $\rho\colon \widehat{A} \to B$ such that $\rho \circ \psi_A = \phi$.

Indeed, recall from Remark 1.5 that we have a homomorphism $\widehat{\phi}\colon \widehat{A} \to \widehat{B}$ such that $\widehat{\phi} \circ \psi_A = \psi_B \circ \phi$. Since ψ_B is an isomorphism by assumption, we may take $\rho = \psi_B^{-1} \circ \widehat{\phi}$ and this clearly satisfies the required condition.

In order to prove uniqueness, note that if $\widehat{\phi}$ is as in the statement, then $\widehat{\phi}$ induces for every n a morphism $\phi_n\colon \widehat{A}/I^n \cdot \widehat{A} \to B/J^nB$, whose composition with the isomorphism $A/I^n \to \widehat{A}/I^n\widehat{A}$ is the morphism $A/I^n \to B/J^n$ induced by ϕ. Since we have $\psi_B \circ \widehat{\phi} = \varprojlim_{n\geq 1} \phi_n$, we obtain the uniqueness of $\widehat{\phi}$.

Remark 2.7. If A is Noetherian and I is an ideal in A, then the completion $\widehat{A}$ is again Noetherian. We do not use this property, so we refer to [Mat89, Theorem 8.12] for the proof.

Remark 2.8. An important case is that when $(A, \mathfrak{m})$ is a local Noetherian ring and $I = \mathfrak{m}$. Note that in this case the morphism $\psi_A\colon A \to \widehat{A}$ is injective (see Remark 1.4). Note also that $\mathfrak{m}\widehat{A}$ is a maximal ideal, with $\widehat{A}/\mathfrak{m}\widehat{A} \simeq A/\mathfrak{m}$ (see Corollary 2.4).

In fact, $\mathfrak{m}\widehat{A}$ is the unique maximal ideal of $\widehat{A}$. In order to see this, it is enough to show that if $u \in \widehat{A} \smallsetminus \mathfrak{m}\widehat{A}$, then $1-u$ is invertible. This follows from the fact that $\widehat{A} \simeq \varprojlim_{n\geq 1} \widehat{A}/(\mathfrak{m}\widehat{A})^n$ and if we put $a_n = \sum_{j=0}^{n-1} u^j$ for every $n \geq 1$, then the element in $\widehat{A}$ corresponding to $(a_n)_{n\geq 1}$ is an inverse of $1-u$.

Remark 2.9. We did not mention the topology on the ring A associated to the ideal I, since we do not need it. However, for the interested reader, we mention the notions of Cauchy sequences and convergent sequences that come out of the topological considerations. Given a ring A, the ideal I in A, and an A-module M, we say that a sequence $(x_n)_{n\geq 1}$ of elements in M is a *Cauchy sequence* if for every m, there is N such that $x_n - x_{n+1} \in I^mM$ for all $n \geq N$. The sequence has a *limit* $x \in M$ if for every m, there is N such that $x_n - x \in I^mM$ for all $n \geq N$. One can show that M is complete if and only if every Cauchy sequence in M has a limit and this limit is unique. We leave the proof as an exercise for the reader.

References

[Mat89] H. Matsumura, Commutative ring theory, second edition, Cambridge Studies in Advanced Mathematics, 8, Cambridge University Press, Cambridge, 1989. 1, 5