Review sheet 1: Associated primes and primary decomposition

We give a brief treatment of associated primes and primary decomposition.

1. Associated primes and zero-divisors

When dealing with associated primes, it is convenient to work more generally with modules, instead of just with the ring itself. Let us fix a Noetherian ring R.

Definition 1.1. If M is a finitely generated R-module, an *associated prime* of M is a prime ideal $\mathfrak{p}$ in R such that

$$\mathfrak{p} = \mathrm{Ann}_R(u) \quad \text{for some} \quad u \in M, u \neq 0.$$

The set of associated primes of M is denoted $\mathrm{Ass}(M)$ (we write $\mathrm{Ass}_R(M)$ if the ring is not understood from the context).

Recall that if M is an R-module, an element $a \in R$ is a *zero-divisor* of M if $au = 0$ for some $u \in M \smallsetminus \{0\}$; otherwise a is a *non-zero-divisor* of M. Note that for $M = R$, we recover the usual notion of zero-divisor in R. The third assertion in the next proposition is the main reason why associated primes are important:

Proposition 1.2. *If M is a finitely generated R-module, then the following hold:*

i) *The set $\mathrm{Ass}(M)$ is finite.*
ii) *If $M \neq 0$, then $\mathrm{Ass}(M)$ is non-empty.*
iii) *The set of zero-divisors of M is equal to*

$$\bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}.$$

We begin with the following easy lemma:

Lemma 1.3. *Given an exact sequence of R-modules*

$$0 \to M' \to M \to M'' \to 0,$$

we have

$$\mathrm{Ass}(M') \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(M') \cup \mathrm{Ass}(M'').$$

Proof. The first inclusion is obvious, hence we only prove the second one. Suppose that $\mathfrak{p} \in \mathrm{Ass}(M)$, and let us write $\mathfrak{p} = \mathrm{Ann}_R(u)$, for some nonzero $u \in M$. If $u \in M'$, then clearly $\mathfrak{p} \in \mathrm{Ass}(M')$. Otherwise, the image $\overline{u}$ of u in M'' is non-zero and it is clear that $\mathfrak{p} \subseteq \mathrm{Ann}_R(\overline{u})$. If this is an equality, then $\mathfrak{p} \in \mathrm{Ass}_R(M'')$, hence let us assume that there is $a \in \mathrm{Ann}_R(\overline{u}) \smallsetminus \mathfrak{p}$. In this case $au \in M' \smallsetminus \{0\}$, and the fact that $\mathfrak{p}$ is prime implies that the obvious inclusion $\mathrm{Ann}_R(u) \subseteq \mathrm{Ann}_R(au)$ is an equality. Therefore $\mathfrak{p} \in \mathrm{Ass}(M')$. $\square$

Proof of Proposition 1.2. We may assume that M is nonzero, as otherwise all assertions are trivial. Consider the set $\mathcal{P}$ consisting of the ideals of R of the form $\mathrm{Ann}_R(u)$, for some $u \in M \smallsetminus \{0\}$. Since R is Noetherian, there is a maximal element $\mathfrak{p} \in \mathcal{P}$. We show that in this case $\mathfrak{p}$ is a prime ideal, so that $\mathfrak{p} \in \mathrm{Ass}(M)$.

By assumption, we can write $\mathfrak{p} = \mathrm{Ann}_R(u)$, for some $u \in M \smallsetminus \{0\}$. Since $u \neq 0$, we have $\mathfrak{p} \neq R$. If $b \in R \smallsetminus \mathfrak{p}$, then $bu \neq 0$ and we clearly have

$$\mathrm{Ann}_R(u) \subseteq \mathrm{Ann}_R(bu).$$

By the maximality of $\mathfrak{p}$, we conclude that this is an equality, hence for every $a \in R$ such that $ab \in \mathfrak{p}$, we have $a \in \mathfrak{p}$.

In particular, this proves ii). We thus know that if M is non-zero, then we can find $u \in M \smallsetminus \{0\}$ such that $\mathrm{Ann}_R(u) = \mathfrak{p}_1$ is a prime ideal. The map $R \to M$, $a \to au$ induces thus an injection $R/\mathfrak{p} \hookrightarrow M$, so that we have a short exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0,$$

with $M_1 \simeq R/\mathfrak{p}_1$. Note now that since $\mathfrak{p}_1$ is a prime ideal in R, then we clearly have $\mathrm{Ass}(R/\mathfrak{p}_1) = \{\mathfrak{p}_1\}$, and the lemma implies

$$\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M/M_1) \cup \{\mathfrak{p}\}.$$

Therefore in order to prove that $\mathrm{Ass}(M)$ is finite it is enough to show that $\mathrm{Ass}(M/M_1)$ is finite. If $M_1 \neq 0$, we can repeat this argument and find $M_1 \subseteq M_2$ such that $M_2/M_1 \simeq R/\mathfrak{p}_2$, for some prime ideal $\mathfrak{p}_2$ in R. Since M is a Noetherian module, this process must terminate, hence after finitely many steps we conclude that $\mathrm{Ass}_R(M)$ is finite.

We now prove the assertion in iii). It is clear from definition that for every $\mathfrak{p} \in \mathrm{Ass}(M)$, the ideal $\mathfrak{p}$ is contained in the set of zero-divisors of M. On the other hand, if $a \in R$ is a zero-divisor, then $a \in I$, for some $I \in \mathcal{P}$. If we choose a maximal $\mathfrak{p}$ in $\mathcal{P}$ that contains I, then we have seen that $\mathfrak{p} \in \mathrm{Ass}_R(M)$, hence a lies in the union of the associated primes of M. This completes the proof of the proposition. $\square$

We record in the next corollary a useful assertion that we obtained in the above proof.

Corollary 1.4. *If M is a finitely generated R-module, then there is a sequence of submodules*

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$$

such that $M_i/M_{i-1} \simeq R/\mathfrak{p}_i$ for $1 \leq i \leq r$, where each $\mathfrak{p}_i$ is a prime deal in R.

Remark 1.5. The results in Proposition 1.2 are often applied as follows: if an ideal I in R has no non-zero-divisors on M, then it is contained in the union of the associated primes. Since there are finitely such prime ideals, the Prime Avoidance lemma implies that I is contained in one of them. Therefore there is $u \in M$ non-zero such that $I \cdot u = 0$.

Remark 1.6. If M is a finitely generated R-module, then for every multiplicative system S in R, if we consider the finitely generated $S^{-1}R$-module $S^{-1}M$, we have

$$\mathrm{Ass}_{\mathrm{S^{-1}R}}(S^{-1}M) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}(M), S \cap \mathfrak{p} \neq 0\}.$$

Indeed, if $\mathfrak{p} = \mathrm{Ann}_R(u)$ and $\mathfrak{p} \cap S = \emptyset$, then $S^{-1}\mathfrak{p} = \mathrm{Ann}_{S^{-1}R}\left(\frac{u}{1}\right)$. Conversely, if $S^{-1}\mathfrak{p} = \mathrm{Ann}_{S^{-1}R}\left(\frac{v}{s}\right)$, for some prime ideal $\mathfrak{p}$ in R, with $\mathfrak{p} \cap S = \emptyset$, then it is easy to see that $\mathfrak{p} = \mathrm{Ann}_R(v)$.

Remark 1.7. Let M be a finitely generated R-module and $I = \mathrm{Ann}_R(M)$. It is clear from definition that if $\mathfrak{p} \in \mathrm{Ass}(M)$, then $I \subseteq \mathfrak{p}$. Moreover, we have

$$\mathrm{Ass}_{R/I}(M) = \{\mathfrak{p}/I \mid \mathfrak{p} \in \mathrm{Ass}_R(M)\}.$$

We recall the easy fact that since M is a finitely generated R-module, for every prime ideal $\mathfrak{p}$ in R, we have $M_{\mathfrak{p}} \neq 0$ if and only if $I \subseteq \mathfrak{p}$. We note that every prime ideal in R that contains I and is minimal with this property lies in $\mathrm{Ass}_R(M)$. Indeed, the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is nonzero, hence $\mathrm{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ is non-empty by Proposition 1.2. However, there is a unique prime ideal in $R_{\mathfrak{p}}$ that contains $\mathrm{Ann}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = I_{\mathfrak{p}}$, namely $\mathfrak{p}R_{\mathfrak{p}}$. Using again the previous remark, we see that $\mathfrak{p} \in \mathrm{Ass}_R(M)$. The primes in $\mathrm{Ass}_R(M)$ that are not minimal over $\mathrm{Ann}_R(M)$ are called *embedded primes*.

Example 1.8. If I is a radical ideal in R, then it is easy to see that the set of zero-divisors in R/I is the union of the minimal prime ideals containing I. We deduce using Proposition 1.2 and the Prime Avoidance lemma that every $\mathfrak{p} \in \mathrm{Ass}_R(R/I)$ is a minimal prime containing I.

2. Primary decomposition

We discussing primary decomposition and its connection to associated primes. Since we will only need this for ideals, for the sake of simplicity, we stick to this case.

Definition 2.1. An ideal $\mathfrak{q}$ in R is *primary* if whenever $a, b \in R$ are such that $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$, then $b \in \mathrm{rad}(\mathfrak{q})$. It is straightforward to see that in this case $\mathfrak{p} := \mathrm{rad}(\mathfrak{q})$ is a prime ideal; one also says that $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal. A *primary decomposition* of an ideal I is an expression

$$I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n,$$

where all $\mathfrak{q}_i$ are primary ideals.

Remark 2.2. It follows from definition that if $I \subseteq \mathfrak{q}$ are ideals in R, then $\mathfrak{q}/I$ is a primary ideal in R/I if and only if $\mathfrak{q}$ is a primary ideal in R.

Proposition 2.3. *If $\mathfrak{q}$ is an ideal in R, then $\mathfrak{q}$ is a primary ideal if and only if $\mathrm{Ass}_R(R/\mathfrak{q})$ has only one element. Moreover, in this case the only associated prime of $R/\mathfrak{q}$ is $\mathrm{rad}(\mathfrak{q})$.*

Proof. Suppose first that $\mathfrak{q}$ is $\mathfrak{p}$-primary. Note that $\mathfrak{p}$ is the only minimal prime containing $\mathfrak{q}$, hence $\mathfrak{p} \in \mathrm{Ass}(R/\mathfrak{q})$ by Remark 1.7. On the other hand, since $\mathfrak{q}$ is $\mathfrak{p}$-primary, it follows that every zero-divisor of $R/\mathfrak{q}$ lies in $\mathfrak{p}$. Since the set of zero-divisors of $R/\mathfrak{q}$ is the union of the associated primes of $R/\mathfrak{q}$ by Proposition 1.2, and each of these associated primes contains $\mathrm{Ann}_R(R/\mathfrak{q}) = \mathfrak{q}$, we conclude that $\mathfrak{p}$ is the only element of $\mathrm{Ass}_R(R/\mathfrak{q})$.

Conversely, suppose that $\mathrm{Ass}_R(R/\mathfrak{q})$ has only one element $\mathfrak{p}$. In this case, it follows from Remark 1.7 that $\mathfrak{p}$ is the unique minimal prime containing $\mathfrak{q}$, hence $\mathfrak{p} = \mathrm{rad}(\mathfrak{p})$.

Moreover, it follows from Proposition 1.2 that the set of non-zero-divisors of $R/\mathfrak{q}$ is equal to $\mathfrak{p}$, which implies, by definition, that $\mathfrak{q}$ is a primary ideal. $\square$

Proposition 2.4. *Every ideal I in R has a primary decomposition.*

Proof. After replacing R by R/I, we may assume that $I = 0$. We claim that for every $\mathfrak{p} \in \mathrm{Ass}(R)$, there is a primary ideal $\mathfrak{q}$ in R such that $\mathfrak{p} \not\in \mathrm{Ass}(\mathfrak{q})$. Indeed, consider the ideals J in R such that $\mathfrak{p} \not\in \mathrm{Ass}(J)$ (the set is non-empty since it contains 0) and since R is Noetherian, we may choose an ideal $\mathfrak{q}$ which is maximal with this property. Note that $\mathfrak{q} \neq R$, hence $\mathrm{Ass}(R/\mathfrak{q})$ is non-empty. By Proposition 2.3, in order to show that $\mathfrak{q}$ is a primary ideal, it is enough to show that for every prime ideal $\mathfrak{p}' \neq \mathfrak{p}$, we have $\mathfrak{p}' \not\in \mathrm{Ass}(R/\mathfrak{q})$. If $\mathfrak{p}' \in \mathrm{Ass}(R/\mathfrak{q})$, then we obtain an ideal $\mathfrak{q}' \supseteq \mathfrak{q}$ such that $\mathfrak{q}'/\mathfrak{q} \simeq R/\mathfrak{p}'$. We assumed $\mathfrak{q}' \neq \mathfrak{q}$, while Lemma 1.3 implies

$$\mathrm{Ass}(\mathfrak{q}') \subseteq \mathrm{Ass}(\mathfrak{q}) \cup \mathrm{Ass}(\mathfrak{q}'/\mathfrak{q}) = \mathrm{Ass}(\mathfrak{q}) \cup \{\mathfrak{p}'\},$$

hence $\mathfrak{p} \not\in \mathrm{Ass}(\mathfrak{q}')$, contradicting the maximality of $\mathfrak{q}$.

We thus conclude that if $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the associated primes of R, we can find primary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ such that $\mathfrak{p}_i \not\in \mathrm{Ass}(\mathfrak{q}_i)$ for all i. If $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$, then $\mathrm{Ass}(\mathfrak{a}) \subseteq \mathrm{Ass}(R)$ and at the same time $\mathrm{Ass}(\mathfrak{a}) \subseteq \mathrm{Ass}(\mathfrak{q}_i)$ for all i, hence $\mathfrak{p}_i \not\in \mathrm{Ass}(\mathfrak{a}_i)$. This implies that $\mathfrak{a}$ has no associated primes, hence $\mathfrak{a} = 0$. $\square$

Remark 2.5. Note that if $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are $\mathfrak{p}$-primary ideals, then $\mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ is a $\mathfrak{p}$-primary ideal. It is thus straightforward to see that given any ideal I and any primary decomposition $I = \mathfrak{q}_1, \ldots \cap \mathfrak{q}_r$, we can obtain a *minimal* such decomposition, in the sense that the following conditions are satisfied:

i) We have $\mathrm{rad}(\mathfrak{q}_i) \neq \mathrm{rad}(\mathfrak{q}_j)$ for all i and j, and
ii) For every i, with $1 \leq i \leq r$, we have $\bigcap_{j \neq i} \mathfrak{q}_j \neq I$.

Given such a reduced primary decomposition, if $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i)$, then $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the distinct associated primes of R/I. Indeed, the injective morphism

$$R/I \hookrightarrow \bigoplus_{i=1}^{r} R/\mathfrak{q}_i$$

implies that $\mathrm{Ass}(R/I) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. On the other hand, for every i, there is $u \in \bigcap_{j \neq i} \mathfrak{q}_j$ such that $u \not\in \mathfrak{q}_i$. Moreover, after multiplying u by a suitable element in $\mathfrak{p}_i^m$, for some non-negative integer m, we may assume that $u \cdot \mathfrak{p}_i \subseteq \mathfrak{q}_i$. In this case, $\mathfrak{p}_i$ is the annihilator of the image of u in R/I, hence $\mathfrak{p}_i \in \mathrm{Ass}(R/I)$.

Remark 2.6. In general, the primary ideals in a minimal primary decomposition of I are not unique. However, if $\mathfrak{p}$ is a minimal prime containing I, then the corresponding $\mathfrak{p}$-primary ideal $\mathfrak{q}$ in a primary decomposition of I is unique. Indeed, it is easy to check that $I \cdot R_{\mathfrak{p}} = \mathfrak{q} \cdot R_{\mathfrak{p}}$ and deduce, using that $\mathfrak{q}$ is $\mathfrak{p}$-primary, that $\mathfrak{q} = I \cdot R_{\mathfrak{p}} \cap R$.