Problem session 9

The goal of the first problem is to compute the cohomology of projective hypersurfaces, and more generally, to compute the cohomology of $\mathcal{O}_D(m)$, where D is an effective Cartier divisor on $\mathbb{P}^n$. We begin by setting up some notation.

Let $n \geq 1$ and let D be an effective Cartier divisor on $\mathbb{P}^n$. Recall that we have an isomorphism $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$ that maps $\mathcal{O}_{\mathbb{P}^n}(m)$ to m (see Example 9.3.4 in the notes). Suppose that D *has degree* d, that is, $\mathcal{O}_{\mathbb{P}^n}(D) \simeq \mathcal{O}_{\mathbb{P}^n}(d)$; equivalently, if $D = \sum_{i=1}^r a_i D_i$, where D_i is an irreducible hypersurface in $\mathbb{P}^n$ of degree d_i, then $d = \sum_{i=1}^r a_i d_i$.

Recall that by Proposition 9.4.24 in the notes, effective Cartier divisors of degree d on $\mathbb{P}^n$ are in bijection with sections of $\Gamma\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)\big) \simeq S_d$, up to multiplication by non-zero elements of k, where $S = k[x_0, \ldots, x_n]$. We write $f_D \in S_d$ for such a polynomial corresponding to D. Note that if D is a hypersurface in $\mathbb{P}^n$ (that is, it is a reduced divisor, that we identify with its support), then f_D is a generator for the principal radical ideal corresponding to this hypersurface. In general, if $D = \sum_{i=1}^r a_i D_i$, where D_i is an irreducible hypersurface in $\mathbb{P}^n$, with corresponding radical ideal generated by f_i, then we can take $f_D = \prod_{i=1}^r f_i^{a_i}$.

Problem 1. Let $n \geq 2$ and let D be an effective Cartier divisor of degree d on $\mathbb{P}^n$. The goal is to compute $H^i\big(\mathbb{P}^n, \mathcal{O}_D(m)\big)$ for all m.

i) Show that
$$H^i\big(\mathbb{P}^n, \mathcal{O}_D(m)\big) = 0 \quad \text{for} \quad 1 \leq i \leq n-2, m \in \mathbb{Z}.$$

ii) Show that
$$H^0\big(\mathbb{P}^n, \mathcal{O}_D(m)\big) \simeq (S/Sh)_m,$$
where $S = k[x_0, \ldots, x_n]$ and $h \in S_d$ is an equation defining D. In particular, we have
$$\dim_k H^0\big(\mathbb{P}^n, \mathcal{O}_D(m)\big) = \binom{m+n}{n} - \binom{m+n-d}{n} \quad \text{for} \quad m \geq 0,$$
with the convention that the second binomial coefficient is 0 for $m < d$.

iii) Finally, show that for every m, we have an exact sequence
$$0 \to H^{n-1}\big(\mathbb{P}^n, \mathcal{O}_D(m)\big) \to H^n\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m-d)\big) \to H^n\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)\big) \to 0.$$
In particular, we have
$$H^{n-1}\big(\mathbb{P}^n, \mathcal{O}_D(m)\big) = 0 \quad \text{for} \quad m \geq -n+d, \quad \text{and}$$
$$H^{n-1}\big(\mathbb{P}^n, \mathcal{O}_D(d-n-1)\big) \simeq k.$$

We have seen in class that if X is a projective variety, then for every quasi-coherent sheaf $\mathcal{F}$ on X, we have $\dim_k H^i(X, \mathcal{F}) < \infty$ for all i. We will see next time that the same holds if we only assume that X is complete. The *geometric genus* of a smooth complete variety X is given by
$$p_g(X) = \dim_k \Gamma(X, \omega_X).$$

Problem 2. Show that the geometric genus is a birational invariant: if X and Y are birational smooth complete varieties, then $p_g(X) = p_g(Y)$.

Problem 3. Compute the geometric genus of a smooth hypersurface in $\mathbb{P}^n$, with $n \geq 2$.