Problem session 9

We consider a projective space $\mathbb{P}^n$ and let S be its homogeneous coordinate ring. Recall that a hypersurface in $\mathbb{P}^n$ is a closed subvariety of $\mathbb{P}^n$ whose corresponding radical homogeneous ideal is of the form (F), for some nonzero homogeneous polynomial of positive degree. If $\deg(F) = d$, then the hypersurface has *degree d*.

Note that two polynomials F and G define the same hypersurface if and only if there is $\lambda \in k^*$ such that $F = \lambda G$. Let $\mathbb{P}^{N_d}$ be the projective space parametrizing lines in the vector space S_d, hence $N_d = \binom{n+d}{n} - 1$.

Problem 1. Show that the subset $\mathcal{H}_d$ of $\mathbb{P}^{N_d}$ consisting of those $[F]$ such that the ideal (F) is radical is a non-empty open subset of $\mathbb{P}^{N_d}$.

Problem 2. Consider the incidence correspondence

$$\mathcal{Z}_d := \left\{ \big(p, [F]\big) \in \mathbb{P}^n \times \mathbb{P}^{N_d} \mid F(p) = 0 \right\}.$$

Show that $\mathcal{Z}$ is an irreducible closed subvariety of $\mathbb{P}^n \times \mathbb{P}^{N_d}$, of dimension $N_d + n - 1$.

The next goal is to discuss linear subspaces on projective hypersurfaces. Given $r < n$, let $G = G(r+1, n+1)$ be the Grassmann variety parametrizing the r-dimensional linear subspaces in $\mathbb{P}^n$. Consider the incidence correspondence $I \subseteq \mathbb{P}^{N_d} \times G$ consisting of pairs $\big([F], [\Lambda]\big)$ such that F vanishes on Λ.

Problem 3. Show that I is a closed subset of $\mathbb{P}^{N_d} \times G$.

Problem 4. Show that the projective variety I is irreducible, of dimension

$$(r+1)(n-r) + \binom{n+d}{d} - \binom{r+d}{d} - 1.$$

Problem 5. Show that every hypersurface in $\mathbb{P}^3$ defined as the zero-locus of a degree 3 polynomial, contains a line.