Problem session 8

Problem 1. Show that if $U = \mathbf{A}^2 \smallsetminus \{(0,0)\}$, then we have an isomorphism

$$H^1(U, \mathcal{O}_U) \simeq \bigoplus_{i,j<0} kx^i y^j.$$

In particular, $H^1(U, \mathcal{O}_U)$ has infinite dimension over k.

Problem 2.

i) Show that if X is an algebraic variety and $Z \subseteq X$ is a closed subset with $\dim(Z) = d$, then there are finitely many affine open subsets $(U_i)_{i \in I}$ in X such that $Z \subseteq \bigcup_{i \in I} U_i$ and for every distinct $i_0, \ldots, i_{d+1} \in I$, we have $Z \cap U_{i_0} \cap \ldots \cap U_{i_{d+1}} = \emptyset$.

ii) Deduce that if X is an algebraic variety of dimension n, then for every quasi-coherent sheaf $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}) = 0$ for all $i > n$. Similarly, if $\mathcal{F}$ is a coherent sheaf on X with $\dim\big(\mathrm{Supp}(\mathcal{F})\big) = r$, then $H^i(X, \mathcal{F}) = 0$ for all $i > r$.

Problem 3.

i) Give an argument that does not involve spectral sequences for the following assertion: if $f\colon X \to Y$ is an affine morphism of algebraic varieties, then for every quasi-coherent sheaf $\mathcal{F}$ on X, we have

$$H^i\big(Y, f_*(\mathcal{F})\big) \simeq H^i(X, \mathcal{F}) \quad \text{for every} \quad i \geq 0.$$

ii) More generally, show that given two morphisms of algebraic varieties $f\colon X \to Y$ and $g\colon Y \to Z$ and a quasi-coherent sheaf $\mathcal{F}$ on X, if f is an affine morphism, then we have isomorphisms

$$R^p g_*\big(f_*(\mathcal{F})\big) \simeq R^p(g \circ f)_*(\mathcal{F}) \quad \text{for all} \quad p \geq 0.$$

If we assume instead that g is affine, then we have isomorphisms

$$g_*\big(R^p f_*(\mathcal{F})\big) \simeq R^p(g \circ f)_*(\mathcal{F}) \quad \text{for all} \quad p \geq 0.$$

Problem 4. Suppose that we have a spectral sequence

$$E_2^{p,q} \Rightarrow_p \mathcal{H}^{p+q}.$$

i) Show that if $E_2^{p,q} = 0$ whenever $p < 0$, then for every n we have canonical morphisms

$$\mathcal{H}^n \to E_\infty^{0,n} \to E_2^{0,n},$$

where the first morphism is surjective and the second one is injective.

ii) Similarly, if $E_2^{p,q} = 0$ whenever $q < 0$, then for every n we have canonical morphisms

$$E_2^{n,0} \to E_\infty^{n,0} \to \mathcal{H}^n,$$

where the first morphism is surjective and the second one is injective.

iii) For example, we deduce from the Leray spectral sequence that if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of ringed spaces and $\mathcal{F}$ is an $\mathcal{O}_X$-module on X, then we have canonical morphisms

$$R^n g_*\big(f_*(\mathcal{M})\big) \to R^n(g \circ f)_*(\mathcal{M}) \to g_*\big(R^n f_*(\mathcal{M})\big).$$