Problem session 8

Problem 1. The Grassmann variety $G(r,n)$ parametrizes r-dimensional linear subspaces in $V = k^{n+1}$. If W is such a linear subspace, we denote by $[W]$ the corresponding point of $G(r,n)$. Put a structure of algebraic prevariety on $G(r,n)$, as follows.

i) Show that by fixing a basis on V (e.g. the standard basis), we can identify $G(r,n)$ with the quotient of the set of $r \times n$-matrices with entries in k, of maximal rank, modulo the action of $GL_r(k)$, given by left multiplication.
ii) Show that by choosing nonzero maximal minors of this matrix, we can cover $G(r,n)$ by subsets that are in bijection with $\mathbb{A}^{r(n-r)}$. Show that one can put a structure of prevariety on $G(r,n)$ such that these bijections are isomorphisms. Note that $G(r,n)$ is irreducible, of dimension $r(n-r)$.

Problem 2. Show that $G(r,n)$ is a projective variety by showing that the map $G(r,n) \to \mathbb{P}^{\binom{n}{r}-1}$ that maps a subspace W to the line $\wedge^r W$ of $\wedge^r V \simeq k^{\binom{n}{r}}$ is a closed immersion. This is the *Plücker* embedding of $G(r,n)$.

Remarks.

i) If $r = 1$, we recover $\mathbb{P}^{n-1}$, while if $r = n-1$, we recover the "dual" projective space, parametrizing the hyperplanes in $\mathbb{P}^{n-1}$.
ii) In general, we have an isomorphism $G(r,n) \simeq G(n-r,n)$ that maps a subspace $W \subseteq V = k^n$ to the kernel of $k^n \simeq V^* \to W^*$.
iii) The group of linear automorphisms $GL_n(V)$ has an induced transitive action on $G(r,n)$.
iv) Note that the set of r-dimensional linear subspaces of $\mathbb{P}^n$ can be identified to $G(r+1,n+1)$.

Problem 3. With the above identification, consider the following *incidence correspondence*:

$$\mathcal{Z} = \left\{ \big(q,[V]\big) \in \mathbb{P}^n \times G \mid q \in V \right\},$$

where $G = G(r+1,n+1)$ is the Grassman variety parametrizing r-dimensional linear subspaces in $\mathbb{P}^n$.

i) Show that $\mathcal{Z}$ is a closed subset of $\mathbb{P}^n \times G$.
ii) Consider the morphisms $\pi\colon \mathcal{Z} \to \mathbb{P}^n$ and $\pi_2\colon \mathcal{Z} \to G$ induced by the projections onto the two components. Describe the fibers of these two morphisms and use this to show that $\mathcal{Z}$ is irreducible, with $\dim(\mathcal{Z}) = r + (r+1)(n-r)$.

Problem 4. Given a closed subset X in $\mathbb{P}^n$, let

$$M_r(X) = \{[V] \in G = G(r+1,n+1) \mid V \cap X \neq \emptyset\}.$$

i) Show that $M_r(X)$ is a closed subset of G.
ii) Show that $M_r(X)$ is irreducible if X is irreducible.
iii) Show that $\operatorname{codim}_G\big(M_r(X)\big) = n - r - d$ for $0 \leq r \leq n - d$.