Problem session 6

Problem 1. Show that if G is an irreducible linear algebraic group acting on a variety X, then every irreducible component of X is invariant under the G-action.

Problem 2. Show that if X is a connected, complete variety, then $\Gamma(X, \mathcal{O}_X) = k$. Deduce that a complete variety is also affine if and only if it is a finite set of points.

Problem 3. Show that if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of algebraic varieties, with $g \circ f$ proper, then f is proper.

Problem 4. Let $X \subseteq \mathbb{P}^n$ be a closed subvariety and $x \in X$ corresponds to the homogeneous prime ideal $\mathfrak{p} \subseteq S = k[x_0, \ldots, x_n]$. Show that there is a canonical isomorphism

$$\mathcal{O}_{X,x} \simeq S_{(\mathfrak{q})}.$$

Problem 5. Thinking of $\mathbb{P}^{n-1}$ as the set of lines in $\mathbb{A}^n$, define the *blow-up of* $\mathbb{A}^n$ *at 0* as the set

$$\mathrm{Bl}_0(\mathbb{A}^n) := \{(P, \ell) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid P \in \ell\}.$$

1) Show that $\mathrm{Bl}_0(\mathbb{A}^n)$ is a closed subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$.
2) Show that the restriction of the projection onto the first component gives a morphism $\pi\colon \mathrm{Bl}_0(\mathbb{A}^n) \to \mathbb{A}^n$ that is an isomorphism over $\mathbb{A}^n \smallsetminus \{0\}$.
3) Show that $\pi^{-1}(0) \simeq \mathbb{P}^{n-1}$.
4) Show that π is a proper morphism.