Problem session 6

Recall that a topological space X is *paracompact* if it is Hausdorff and for every open cover $X = \bigcup_{i\in I} U_i$ there is a locally finite open cover $X = \bigcup_{j\in J} V_j$ that refines it. A useful property is that if $X = \bigcup_{j\in J} V_j$ is a locally finite open cover of a paracompact space X, then there is another open cover $X = \bigcup_{j\in J} W_j$ such that $\overline{W_j} \subseteq V_j$ for all $j \in J$.

Recall that if $\mathcal{F}$ is a sheaf on a topological space X and Z is an arbitrary subset of X, with $i\colon Z \hookrightarrow X$ the inclusion map, then we put

$$\Gamma(Z, \mathcal{F}) := \Gamma\big(Z, i^{-1}(\mathcal{F})\big).$$

A sheaf $\mathcal{F}$ on X is *soft* if for every closed subset Z of X, the restriction map

$$\Gamma(X, \mathcal{F}) \to \Gamma(Z, \mathcal{F})$$

is surjective.

Problem 1. Let X be a paracompact topological space and $\mathcal{F}$ a sheaf of Abelian groups on X.

i) Show that for every closed subset Z of X and every $s \in \mathcal{F}(Z)$, there is an open subset U containing Z and $t \in \mathcal{F}(U)$ such that $t|_Z = s$.
ii) Deduce that if $\mathcal{F}$ is flasque, then it is soft.

Problem 2. Let X be a paracompact topological space and $\mathcal{F}$ a presheaf of Abelian groups on X that satisfies the following condition: for every open cover $X = \bigcup_{i\in I} U_i$ and for every $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all i and j, there is $s \in \mathcal{F}(X)$ such that $s|_{U_i} = s_i$ for all i. Show that if $\mathcal{F} \to \mathcal{F}^+$ is the canonical morphism to the associated sheaf, then the morphism $\mathcal{F}(X) \to \mathcal{F}^+(X)$ is surjective.

Problem 3. Let X be a topological space and $\mathcal{F}$ a sheaf of Abelian groups on X. For a section $s \in \Gamma(X, \mathcal{F})$, its *support* $\mathrm{Supp}(s)$ is the set of points $x \in X$ such that $s_x \in \mathcal{F}_x$ is non-zero.

i) Show that by associating to a sheaf $\mathcal{F}$ the group of sections with support in Z:

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \Gamma(X, \mathcal{F}) \mid \mathrm{Supp}(s) \subseteq Z\}$$

we get a left exact functor from the category of sheaves of Abelian groups to the category of Abelian groups.
ii) If X is an affine algebraic variety and $\mathcal{F} = \widetilde{M}$, for some $\mathcal{O}(X)$-module M, describe $\Gamma_Z(X, \mathcal{F})$.