Problem session 6

We know that the functor mapping X to $\mathcal{O}(X)$ gives an equivalence of categories between the category of affine varieties over k and the category of reduced, finite type k-algebras. The following exercise gives an explicit construction of the inverse functor. This point of view is useful in several instances.

Problem 1. Recall that if R is any commutative ring, then we have the maximal spectrum $\mathrm{MaxSpec}(R)$, a topological space with the underlying space consisting of all maximal ideals in R (see HW #1, though this was denoted there by $\mathrm{Specm}(R)$). Suppose now that R is an algebra of finite type over an algebraically closed field k. Recall that in this case, for every $\mathfrak{m} \in \mathrm{MaxSpec}(R)$, the canonical homomorphism $k \to R/\mathfrak{m}$ is an isomorphism. For every open subset U of $\mathrm{MaxSpec}(R)$, let $\mathcal{O}(U)$ be the set of functions $s\colon U \to k$ such that for every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x and $a, b \in R$ such that for every $\mathfrak{m} \in U_x$, we have

$$b \not\in \mathfrak{m} \quad \text{and} \quad s(\mathfrak{m}) = \overline{a} \cdot \overline{b}^{-1},$$

where we denote by $\overline{u} \in k \simeq R/\mathfrak{m}$ the class of $u \in R$.

1) Show that $\mathcal{O}$ is a sheaf such that the pair $\big(\mathrm{MaxSpec}(R), \mathcal{O}\big)$ defines an element in $\mathcal{T}op_k$ that we denote $\mathrm{Aff}(R)$.
2) Show that given a homomorphism of reduced, finite type k-algebras $R \to S$, we have an induced morphism $\mathrm{Aff}(S) \to \mathrm{Aff}(R)$ such that we obtain a functor Aff from the category of reduced, finite type k-algebras to $\mathcal{T}op_k$.
3) Show that for every R as above, $\mathrm{Aff}(R)$ is an affine variety. Moreover, the functor Aff is an inverse of the functor from the category of affine variety to the category of reduced, finite type k-algebras, that maps X to $\mathcal{O}(X)$.

Problem 2. Show that if $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ are locally closed (open, closed) immersions of prevarieties, then the morphism

$$X_1 \times X_2 \to Y_1 \times Y_2, \quad (x_1, x_2) \to \big(f_1(x_1), f_2(x_2)\big)$$

is a locally closed (respectively, open, closed) immersion.

Problem 3. Show that if I is a homogeneous ideal in a graded ring S, then the following hold:

i) The ideal I is radical if and only if for every *homogeneous* element $f \in S$, with $f^m \in I$ for some $m \geq 1$, we have $f \in I$.
ii) The radical $\mathrm{rad}(I)$ of I is a homogeneous ideal.

Problem 4 Show that if I is a homogeneous ideal in a graded ring S, then I is a prime ideal if and only if for every homogeneous elements $f, g \in S$ with $fg \in I$, we have $f \in I$ or $g \in I$. Deduce that a closed subset Z of $\mathbb{P}^n$ is irreducible if and only if $I(Z)$ is a prime ideal. In particular, $\mathbb{P}^n$ is irreducible.

Problem 5. Let $f\colon X \dashrightarrow Y$ be a rational map between the irreducible varieties X and Y. The *graph* Γ_f of f is defined as follows. If U is an open subset of X such that f is defined on U, then the graph of $f|_U$ is well-defined, and it is a closed subset of $U \times Y$. By definition, Γ_f is the closure of the graph of $f|_U$ in $X \times Y$.

i) Show that the definition is independent of the choice of U.
ii) Let $p\colon \Gamma_f \to X$ and $q\colon \Gamma_f \to Y$ be the morphisms induced by the two projections. Show that p is a birational morphism, and that q is birational if and only if f is.
iii) Show that if the fiber $p^{-1}(x)$ does not consist of only one point, then f is not defined at $x \in X$.

Problem 6. Show that $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$.