Problem session 5

Problem 1. Let X be an irreducible, normal variety. Show that for two divisors D and E, we have $\mathcal{O}_X(D) \simeq \mathcal{O}_X(E)$ if and only if D and E are linearly equivalent.

Problem 2. Let X be an irreducible variety which is smooth in codimension 1, U an open subset, and suppose that the irreducible components of $X \smallsetminus U$ that have codimension 1 in X are $Z_1, \ldots, Z_s$. Show that we have a short exact sequence

$$\mathbb{Z}^s \xrightarrow{\alpha} \mathrm{Cl}(X) \xrightarrow{\beta} \mathrm{Cl}(U) \longrightarrow 0,$$

where

$$\alpha(m_1, \ldots, m_s) = [m_1 Z_1 + \ldots + m_s Z_s] \quad \text{and} \quad \beta\big([D]\big) = [D|_U].$$

Problem 3. Let Y be a hypersurface in $\mathbb{P}^n$. If the irreducible components of Y are $Y_1, \ldots, Y_r$ and $\deg(Y_i) = d_i$, show that $\mathrm{Cl}(\mathbb{P}^n \smallsetminus Y) \simeq \mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of $d_1, \ldots, d_r$.

Problem 4. Let X be the affine cone over a smooth quadric surface in $\mathbb{P}^3$, that is, $X \subseteq \mathbb{A}^4$ is defined by the equation $x_1x_2 - x_3x_4 = 0$. Note that X is smooth in codimension 1 (in fact, it is normal, see the previous problem set). Show that if L_1 and L_2 are the lines in X given by $x_1 = x_3 = 0$ and, respectively $x_1 = x_4 = 0$, then L_1 and L_2 are prime divisors on X which are not Cartier, but $L_1 + L_2$ is Cartier.