Problem session 5

Whenever dealing with sheaves, we assume that they take values in a subcategory $\mathcal{C}$ of the category of all sets, such that a morphism in $\mathcal{C}$ is an isomorphism if and only if it is bijective.

Problem 1. (Glueing morphisms of sheaves). Let X be a topological space and $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X (of objects in some category $\mathcal{C}$). If we have an open cover $X = \bigcup_{i\in I} U_i$ and for every $i \in I$ we have a morphism of sheaves $\phi_i \colon \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ such that for every $i, j \in I$ we have $\phi|_{U_i\cap U_j} = \phi_j|_{U_i\cap U_j}$, then there is a unique morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ such that $\phi|_{U_i} = \phi_i$ for all $i \in I$.

Problem 2. (Glueing sheaves). Let X be a topological space and suppose that $X = \bigcup_{i\in I} U_i$ is an open cover. Suppose that for every $i \in I$ we have a sheaf $\mathcal{F}_i$ on U_i (of objects in some category $\mathcal{C}$) and for every $i, j \in I$ we have isomorphisms

$$\phi_{j,i} \colon \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}, \quad \text{where} \quad U_{ij} = U_i \cap U_j$$

that satisfy the following compatibility relations:

i) We have $\phi_{i,i} = \mathrm{Id}$ for every $i \in I$, and
ii) We have
$$\phi_{k,j}|_{U_{ijk}} \circ \phi_{j,i}|_{U_{ijk}} = \phi_{k,i}|_{U_{ijk}} \quad \text{for all} \quad i, j, k \in I,$$
where $U_{ijk} = U_i \cap U_j \cap U_k$. In this case there is a sheaf $\mathcal{F}$ on X with isomorphisms $\phi_i \colon \mathcal{F}|_{U_i} \to \mathcal{F}_i$ for all $i \in I$, such that

$$\phi_{j,i} \circ \phi_i|_{U_{ij}} = \phi_j|_{U_{ij}} \quad \text{for all} \quad i, j \in I. \tag{1}$$

Moreover, if $\mathcal{G}$ is another such sheaf, with isomorphisms $\psi_i \colon \mathcal{G} \to \mathcal{F}|_{U_i}$ for every $i \in I$ that satisfy the compatibility conditions (1), then there is a unique morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ such that $\psi_i \circ \alpha|_{U_i} = \phi_i$ for all $i \in I$.

Problem 3. (Gluing prevarieties). Let $X_1, \ldots, X_r$ be prevarieties and for every i and j, suppose that we have open subvarieties $U_{i,j} \subseteq X_i$ and isomorphisms $\phi_{i,j} \colon U_{i,j} \to U_{j,i}$ such that

i) We have $U_{i,i} = X_i$ and $\phi_{i,i} = \mathrm{Id}_{X_i}$ for every i, and
ii) $\phi_{j,k} \circ \phi_{i,j} = \phi_{i,k}$ on $U_{i,j} \cap \phi_{i,j}^{-1}(U_{j,k}) \subseteq U_{i,k}$.

In this case, there is a prevariety X, an open cover $X = U_1 \cup \ldots \cup U_r$, and isomorphisms $f_i \colon U_i \to X_i$ such that for every i and j, we have

$$U_i \cap U_j = f_i^{-1}(U_{i,j}) \quad \text{and} \quad \phi_{i,j} \circ f_i = f_j \quad \text{on} \quad U_i \cap U_j.$$

Moreover, if Y is another such prevariety with an open cover $Y = V_1 \cup \ldots \cup V_r$ and isomorphisms $g_i \colon V_i \to X_i$ that satisfy the same compatibility conditions, then there is a unique isomorphism $h\colon X \to Y$ such that $h(U_i) = V_i$ and $g_i \circ h|_{U_i} = f_i$ for $1 \le i \le r$.

Problem 4. Let X and Y be two copies of $\mathbb{A}^1$ and let $U \subseteq X$ and $V \subseteq Y$ be the complement of the origin. We can apply the previous exercise to construct a prevariety W_1 by glueing X and Y along the isomorphism $U \to V$ given by the identity. This prevariety is *the affine line with the origin doubled*. On the other hand, we can glue X and Y along the isomorphism $U \to V$ corresponding to the k-algebra isomorphism

$$k[x, x^{-1}] \to k[x, x^{-1}], \quad x \to x^{-1}.$$

Show that the first example is not separated, while the second one is separated.

Problem 5. Show that if X, Y, and Z are algebraic prevarieties and $X \xrightarrow{f} Y \xrightarrow{g} Z$ are locally closed (open, closed) immersions, then $g \circ f$ is a locally closed (respectively open, closed) immersion.