Problem set 4

Problem 1. Show that if $\phi\colon R \to S$ is a finite homomorphism of Noetherian rings, with $(R, \mathfrak{m})$ a local ring, then an S-module M has finite length over S if and only if it has finite length over R, and in this case, if $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ are the maximal ideals in S, then

$$\ell_R(M) = \sum_{i=1}^{r} \ell_{S_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \cdot [S/\mathfrak{q}_i : R/\mathfrak{m}].$$

Problem 2. Let $f\colon R \hookrightarrow S$ be a finite, injective homomorphism, where $(R, \mathfrak{m})$ is a DVR, with discrete valuation v, and S is a domain, and denote by $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ the maximal ideals of S. Show that if $K = \mathrm{Frac}(R)$ and $L = \mathrm{Frac}(S)$, then for every non-zero $b \in S$, we have

$$v\big(N_{L/K}(b)\big) = \sum_{i=1}^{r} \ell_{S_{\mathfrak{q}_i}}\big(S_{\mathfrak{q}_i}/(b)\big) \cdot [S/\mathfrak{q}_i : R/\mathfrak{m}].$$

Problem 3. Let S be an integral, finitely generated semigroup. Recall that S is saturated if for every $u \in S^{\mathrm{gp}}$ such that $mu \in S$ for some $m \geq 1$, then $u \in S$. Show that $k[S]$ is integrally closed if and only if S is saturated.

Problem 4. Let $f \in k[x_1, \ldots, x_r, y_1, \ldots, y_s]$ be an irreducible polynomial of the form

$$f = x_1^{a_1} \cdots x_r^{a_r} - y_1^{b_1} \cdots y_s^{b_s}$$

for non-negative integers $a_1, \ldots, a_r, b_1 \ldots, b_s$. Let $Z \subseteq \mathbf{A}^{r+s}$ be the hypersurface defined by f.

i) Show that if Z is normal, then either $a_i \in \{0, 1\}$ for $1 \leq i \leq r$ or $b_j \in \{0, 1\}$ for all $1 \leq j \leq s$.
ii) Show that conversely, if $a_i \in \{0, 1\}$ for all i, then f is an irreducible polynomial that defines a normal hypersurface in $\mathbf{A}^{r+s}$.