Problem session 4

Definition 0.1. A subset of a topological space X is *constructible* if it is a finite union of locally closed subsets.

Problem 1. If X is a topological space, the set of constructible subsets of X is the smallest set that contains the open subsets of X and is closed under finite unions, finite intersections, and complements.

Problem 2. Give an example of an affine variety X and a constructible subset A of X which is not locally closed.

The notion of constructible subsets is important due to the following theorem of Chevalley:

Problem 3. If $f\colon X \to Y$ is a morphism between quasi-affine varieties, the image $f(X)$ is constructible. More generally, for every constructible subset A of X, its image $f(A)$ is constructible.

Problem 4.

i) Show that if Y is a topological space and A is a constructible subset of Y, then there is a subset V of A that is open and dense in $\overline{A}$ (in particular, V is locally closed in Y.
ii) Use part i) and Chevalley's theorem to show that if G is a linear algebraic group having an algebraic action on the quasi-affine variety X, then every orbit is a locally closed subset of X. Deduce that there are closed orbits.