Problem set 3

Problem 1. Let X be an algebraic variety and let X_{sm} be the set of smooth points of X. We have seen last semester that if V is a closed irreducible subset of X such that $V \cap X_{\mathrm{sm}} \neq \emptyset$, then X is smooth at V (that is, the local ring $\mathcal{O}_{X,V}$ is regular). Prove now the converse: if X is smooth at V, then $V \cap X_{\mathrm{sm}} \neq \emptyset$.

Problem 2. We have seen in the last problem session that on $\mathbb{P}^n$ we have an injective morphism of vector bundles

$$\mathcal{O}_{\mathbb{P}^n}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}.$$

i) Deduce that we have a surjection

$$\psi\colon \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}.$$

ii) Show that there is an exact sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

iii) Deduce that we have $\omega_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

iv) Show that the exact sequence in ii) is not split.

Problem 3. Show that if X and Y are algebraic varieties and $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are the two projections, then there is an isomorphism

$$\Omega_{X\times Y} \simeq p^*(\Omega_X) \oplus_{\mathcal{O}_{X\times Y}} q^*(\Omega_Y).$$