Problem session 3

Problem 1. Show that if X and Y are affine toric varieties, with tori $T_X \subseteq X$ and $T_Y \subseteq Y$, then $X \times Y$ has a natural structure of toric variety, with torus $T_X \times T_Y$. Describe the semigroup corresponding to $X \times Y$ in terms of the semigroups of X and Y.

Problem 2. Let S be the sub-semigroup of $\mathbb{Z}^3$ generated by e_1, e_2, e_3 and $e_1 + e_2 - e_3$. These generators induce a surjective morphism $f\colon k[\mathbb{N}^4] = k[t_1, \ldots, t_4] \to k[S]$. Show that the kernel of f is generated by $t_1t_2 - t_3t_4$. We have $S^{\mathrm{gp}} = \mathbb{Z}^3$, the embedding of $T = (k^*)^3 \hookrightarrow X$ is given by $(\lambda_1, \lambda_2, \lambda_3) \to (\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2/\lambda_3)$, and the action of T on X is induced via this embedding by component-wise multiplication.

Problem 3. Given an integral semigroup S, show that there is an injective semigroup homomorphism $\iota\colon S \hookrightarrow S^{\mathrm{gp}}$, where S^{gp} is a finitely generated Abelian group, that satisfies the following universal property: given any semigroup homomorphism $h\colon S \to A$, where A is an Abelian group, there is a unique group homomorphism $g\colon S^{\mathrm{gp}} \to A$ such that $g \circ \iota = h$. Hint: if $S \hookrightarrow M$ is an injective semigroup homomorphism, where M is a finitely generated, free Abelian group, then show that one can take S^{gp} to be the subgroup of M generated by S. Note that it follows from this description that S^{gp} is finitely generated (since M is) and S^{gp} is generated as a group by S.

Problem 4. Let $X \subset \mathbb{A}^n$ be a hypersurface defined by an equation $f(x_1, \ldots, x_n) = 0$, where $f = f_{d-1} + f_d$, with f_{d-1} and f_d nonzero, homogeneous of degrees $d-1$ and d, respectively. Show that if X is irreducible, then X is birational to $\mathbb{A}^{n-1}$.

Problem 5. Show that if X is a quasi-affine variety such that $\mathcal{O}(X) = k$, then X consists of only one point.

Problem 6. Consider the hypersurface Z in $\mathbb{A}^2$ defined by

$$x^2 + y^2 = 1.$$

Find the domain of the rational function $\frac{1-y}{x}$ on Z.