Problem set 2

Problem 1. Given $n \geq 1$, consider the blow-up of $\mathbb{A}^{n+1}$ at the origin:

$$\mathrm{Bl}_0(\mathbb{A}^{n+1}) := \left\{ \big(P, \ell\big) \in \mathbb{A}^{n+1} \times \mathbb{P}^n \mid P \in \ell \right\}.$$

The first projection is the blow-up map of $\mathbb{A}^{n+1}$. Let us consider now the morphism $q\colon \mathrm{Bl}_0(\mathbb{A}^{n+1}) \to \mathbb{P}^n$ induced by the second projection $f\colon \mathbb{A}^{n+1} \times \mathbb{P}^n \to \mathbb{P}^n$.

i) Show that q gives a geometric vector bundle of rank 1, in fact a subbundle of the trivial rank $(n+1)$ bundle given by f. This is the *tautological subbundle* on $\mathbb{P}^n$. The sheaf of sections of this bundle is denoted $\mathcal{O}_{\mathbb{P}^n}(-1)$ and its dual by $\mathcal{O}_{\mathbb{P}^n}(1)$, while the corresponding m^{th} tensor powers (for $m > 0$) are denoted by $\mathcal{O}_{\mathbb{P}^n}(-m)$ and $\mathcal{O}_{\mathbb{P}^n}(m)$, respectively.

ii) Show that

$$\Gamma\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)\big) \simeq k[y_0, \ldots, y_n]_m,$$

where the right-hand side is 0 for $m < 0$.

If $\mathcal{E}$ is a vector bundle on X, then the *determinant* $\det(\mathcal{E})$ is obtained by taking on each connected component of X, the top exterior power of $\mathcal{E}$. This is a line bundle on X.

Problem 2. Consider an exact sequence of vector bundles on the algebraic variety X:

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0.$$

i) For every $p \geq 0$ and for every $0 \leq i \leq p$, let $\mathcal{F}_i$ be the image of the composition

$$S^i(\mathcal{E}') \otimes_{\mathcal{O}_X} S^{p-i}(\mathcal{E}) \to S^i(\mathcal{E}) \otimes_{\mathcal{O}_X} S^{p-i}(\mathcal{E}) \to S^p(\mathcal{E}).$$

Show that for every $0 \leq i \leq p$, we have a sequence of subbundles of $S^p(\mathcal{E})$

$$0 = \mathcal{F}_{p+1} \hookrightarrow \mathcal{F}_p \hookrightarrow \mathcal{F}_{p-1} \hookrightarrow \ldots \hookrightarrow \mathcal{F}_0 = S^p(\mathcal{E})$$

and we have canonical isomorphisms

$$\mathcal{F}_i/\mathcal{F}_{i+1} \simeq S^i(\mathcal{E}') \otimes_{\mathcal{O}_X} S^{p-i}(\mathcal{E}'') \quad \text{for} \quad 0 \leq i \leq p.$$

ii) Similarly, show that for every $p \geq 0$ and $0 \leq i \leq p$, we have a sequence of subbundles

$$0 = \mathcal{G}_{p+1} \hookrightarrow \mathcal{G}_p \hookrightarrow \ldots \hookrightarrow \mathcal{G}_0 = \wedge^p \mathcal{E}$$

such that we have canonical isomorphisms

$$\mathcal{G}_i/\mathcal{G}_{i+1} \simeq \wedge^i \mathcal{E}' \otimes_{\mathcal{O}_X} \wedge^{p-i} \mathcal{E}'' \quad \text{for} \quad 0 \leq i \leq p.$$

In particular, we have a canonical isomorphism

$$\det(\mathcal{E}) \simeq \det(\mathcal{E}') \otimes_{\mathcal{O}_X} \det(\mathcal{E}'').$$