Problem session 2

Problem 1. Let X be a Noetherian topological space and Y a subset X. Show that if $Y = Y_1 \cup \ldots \cup Y_r$ is the irreducible decomposition of Y, then $\overline{Y} = \overline{Y_1} \cup \ldots \cup \overline{Y_r}$ is the irreducible decomposition of $\overline{Y}$.

Problem 2. If X is an affine algebraic variety, and if $u \in \mathcal{O}(X)$, then we denote by $D_X(u)$ the open subset of X

$$D_X(u) = \{x \in X \mid u(x) \neq 0\}$$

(we have seen in class that this is again an affine variety). Suppose that $f\colon X \to Y$ is a morphism of affine algebraic varieties, and denote by $f^\sharp\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ the induced ring homomorphism, that takes $\phi \in \mathcal{O}(Y)$ to $\phi \circ f$. Show that if $u \in \mathcal{O}(Y)$, then

i) We have $f^{-1}(D_Y(u)) = D_X(w)$, where $w = f^\sharp(u)$.
ii) The induced ring homomorphism

$$\mathcal{O}(D_Y(u)) \to \mathcal{O}(D_X(w))$$

can be identified with the homomorphism

$$\mathcal{O}(Y)_u \to \mathcal{O}(X)_w$$

induced by $f^\sharp$ by localization.

Problem 3.

Let X and Y be quasi-affine varieties. You know that if $f\colon X \to Y$ is a morphism, and if $p \in X$ is a point such that $f(p) = q$, then f induces a local ring homomorphism $\phi\colon \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ (this is a special case of a problem on the 2nd homework set).

i) Show that if $f'\colon X \to Y$ is another morphism with $f'(p) = q$, and induced homomorphism $\phi'\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, then $\phi = \phi'$ if and only if there is an open neighborhood U of p such that $f|_U = g|_U$.
ii) Show that given any local morphism of local k-algebras $\psi\colon \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$, there is an open neighborhood W of p, and a morphism $g\colon W \to Y$ with $g(p) = q$, and inducing ψ.
iii) Deduce that $\mathcal{O}_{X,p}$ and $\mathcal{O}_{Y,q}$ are isomorphic as k-algebras if and only if there are open neighborhoods W of p and V of q, and an isomorphism $h\colon W \to V$, with $h(p) = q$.

Problem 4. Let X be a quasiaffine variety, and let $X_1, \ldots, X_r$ be its irreducible components. Show that if $k(X)$ denotes the k-algebra of rational functions on X, there is a canonical isomorphism

$$k(X) \simeq k(X_1) \times \cdots \times k(X_r).$$

Problem 5. Let X be the nodal curve

$$X = \{(u, v) \in \mathbf{A}^2 \mid u^2 = v^2(v + 1)\}.$$

Show that X is birational to $\mathbf{A}^1$.

Problem 6. Show that a rational map $f\colon X \dashrightarrow Y$ between the irreducible quasiaffine varieties X and Y is birational if and only if there are open subsets $U \subseteq X$ and $V \subseteq Y$ such that f induces an isomorphism $U \simeq V$.