Problem session 1

Problem 1 Show that if $f\colon X \to Y$ is a morphism of algebraic varieties and $\mathcal{M}$ is a quasi-coherent sheaf on X, then $f_*(\mathcal{M})$ is a quasi-coherent sheaf on Y.

Problem 2. Let X be an algebraic variety. Given a quasi-coherent sheaf $\mathcal{M}$ on X and quasi-coherent subsheaves $\mathcal{M}_i$ of $\mathcal{M}$, for $i \in I$, consider the subsheaf $\mathcal{F}$ of $\mathcal{M}$ given by

$$\Gamma(U, \mathcal{F}) = \bigcap_{i \in I} \Gamma(U, \mathcal{F}_i).$$

Show that if I is infinite, $\mathcal{F}$ might not be quasi-coherent.

Problem 3. Let X be an algebraic variety, $x \in X$ a point, and $\mathcal{F}$ the presheaf on X given by

$$\mathcal{F}(U) = k \quad \text{if} \quad x \in U$$

and $\mathcal{F}(U) = 0$, otherwise (with the nonzero restriction maps being id_k). Show that $\mathcal{F}$ is a sheaf and that one can put an $\mathcal{O}_X$-module structure on $\mathcal{F}$ such that it is a coherent sheaf.

Problem 4. Let X be an algebraic variety. Show that if $(\mathcal{M}_i)_{i \in I}$ is a family of quasi-coherent sheaves, then $\bigoplus_{i \in I} \mathcal{M}_i$ is quasi-coherent.

Problem 5.

i) Show that if X is an algebraic variety, then every locally free sheaf $\mathcal{F}$ on X is coherent (recall that a *locally free sheaf* is an $\mathcal{O}_X$-modules such that for every $x \in X$, there is an open neighborhood U of X such that $\mathcal{F}|_U \simeq \mathcal{O}_U^{\oplus r}$ for some r).
ii) Show that if $f\colon X \to Y$ is a morphism of algebraic varieties and $\mathcal{F}$ is a locally free sheaf on Y, then $f^*(\mathcal{F})$ is locally free. On the other hand, show that if $\mathcal{M}$ is locally free on X, then $f_*(\mathcal{M})$ might not be locally free on Y.

Problem 6. Show that if $f\colon X \to Y$ is a morphism of algebraic varieties, then for every locally free sheaf $\mathcal{E}$ on Y and for every $\mathcal{O}_X$-module $\mathcal{F}$ on X, we have a canonical isomorphism

$$f_*\big(f^*(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{F}\big) \simeq \mathcal{E} \otimes_{\mathcal{O}_Y} f_*(\mathcal{F})$$

(this is known as the *projection formula*).