Problem session 1

Problem 1. Describe the closed algebraic subsets of $\mathbf{A}^1$.

Problem 2. Let Y be the algebraic subset of $\mathbf{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find the corresponding prime ideals.

Problem 3. For m and $n \geq 1$, let us identify $\mathbf{A}^{mn}$ with the set of all matrices $B \in M_{m,n}(k)$. Show that the set

$$M^r_{m,n}(k) := \{B \in M_{m,n}(k) \mid \mathrm{rk}(B) \leq r\}$$

is an algebraic set of $M_{m,n}(k)$.

Problem 4. Show that the following subset of $\mathbf{A}^3$

$$W_1 = \{(t, t^2, t^3) \mid t \in k\}$$

is an algebraic subset, and describe $I(W_1)$. Can you do the same for

$$W_2 = \{(t^2, t^3, t^4) \mid t \in k\}?$$

How about

$$W_3 = \{(t^3, t^4, t^5) \mid t \in k\}?$$

Problem 5. Show that if X_1 and X_2 are algebraic subsets of $\mathbf{A}^n$, then

$$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}.$$