Problem session 13

This problem set treats two applications of Generic Smoothness.

Problem 1. Prove the following version of Bertini's theorem, due to Kleiman: let X be a smooth, irreducible variety, over an algebraically closed field of characteristic 0. If $\mathcal{L}$ is a line bundle on X and $V \subseteq \Gamma(X, \mathcal{L})$ is a finite-dimensional vector space that generates $\mathcal{L}$, then there is a non-empty open subset U of $\mathbb{P}(V^\vee)$ such that every effective Cartier divisor corresponding to a point in U is smooth (that is, it is reduced, with smooth support).

The second problem is concerned with the *reflexivity of the dual variety*. Let us begin with a definition. Let $X \subseteq \mathbb{P}^n$ be a closed subvariety and consider the subset of $\mathbb{P}^n \times (\mathbb{P}^n)^*$ given by

$$I_0 := \{(p, H) \in \mathbb{P}^n \times (\mathbb{P}^n)^* \mid p \in X_{\mathrm{sm}}, H \supseteq T_p X\}.$$

Let I be the closure of I_0 in $\mathbb{P}^n \times (\mathbb{P}^n)^*$ and let $p\colon I \to X$ and $q\colon I \to (\mathbb{P}^n)^*$ be the maps induced by the two projections.

Let $V = \Gamma(\mathbb{P}^n, \mathcal{O}(1))^\vee$, so that $\mathbb{P}^n$ parametrizes lines in V and $(\mathbb{P}^n)^*$ parametrizes hyperplanes in V (we thus have surjective maps $V \smallsetminus \{0\} \to \mathbb{P}^n$ and $V^\vee \smallsetminus \{0\} \to (\mathbb{P}^n)^*$, and for $v \in V \smallsetminus \{0\}$ and $u \in V^\vee \smallsetminus \{0\}$, we write $[v]$ and $[u]$ for the images in $\mathbb{P}^n$ and, respectively, $(\mathbb{P}^n)^*$).

Problem 2.

i) Show that I is irreducible, of dimension $n-1$. The subvariety $q(I) \subseteq (\mathbb{P}^n)^*$ is the *dual variety* of X, denoted X^*.

ii) Note that we have canonical isomorphisms

$$T_{[v]}\mathbb{P}^n \simeq V/k \cdot v \quad \text{and} \quad T_{[u]}(\mathbb{P}^n)^* \simeq V^\vee/k \cdot u.$$

Show that for every $\big([v], [u]\big) \in I$, we have

$$T_{([v],[u])} I \subseteq \Big\{\big([a], [b]\big) \in V/k \cdot v \times V^\vee/k \cdot u \mid \langle b, v\rangle + \langle u, a\rangle = 0\Big\},$$

where $\langle -, -\rangle\colon V^\vee \times V \to k$ is the duality pairing.

iii) Use the assertion in ii) and Generic Smoothness, to prove the following: if we write I_X instead of I and, similarly, I_{X^*} for the corresponding incidence variety associated to X^*, then the isomorphism $\mathbb{P}^n \times (\mathbb{P}^n)^* \to (\mathbb{P}^n)^* \times \mathbb{P}^n$, $\big([v], [u]\big) \to \big([u], [v]\big)$ maps I_X onto I_{X^*}. In particular, we have $(X^*)^* = X$.