Problem session 12

Problem 1. Show that if R is a Noetherian local ring, M is a non-zero finitely generated R-module, and $x_1, \ldots, x_n$ generate an ideal $\mathfrak{a}$ contained in the maximal ideal, then $r = \mathrm{depth}(\mathfrak{a}, M) \leq n$ and

$$\mathcal{H}_i\big(K(\underline{x}; M)\big) = 0 \quad \text{for} \quad i > n - r, \quad \text{while} \quad \mathcal{H}_{n-r}\big(K(\underline{x}, M)\big) \neq 0.$$

Problem 2. Deduce the following consequences:

• If $(R, \mathfrak{m})$ is a Noetherian local ring, M is a finitely generated non-zero R-module, and $\mathfrak{a} = (x_1, \ldots, x_n) \subseteq \mathfrak{m}$, then $\mathrm{depth}(\mathfrak{a}, M) = n$ if and only if $x_1, \ldots, x_n$ form an M-regular sequence.

• If $(A, \mathfrak{m})$ is a Noetherian local ring and $\phi\colon A \to B$ is a finite homomorphism, then for every finitely generated B-module M and every ideal $\mathfrak{a} \subseteq \mathfrak{m}$, we have

$$\mathrm{depth}(\mathfrak{a}, M) = \mathrm{depth}(\mathfrak{a}B, M)$$

(this is a result that we used in class for $\mathfrak{a} = \mathfrak{m}$ and $M = B$).

Let $\mathcal{E}$ be a locally free sheaf of rank r on a variety X, and let $s \in \Gamma(X, \mathcal{E})$. We may consider s as a morphism $\mathcal{E}^\vee \to \mathcal{O}_X$ and denote by $\mathcal{I}(s)$ the image of this morphism and by $V(s)$ the closed subset that is the zero locus of $\mathcal{I}(s)$. The section s is a *regular section* if for every $x \in V(s)$, we have $\mathrm{depth}\big(\mathcal{I}(s)_x\big) = r$. Equivalently, if we choose an isomorphism $\mathcal{E}_x \simeq \mathcal{O}_{X,x}^{\oplus r}$, such that s_x corresponds to $(a_1, \ldots, a_r)$, then $a_1, \ldots, a_r$ is an $\mathcal{O}_{X,x}$-regular sequence.

Problem 3. Suppose that $\mathcal{E}$ is a locally free sheaf of rank r on a variety X, and let $s \in \Gamma(X, \mathcal{E})$ be a regular section. Show that if $\mathcal{I}(s)$ is a radical sheaf of ideals defining the closed subvariety Y, then the normal sheaf $N_{Y/X}$ is isomorphic to $\mathcal{E}|_Y$.