Problem session 11

Problem 1. Let $f\colon X \to Y$ be a morphism of algebraic varieties and $\mathcal{L}, \mathcal{M}$ line bundles on X. Show that the following hold:

i) If $\mathcal{L}$ is very ample over Y, then $\mathcal{L}$ is globally generated.
ii) If $\mathcal{L}$ is very ample over Y and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is very ample over Y.
iii) If $\mathcal{L}$ is very ample over Y, then $\mathcal{L}^d$ is very ample over Y for all $d > 0$.

Problem 2. Let X be a complete variety and $\mathcal{L}$ an ample line bundle on X. Show that if $f\colon W \to X$ is a finite morphism, then $f^*(\mathcal{L})$ is ample.

Remark. In particular, the above problem implies that if $f\colon W \to X$ is a finite morphism of algebraic varieties and X is projective, then W is projective. For example, this implies that the normalization of any projective variety is again a projective variety.

Problem 3. Prove now the following converse to the previous problem: if X is a complete variety, $f\colon W \to X$ is a finite, surjective morphism, and $\mathcal{L}$ is a line bundle on X such that $f^*(\mathcal{L})$ is ample, then $\mathcal{L}$ is ample.

Problem 4. We have seen that every line bundle on $\mathbb{P}^m \times \mathbb{P}^n$ is of the form

$$\mathcal{O}(a,b) := \mathrm{pr}_1^*\big(\mathcal{O}_{\mathbb{P}^m}(a)\big) \otimes \mathrm{pr}_2^*\big(\mathcal{O}_{\mathbb{P}^n}(b)\big).$$

Describe for which values of a and b the line bundle $\mathcal{O}(a,b)$ is ample or very ample.

Problem 5. Show that if $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ are $\mathcal{O}_X$-modules on X, with $\mathcal{E}$ locally free, then we have functorial isomorphisms

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}) \simeq \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}) \simeq \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{E}^\vee$$

and

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}).$$