Problem session 10

Problem 1. Let X be an algebraic variety and $\mathcal{F}$ a coherent sheaf on X. An irreducible closed subset Y of X is an *associated variety* of $\mathcal{F}$ if for an affine open subset U of X with $U \cap Y \neq \emptyset$, the prime ideal corresponding to $Y \cap U$ in $\mathcal{O}(U)$ lies in $\mathrm{Ass}_{\mathcal{O}(U)}\big(\mathcal{F}(U)\big)$. We denote the set of associated subvarieties of $\mathcal{F}$ by $\mathrm{Ass}(\mathcal{F})$.

i) Show that the above condition is independent of the chosen subset U.
ii) Show that $\mathrm{Ass}(\mathcal{F})$ is a finite set containing the irreducible components of $\mathrm{Supp}(\mathcal{F})$.
iii) Show that if X is irreducible and D is an effective Cartier divisor on X, then we have a short exact sequence

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \to \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_D \to 0.$$

Problem 2. Show that if $\mathcal{F}$ is a coherent sheaf on a projective space $\mathbb{P}^n$, for a positive integer n, then there is a polynomial $P_{\mathcal{F}} \in \mathbb{Q}[t]$ such that

$$P_{\mathcal{F}}(m) = \chi\big(\mathcal{F}(m)\big) \quad \text{for all} \quad m \in \mathbb{Z}.$$

Moreover, if $\mathcal{F} \neq 0$, then $\deg(P_{\mathcal{F}}) = \dim\big(\mathrm{Supp}(\mathcal{F})\big)$.

The polynomial $P_{\mathcal{F}}$ in the above problem is the *Hilbert polynomial* of $\mathcal{F}$. If X is a closed subvariety of $\mathbb{P}^n$, then we write P_X for $P_{\mathcal{O}_X}$; this is the *Hilbert polynomial* of X (note that this depends not just on X, but also on the embedding in $\mathbb{P}^n$).

Problem 3. Let $S = k[x_0, \ldots, x_n]$ and M a finitely generated, graded S-module. Show that there is a polynomial P_M such that

$$P_M(j) = \dim_k M_j \quad \text{for} \quad j \gg 0.$$

This is the *Hilbert polynomial* of M.

Problem 4. Compute the Hilbert polynomial for the following subvarieties of $\mathbb{P}^n$:

i) A linear subspace of $\mathbb{P}^n$.
ii) A hypersurface of degree d in $\mathbb{P}^n$.

Problem 5. Let $\mathcal{F}$ be a non-zero coherent sheaf on $\mathbb{P}^n$. Show that if $\dim\big(\mathrm{Supp}(\mathcal{F})\big) = d$, then the top degree term in $P_{\mathcal{F}}(t)$ is $\frac{e}{d!}t^d$, where e is a positive integer called the *degree* of $\mathcal{F}$. If X is a closed subvariety of $\mathbb{P}^n$, then the *degree* of X is the degree of $\mathcal{O}_X$. Show that for hypersurfaces, this notion of degree agrees with our old notion.