Problem set 10

Problem 1. Let p be a smooth point on a variety X. If $f_1, \ldots, f_r$ are regular functions defined in an open neighborhood of p, vanishing at p, and whose images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent, where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}_{X,p}$, show there is an affine open neighborhood U of x such that the following conditions hold:

i) We have $f_1, \ldots, f_r \in \mathcal{O}(U)$.
ii) We have a closed subvariety Y of X with $I_U(Y \cap U) = (f_1, \ldots, f_r)$.
ii) The subvariety Y is smooth at p.

The next result describes the behavior of smooth closed subvarieties of a smooth variety.

Problem 2. Let X be an algebraic variety and Y a closed subvariety of X. If $p \in Y$ is a point that is smooth on both Y and X, show that after replacing X with a suitable affine open neighborhood of p, the following conditions hold:

i) The ideal $I = I_X(Y)$ is generated by r elements, where $r = \dim_p(X) - \dim_p(Y)$; in fact these elements can be chosen such that their images in $\mathcal{O}_{X,p}$ are part of a regular system of parameters[1].
ii) If $R = \mathcal{O}(X)$, then the generators of I induce an isomorphism

$$R/I[x_1, \ldots, x_r] \simeq \bigoplus_{j \geq 0} I^j/I^{j+1} =: \mathrm{gr}_I(R).$$

Given a smooth variety X and two smooth closed subvarieties Y and Z of X, recall that for every $p \in Y \cap Z$, we may consider T_pY and T_pY as linear subspaces of T_pX. We say that Y and Z intersect transversely, if for every $p \in Y \cap Z$, we have

$$\mathrm{codim}_{T_p(X)}(T_pY \cap T_pZ) = \mathrm{codim}_X^p(Y) + \mathrm{codim}_X^p(Z).$$

Note that p lies on unique irreducible components X' and Y' of X and Y, respectively, and we put $\mathrm{codim}_X^p(Y) = \mathrm{codim}_{X'}(Y')$; a similar definition applies for $\mathrm{codim}_X^p(Z)$.

Problem 3. Show that if X is a smooth variety and Y, Z are smooth closed subvarieties of X that intersect transversely, then $Y \cap Z$ is smooth, and for every $p \in Y \cap Z$, we have

$$\mathrm{codim}_X^p(Y \cap Z) = \mathrm{codim}_X^p(Y) + \mathrm{codim}_X^p(Z) \quad \text{and}$$

$$T_p(Y \cap Z) = T_pY \cap T_pZ.$$

Moreover, for every affine open subset U of X, we have

$$I_U(Y \cap Z \cap U) = I_U(Y \cap U) + I_U(Z \cap U).$$

[1]For every regular local ring $(R, \mathfrak{m})$, a *regular system of parameters* is a minimal set of generators of $\mathfrak{m}$. Note that since R is regular, the length of such a system is equal to $\dim(R)$. If X is a variety and $p \in X$ is a smooth point, we say that some regular functions $f_1, \ldots, f_n$ defined in a neighborhood of p give a regular system of parameters at p if their images in $\mathcal{O}_{X,p}$ give a regular system of parameters.

Problem 4. Show that if $X = \mathrm{MaxSpec}(A)$ is a smooth variety and $f\colon \widetilde{X} \to X$ is the blow-up of X along the radical ideal I, corresponding to the smooth closed subvariety Y of X, then $\widetilde{X}$ is smooth. Show also that if X and Y are irreducible, and I is generated by $r = \mathrm{codim}_X(Y)$ elements $f_1, \ldots, f_r$, then $\widetilde{X}$ is isomorphic to the subvariety of $X \times \mathbb{P}^{r-1}$ defined by the ideal J generated by all differences $f_i y_j - f_j y_i$, for $i \neq j$ (here $y_1, \ldots, y_r$ denote the homogeneous coordinates on $\mathbb{P}^{r-1}$).