Topics on singularities of algebraic varieties
Lecture notes for Math 732, Winter 2022

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CHAPTER 1

Introduction: complex powers and p-adic zeta functions

The goal of this course is to look at several related points of view on singularities of algebraic varieties and discuss the connections between these points of view. In this introduction we describe one question having to do with singularities that has both an Archimedean and a non-Archimedean incarnations. We will see some similar ideas and results come up later in the course. Everything that we discuss in this lecture is covered in a lot more detail in Igusa's book [**Igu00**].

1.1. Complex powers

We begin with the following question of an analytic flavor: given a polynomial function on $\mathbf{R}^n$, what can be said about the complex power of this function, viewed as a distribution on $\mathbf{R}^n$?

More precisely, let $f \in \mathbf{R}[x_1, \ldots, x_n]$ be a nonzero polynomial. For every function $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$ and every $s \in \mathbf{C}$, with $\mathrm{Re}(s) > 0$, consider

$$Z_{f,\varphi}(s) := \int_{\mathbf{R}^n} |f(x)|^s \varphi(x) dx.$$

Recall that for $a \in \mathbf{R}_{>0}$ and $\lambda \in \mathbf{C}$, we have $a^\lambda = \exp\big(\lambda \cdot \log(a)\big)$. Of course, we can ignore those x with $f(x) = 0$, which form a set of measure 0.

Note first that $Z_{f,\varphi}(s)$ is well-defined: if $\mathrm{Re}(s) \geq 0$ and $f(x) \neq 0$, then

$$||f(x)|^s \varphi(x)| = |f(x)|^{\mathrm{Re}(s)} \cdot |\varphi(x)| \leq T^{\mathrm{Re}(s)} \cdot |\varphi(x)|$$

if $|f(x)| \leq T$ on the support of φ (which is compact, by assumption).

Moreover, we have the following:

Proposition 1.1. The function $Z_{f,\varphi}$ is holomorphic in the half-plane $H_0 = \{s \mid \mathrm{Re}(s) > 0\}$.

Proof. We give an argument that exhibits the coefficients of the Taylor expansion at any $s_0 \in H_0$. Note that the Taylor expansion of the function $s \to \exp(s \cdot \log|f(x)|)$ at s_0 gives

$$|f(x)|^s \varphi(x) = \sum_{k \geq 0} \frac{(\log|f(x)|)^k |f(x)|^{s_0} \varphi(x)}{k!} (s - s_0)^k.$$

By assumption, $D := \mathrm{supp}(\varphi)$ is compact. The key point is to show that if $0 < \epsilon < \mathrm{Re}(s_0) = a$, then there is $M > 0$ such that

$$\sup_{x \in D, f(x) \neq 0} \frac{|(\log|f(x)|)^k |f(x)|^{s_0}|}{k!} \leq \frac{M}{\epsilon^k} \quad \text{for all} \quad k \geq 0. \tag{1.1}$$

Indeed, if this holds and if we choose $M'>0$ such that $|\varphi(x)|\leq M'$ for all x, then we conclude that for all s with $|s-s_0|<\epsilon$, we have

$$\sum_{k\geq 0}\int_{\mathbf{R}^n}\left|\frac{(\log|f(x)|)^k|f(x)|^{s_0}\varphi(x)}{k!}(s-s_0)^k\right|dx\leq\sum_{k\geq 0}MM'\cdot\mathrm{vol}(D)\frac{|s-s_0|^k}{\epsilon^k}.$$

Since the right-hand side is convergent, we conclude that

$$Z_{f,\varphi}(s)=\sum_{k\geq 0}\left(\int_{\mathbf{R}^n}\frac{(\log|f(x)|)^k|f(x)|^{s_0}\varphi(x)}{k!}dx\right)(s-s_0)^k.$$

In order to prove (1.1), note that if $T\geq 1$ is such that $|f(x)|\leq T$ for all $x\in D$, then for all $x\in D$ with $f(x)\neq 0$ we have

$$|(\log|f(x)|)^k|f(x)|^{s_0}|=|\log|f(x)||^k|f(x)|^a=\frac{|\log(|f(x)|^a)|^k|f(x)|^a}{a^k}\leq\frac{\sup_{0<y\leq T}|\log(y)|^k y}{a^k}.$$

Note that the function $g(y)=\log(y)^k y$ is positive and increasing on $(1,\infty)$, its only critical point in $(0,1)$ is $y=e^{-k}$, $g(1)=0$, and $\lim_{y\to 0}g(y)=0$. We thus have

$$\sup_{0<y\leq T}|g(y)|=\max\{k^k e^{-k},\log(T)^k T\}.$$

Note that there is k_0 such that $k^k e^{-k}\geq\log(T)^k T$ for all $k\geq k_0$. By Stirling's formula, we have

$$\lim_{k\to\infty}\frac{k!}{\sqrt{2\pi k}\cdot k^k e^{-k}}=1,$$

and it is easy now to see that if $0<\epsilon<a$, then there is $M>0$ such that

$$\sup_{x\in D,f(x)\neq 0}\frac{|\log|f(x)||^k|f(x)|^{s_0}}{k!}\leq\frac{\max\{k^k e^{-k},\log(T)^k T\}}{a^k k!}\leq\frac{M}{\epsilon^k}\quad\text{for all}\quad k\geq 0.$$

This completes the proof of the proposition. □

It was a question of I. Gel'fand (ICM, Amsterdam, 1954) whether $Z_{f,\varphi}$ admits a meromorphic extension to $\mathbf{C}$. In fact, one would like to do this uniformly in φ (more precisely, given any $s_0\in\mathbf{C}$, one would like to find $N=N(s_0)$ such that $(s-s_0)^N Z_{f,\varphi}$ is holomorphic in a neighborhood of s_0 for all φ.

EXAMPLE 1.2. An easy example is the case when $f=x_1^{a_1}\cdots x_r^{a_r}$ for some $a_1,\ldots,a_r\in\mathbf{Z}_{>0}$. In this case note first that an easy application of the change of variable formula implies that

$$Z_{f,\varphi}(s)=\int_{\mathbf{R}^n_{\geq 0}}x_1^{a_1 s}\cdots x_n^{a_n s}\widetilde{\varphi}(x)dx$$

for some $\widetilde{\varphi}\in\mathcal{C}_0^\infty(\mathbf{R}^n)$. Moreover, the argument in the proof of Proposition 1.1 implies that every integral like this gives a holomorphic function of s for $\mathrm{Re}(s)>0$. Integration by parts with respect to x_1 gives for all s with $\mathrm{Re}(s)>0$:

$$Z_{f,\varphi}(s)=-\frac{1}{a_1 s+1}\int_{\mathbf{R}^n_{\geq 0}}x_1^{a_1 s+1}x_2^{a_2 s}\cdots x_r^{a_r s}\frac{\partial\widetilde{\varphi}}{\partial x_1}dx$$

(recall that $\widetilde{\varphi}$ has compact support). Repeating this with respect to $x_2,\ldots,x_r$, we obtain

$$Z_{f,\varphi}(s)=(-1)^n\frac{1}{(a_1 s+1)\cdots(a_r s+1)}\int_{\mathbf{R}^n_{\geq 0}}x_1^{a_1 s+1}\cdots x_r^{a_r s+1}\psi(x)dx$$

for some $\psi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$. Note now that by Proposition 1.1, the integral on the right-hand side is holomorphic in the half-plane $\{s \mid \mathrm{Re}(s) > -\min_i \frac{1}{a_i}\}$. Repeating this procedure, we see that indeed in this case $Z_{f,\varphi}$ admits a meromorphic continuation with poles of order $\leq n$ at the rational numbers $-\frac{j}{a_i}$, for $1 \leq i \leq r$ and $j \in \mathbf{Z}_{>0}$.

A few years after Hironaka's proof of resolution of singularities in [**Hir64**], an affirmative answer to I. Gel'fand's question was given independently by Bernstein-S. Gel'fand [**BG69**] and Atiyah [**Ati70**], based on Hironaka's result. Here is an outline of the argument: consider an embedded resolution of singularities of the hypersurface defined by f. This is a proper, birational morphism $\pi = (\pi_1, \ldots, \pi_n) \colon Y \to \mathbf{A}^n_{\mathbf{R}}$, with Y smooth over $\mathbf{R}$ (so that $Y(\mathbf{R})$ has a corresponding structure of n-dimensional real manifold), with the following property: for every point $P \in Y(\mathbf{R})$, there is a chart U around P and coordinates $y_1, \ldots, y_n$ on U such that

i) $f \circ \pi = u \cdot \prod_{i=1}^n y_i^{a_i}$ on U, for some $a_1, \ldots, a_n \in \mathbf{Z}_{\geq 0}$ and some invertible function u.
ii) $\det(\frac{\partial \pi_i}{\partial y_j})_{1 \leq i,j \leq n} = v \cdot \prod_{i=1}^n y_i^{k_i}$ for some $k_1, \ldots, k_n \in \mathbf{Z}_{\geq 0}$ and some invertible function v.

Since π is birational, the induced map $Y(\mathbf{R}) \to \mathbf{R}^n$ is a diffeomorphism away from measure 0 subsets. The change of variable formula thus gives

$$Z_{f,\varphi}(s) = \int_{Y(\mathbf{R})} |f \circ \pi|^s (\varphi \circ \pi) d\pi_1 \wedge \ldots \wedge d\pi_n. \tag{1.2}$$

Note also that since π is proper, $\pi^{-1}(\mathrm{supp}(\varphi))$ is a compact subset of $Y(\mathbf{R})$ and thus $\varphi \circ \pi$ has compact support.

By assumption, around each point $P \in Y(\mathbf{R})$, we can find an open subset U that satisfies properties i) and ii). In U we can thus write

$$|f \circ \pi|^s (\varphi \circ \pi) d\pi_1 \wedge \ldots \wedge d\pi_n = |u|^s v \cdot \prod_{i=1}^n y_i^{a_i s + k_i} dy.$$

By taking a partition of unity, we write the integral on the right-hand side of (1.2) as a sum of finitely many integrals of the form

$$\int_U |u|^s v \prod_{i=1}^n |y_i|^{a_i s + k_i} \varphi_U(y) dy,$$

for some $\varphi_U \in \mathcal{C}_0^\infty(U)$. Since u and v do not vanish on U, their contribution can be ignored. Then an argument similar to the one in Example 1.2 allows one to conclude that $Z_{f,\varphi}$ can be extended as a meromorphic function to $\mathbf{C}$; moreover, each pole has multiplicity $\leq n$ and is of the form $-\frac{k_i+j}{a_i}$ for some chart U as above, some i, and some $j \in \mathbf{Z}_{\geq 0}$.

In particular, we see that $Z_{f,\varphi}$ always has a holomorphic extension to the half-plane $\{s \mid \mathrm{Re}(s) > -\lambda\}$, where $\lambda = \min_i \frac{k_i+1}{a_i}$. This λ is called the (real) log canonical threshold of f and one can show that it is independent of the resolution π. Its complex version will play an important role in our course.

A second solution to I. Gel'fand's question was given shortly afterwards by Bernstein [**Ber72**], directly extending the integration by parts argument in Example 1.2. This uses what is nowadays called the Bernstein-Sato polynomial of f.

More precisely, Bernstein showed that there is a nonzero polynomial $b(s) \in \mathbf{R}[s]$ and a differential operator $P \in \mathbf{R}[x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n}, s]$ that satisfy the equation

$$b(s)f^s = P(x, \partial_x, s) \bullet f^{s+1}. \tag{1.3}$$

Here one can interpret f^s as a formal symbol on which the partial derivatives act in the expected way:

$$\partial_{x_i} \bullet f^s = s\frac{\partial f/\partial x_i}{f} f^s.$$

It is easy to see that the polynomials $b(s)$ for which there is P satisfying (1.3) form an ideal in $\mathbf{R}[s]$. The monic generator $b_f(s)$ of this ideal is the *Bernstein-Sato polynomial* of f.

EXAMPLE 1.3. If $f = x_1$, then

$$\partial_{x_1} \bullet x_1^{s+1} = (s+1)x_1^s,$$

hence we may take $b(s) = s + 1$. In fact, one can show that $b_f(s) = s + 1$.

EXAMPLE 1.4. For a less trivial example, consider $f = x_1^2 + \ldots + x_n^2$. In this case we have

$$\partial_{x_i} \bullet f^{s+1} = 2(s+1)x_i f^s,$$

hence

$$\left(\sum_{i=1}^{n} \partial_{x_i}^2\right) \bullet f^{s+1} = (s+1)(4s+2n)f^s,$$

hence we may take $b(s) = (s+1)(4s+2n)$. In fact, one can show that in this case $b_f(s) = (s+1)(s+\frac{n}{2})$.

The existence of a nonzero such polynomial $b(s)$ is a deep result proved via D-module theory (in fact, Bernstein developed the theory of D-modules over polynomial rings in [**Ber71**] in order to prove the existence of this polynomial). We will discuss this in detail in the last part of the course.

Let's outline the solution to Gel'fand's problem using the functional equation (1.3). Let $B_+ = \{x \in \mathbf{R}^n \mid f(x) > 0\}$ and $B_- = \{x \in \mathbf{R}^n \mid f(x) < 0\}$, so that $Z_{f,\varphi} = Z_{f,\varphi}^+ + Z_{f,\varphi}^-$, where

$$Z_{f,\varphi}^+(s) = \int_{B_+} f(x)^s \varphi(x)dx \quad \text{and} \quad Z_{f,\varphi}^-(s) = \int_{B_-} (-f(x))^s \varphi(x)dx.$$

We analyze separately the two integrals using (1.3).

The argument in the proof of Proposition 1.1 implies that $Z_{f,\varphi}^+$ and $Z_{f,\varphi}^-$ are holomorphic in the half-plane $H_0 = \{s \mid \mathrm{Re}(s) > 0\}$. Using equation (1.3) and the Stokes theorem, we see that for $s \in H_0$, we have

$$b(s)Z_{f,\varphi}^+(s) = \int_{B_+} (P(x, \partial_x, s) \bullet f(x)^{s+1})\varphi(x)dx = \int_{B_+} f(x)^{s+1}\psi(x)dx, \tag{1.4}$$

for some $\psi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$. Note now that the right-hand side of (1.4) is holomorphic in the half-plane $\{s \mid \mathrm{Re}(s) > -1\}$. We iterate this argument, multiplying by $b(s+1)$, $b(s+2)$, etc. to conclude that indeed $Z_{f,\varphi}^+$ extends meromorphically to $\mathbf{C}$, with each pole of multiplicity $\leq \deg(b)$ and equal to some $\alpha - m$, for some root α of $b(s)$ and some $m \in \mathbf{Z}_{\geq 0}$. The assertion for $Z_{f,\varphi}^-$ follows from the one for $Z_{f,\varphi}^+$ by replacing f with $-f$, noting that equation (1.3) implies that

$$b(s)(-f)^s = Q(x, \partial_x, s) \bullet (-f)^{s+1} \quad \text{for some} \quad Q \in \mathbf{R}[x, \partial_x, s].$$

By comparing the pole candidates for $Z_{f,\varphi}$ that arise via the two solutions, one expects a connection between the roots of $b_f(s)$ and the numerical data associated to a resolution of singularities. Such a result was indeed proved by Kashiwara [**Kas76**] and Lichtin [**Lic89**]. We will discuss this in the last part of the course.

There is also a complex version of the results we discussed in this section. More precisely, if $f \in \mathbf{C}[z_1, \ldots, z_n]$, then for every $\varphi \in \mathbf{C}_0^\infty(\mathbf{C}^n)$, one considers the function

$$Z_{f,\varphi}(s) = \int_{\mathbf{C}^n} |f(z)|^{2s} \varphi(z) dz d\overline{z}.$$

Again, this is holomorphic in the half-plane $\{s \mid \mathrm{Re}(s) > 0\}$ and admits a meromorphic continuation to $\mathbf{C}$, for which the poles can be estimated either using an embedded resolution of singularities or the roots of the Bernstein-Sato polynomial of f. The proofs are entirely similar to the ones we discussed (though slightly more technical), so we don't go into details. We only point out that the different normalization given by the presence of 2 in the exponent is related to the fact that the function $g(x) = \frac{1}{|x|^q}$ is locally integrable on $\mathbf{R}$ if and only if $q < 1$ and is locally integrable on $\mathbf{C}$ if and only if $q < 2$.

1.2. A non-Archimedean analogue: Igusa's p-adic zeta function

We next turn to a problem with an arithmetic flavor. Suppose that p is a fixed prime integer and let $f \in \mathbf{Z}[x_1, \ldots, x_n]$. For every $m \geq 1$, let

$$c_m := \#\{u \in (\mathbf{Z}/p^m\mathbf{Z})^n \mid f(u) = 0\}.$$

The Poincaré series of f (with respect to p) is

$$P_f = P_{f,p} := \sum_{m \geq 0} \frac{c_m}{p^{mn}} T^m \in \mathbf{Q}[\![T]\!].$$

Example 1.5. If the hypersurface H defined by f is smooth, it is a consequence of Hensel's lemma that every solution of f in $(\mathbf{Z}/p\mathbf{Z})^n$ lifts to a solution in $(\mathbf{Z}/p^m\mathbf{Z})^n$ and the number of such lifts is precisely $p^{(m-1)(n-1)}$ (as it is in the case when $f = x_1$). It follows that in this case we have

$$P_f = 1 + c_1 p^{1-n} \sum_{m \geq 1} (p^{-1}T)^m = 1 + c_1 \cdot \frac{p^{-n}T}{1 - p^{-1}T}.$$

Remark 1.6. Note that the trivial bound $c_m \leq p^{mn}$ implies that the radius of convergence of P_f if ≥ 1.

Borevich and Shafarevich [**BS66**] asked whether P_f is always a rational function. Igusa gave a positive answer using p-adic integration. Before explaining his result, we review a few basic facts about p-adic numbers.

Recall that the ring $\mathbf{Z}_p$ of p-adic integers is the completion of the localization of $\mathbf{Z}$ at the prime $p\mathbf{Z}$. It is a DVR with maximal ideal $p\mathbf{Z}_p$ and residue field $\mathbf{Z}_p/p\mathbf{Z}_p \simeq \mathbf{Z}/p\mathbf{Z}$. It is a topological ring, with a basis of neighborhoods of 0 given by $p^m\mathbf{Z}_p$, for $m \in \mathbf{Z}_{\geq 0}$. For every $a \in \mathbf{Z}_p$ we denote by $\mathrm{ord}_p(a)$ the largest m such that $a \in p^m\mathbf{Z}_p$ (with the convention that this is ∞ if $a = 0$). The absolute value function $|\cdot|_p \colon \mathbf{Z}_p \to \mathbf{Q}$ is given by $|a|_p = \frac{1}{p^{\mathrm{ord}_p(a)}}$ (with the convention that

$|0|_p = 0$). For every $a, b \in \mathbf{Z}_p$, we have $|ab|_p = |a|_p \cdot |b|_p$ and the non-Archimedean triangle inequality

$$|a+b|_p \leq \max\{|a|_p, |b|_p\}.$$

Associated to this we have the distance function given by $d(a,b) = |a-b|_p$ and the topology on $\mathbf{Z}_p$ is the one corresponding to this metric.

An important fact is that $\mathbf{Z}_p$ is compact. It also carries a *Haar measure*: this is the unique translation-invariant measure μ_p normalized by $\mu_p(\mathbf{Z}_p) = 1$. We thus have $\mu_p(p^m\mathbf{Z}_p) = \frac{1}{p^m}$. On $\mathbf{Z}_p^n$ we consider the product measure that we still denote by μ_p, hence

$$\mu_p\left(\prod_{i=1}^n p^{m_i}\mathbf{Z}_p\right) = \left(\frac{1}{p}\right)^{m_1+\ldots+m_n}.$$

Given any $f \in \mathbf{Z}[x_1, \ldots, x_n]$ (or, more generally, in $\mathbf{Z}_p[x_1, \ldots, x_n]$), the *Igusa zeta function* of f is the function $Z_{f,p}$ given by

$$Z_{f,p}(s) = \int_{\mathbf{Z}_p^n} |f(x)|_p^s d\mu_p(x).$$

Note the similarity with the function discussed in the previous section (in this setting, too, one could consider an auxilliary function φ, but it is not necessary, since $\mathbf{Z}_p^n$ is compact). As we will see shortly, this is again a holomorphic function on the half-plane $\{s \mid \mathrm{Re}(s) > 0\}$ and it admits a meromorphic extension to $\mathbf{C}$, but unlike in the Archimedean setting, it is of a simple nature: it is a rational function in p^{-s}.

Let's begin by describing $Z_{f,p}(s)$. If we put

$$A_m = \{u \in \mathbf{Z}_p^n \mid \mathrm{ord}_p(f(u)) \geq m\} \quad \text{for} \quad m \geq 0,$$

then

$$Z_{f,p}(s) = \sum_{m \geq 0} \mu_p(A_m \smallsetminus A_{m+1}) \frac{1}{p^{ms}}.$$

On the other hand, note that we can write

$$A_m = \bigsqcup_{i=1}^{c_m} \left(a_i + (p^m\mathbf{Z}_p)^n\right),$$

where $a_1, \ldots, a_{c_m}$ are lifts to $\mathbf{Z}_p^n$ of the solutions of f in $(\mathbf{Z}/p^m\mathbf{Z})^n$. We thus have

$$\mu_p(A_m \smallsetminus A_{m+1}) = \mu_p(A_m) - \mu_p(A_{m+1}) = \frac{c_m}{p^{mn}} - \frac{c_{m+1}}{p^{(m+1)n}}.$$

An easy computation now gives

$$Z_{f,p}(s) = \sum_{m \geq 0} \left(\frac{c_m}{p^{mn}} - \frac{c_{m+1}}{p^{(m+1)n}}\right) p^{-ms} = P_f(p^{-s}) - p^s\left(P_f(p^{-s}) - 1\right) \tag{1.5}$$

$$= P_f(p^{-s})(1-p^s) + p^s.$$

Since the radius of convergence of P_f is ≥ 1 (see Remark 1.6), it follows that $P_f(p^{-s})$ is well-defined whenever $\mathrm{Re}(s) > 0$ and it gives a holomorphic function of s in this half-plane. By (1.5), the same holds for $Z_{f,p}(s)$ in this half-plane.

EXAMPLE 1.7. Suppose that $f = x_1^{a_1} \cdots x_r^{a_r}$, for some $a_1, \ldots, a_r \in \mathbf{Z}_{>0}$. It follows from Fubini's theorem that

$$Z_{f,p}(s) = \int_{\mathbf{Z}_p^n} |x_1|_p^{a_1 s} \cdots |x_r|_p^{a_r s} d\mu_p(x) = \prod_{i=1}^{r} \int_{\mathbf{Z}_p} |x_i|_p^{a_i s} d\mu_p(x_i).$$

However, the case of one variable is easy: we have

$$\int_{\mathbf{Z}_p} |x_i|_p^{a_i s} d\mu_p(x_i) = \sum_{m\geq 0} \mu_p(p^m\mathbf{Z}_p \smallsetminus p^{m+1}\mathbf{Z}_p) p^{-a_i m s} = \left(1 - \frac{1}{p}\right) \cdot \sum_{m \geq 0} p^{-m(a_i s+1)}$$

$$= \left(1 - \frac{1}{p}\right) \cdot (1 - p^{-(a_i s+1)}).$$

We thus conclude that

$$Z_{f,p}(s) = \left(1 - \frac{1}{p}\right)^r \cdot \prod_{i=1}^{r} (1 - p^{-(a_i s+1)}).$$

In particular, we see that $Z_{f,p}$ is a rational function of p^{-s}, and thus meromorphic on $\mathbf{C}$; moreover, each pole has multiplicity $\leq r$ and it is of the form $-\frac{1}{a_i} + \frac{2j\pi\sqrt{-1}}{a_i \log(p)}$ for some i and some $j \in \mathbf{Z}$.

Igusa proved that in fact the same holds for any f (see [**Igu74**], [**Igu75**], and [**Igu78**]).

THEOREM 1.8 (Igusa). *For every $f \in \mathbf{Z}_p[x_1, \ldots, x_n]$, the function $Z_{f,p}$ is a rational function in p^{-s}.*

In light of formula (1.5), we get the following

COROLLARY 1.9. For every $f \in \mathbf{Z}[x_1, \ldots, x_n]$ and every prime p, the Poincaré series $P_{f,p}$ is a rational function.

The approach to Theorem 1.8 follows a similar path to that discussed in the previous section when proving the meromorphic continuation of $Z_{f,\varphi}$ via resolution of singularities. We may assume that f is nonzero and consider an embedded resolution of singularities for the hypersurface defined by f in $\mathbf{A}^n_{\mathbf{Q}_p}$, where $\mathbf{Q}_p$ is the fraction field of $\mathbf{Z}_p$. This is a proper birational morphism $\pi \colon Y \to \mathbf{A}^n_{\mathbf{Q}_p}$ inducing a morphism of p-adic manifolds $\pi \colon Y(\mathbf{Q}_p) \to \mathbf{Q}_p^n$. The key point of using p-adic integrals is that they satisfy a change of variable formula, which allows writing $Z_{f,p}(s)$ as an integral over $\pi^{-1}(\mathbf{Z}_p^n)$. If the morphism π is induced by a morphism $\mathcal{Y} \to \mathbf{A}^n_{\mathbf{Z}_p}$ that also induces an embedded resolution of the hypersurface defined by $\overline{f} \in \mathbf{Z}/p\mathbf{Z}[x_1, \ldots, x_n]$ (this is not the case in general, but it holds if we start with $f \in \mathbf{Z}[x_1, \ldots, x_n]$ and we take p large enough), then the computation of the integral on $\pi^{-1}(\mathbf{Z}_p^n)$ is easy: it essentially comes down to the computation we have done in Example 1.7. The general case is more involved but again it reduces to a computation in the monomial case.

The proof also gives a set of candidate poles for $Z_{f,p}$. With the above notation, we know that for the resolution π, around any point in $Y(\mathbf{Q}_p)$ we can find local coordinates $y_1, \ldots, y_n$ such that

$$f \circ \pi = u y_1^{a_1} \cdots y_n^{a_n} \quad \text{and} \quad \det(\mathrm{Jac}(\pi)) = v y_1^{k_1} \cdots y_n^{k_n},$$

where u and v are invertible functions. As in the Archimedean case, we need to integrate expressions of the form $|y_i|_p^{a_is+k_i}$. As in Example 1.7, we get

$$\int_{\mathbf{Z}_p}|y_i|_p^{a_is+k_i}d\mu_p(y_i)=\sum_{m\geq 0}\left(\frac{1}{p^m}-\frac{1}{p^{m+1}}\right)p^{-m(a_is+k_i)}=\left(1-\frac{1}{p}\right)(1-p^{-(a_is+k_i+1)}).$$

Igusa's proof shows that indeed, $Z_{f,p}$ is a rational function in p^{-s}, a sum of fractions whose denominator is a product of $\leq n$ terms of the form $1-p^{-(a_is+k_i+1)}$. It follows that every pole has multiplicity $\leq n$ and it is of the form $-\frac{k_i+1}{a_i}+\frac{2j\pi\sqrt{-1}}{a_i\log(p)}$, for some i and some $j\in\mathbf{Z}$. In particular, its real part is equal to $-\frac{k_i+1}{a_i}$, for some i.

The following is the major open problem concerning Igusa zeta functions. Note that every $f\in\mathbf{Z}[x_1,\ldots,x_n]$ can be viewed as a polynomial with real (or complex) coefficients and therefore has a Bernstein-Sato polynomial $b_f(s)$.

Conjecture 1.10 (Igusa's strong monodromy conjecture). For every non-constant $f\in\mathbf{Z}[x_1,\ldots,x_n]$ and every prime p, is s_0 is a pole of $Z_{f,p}$, then $\mathrm{Re}(s_0)$ is a root of $b_f(s)$.

A geometric version of the p-adic story was developed about 25 years ago by Kontsevich [**Kon95**], Batyrev [**Ba98**], and Denef-Loeser [**DL99**]. The very rough idea is that one replaces the complete DVR $\mathbf{Z}_p$ and its quotients $\mathbf{Z}/p^m\mathbf{Z}$ by $k[\![t]\!]$ and its quotients $k[t]/(t^m)$. Given an algebraically closed field k of characteristic 0 and a variety X over k one considers the spaces $X\big(k[\![t]\!]\big)$ and $X\big(k[t]/(t^m)\big)$, respectively the *space of arcs* and the space of $(m-1)$-jets of X. Each $X\big(k[t]/(t^m)\big)$ is the set of points of an algebraic variety, and instead of counting the number of points of a finite set (as we did when setting up the Poincaré series P_f) one can define a generating function with coefficients in a Grothendieck ring of algebraic varieties and a motivic zeta function [**DL98**], and the story that one develops has a strong similarity with what we outlined above. We will discuss this in detail in the third part of the course.

1.3. A rough summary of the course

The course will be divided in four parts. The first part will be devoted to invariants of singularities that appear in birational geometry, such as multiplier ideals and log canonical thresholds. We will discuss the connection between these invariants and vanishing theorems and give some applications. While we will see the analytic definition of these objects, which is close in spirit to the material in Section 1.1, our approach to their study will be purely algebraic.

Multiplier ideals and log canonical thresholds can be defined also in positive characteristic, but there are two major obstacles to their study. First, resolution of singularities (which is an important tool in characteristic 0) is not known in this setting. More importantly, vanishing theorems can fail in characteristic p, so multiplier ideals lose their power. However, it turns out that in this setting one can define similar invariants (test ideals and F-pure thresholds) using the Frobenius morphism. Test ideals satisfy similar properties to those enjoyed by multiplier ideals in characteristic 0 and there are deep results and conjectures regarding the connection between these objects via reduction mod p. This will be the subject of the second part of the course.

In the third part we will discuss the spaces of arcs and jets of algebraic varieties and their connection to birational geometry. In particular, we will see how one can describe multiplier ideals and log canonical thresholds using the geometry of certain subsets of the space of arcs. We will also discuss the motivic analogue of Igusa's zeta function and its connection to singularities.

One word about the setting: we will almost exclusively work in an ambient smooth variety. Certain definitions and results (especially those covered in the first two parts of the course) extend to the case when the ambient variety has mild singularities. However, in order to avoid technicalities and since the main ideas already appear in the setting of a smooth ambient variety, we will make this assumption.

CHAPTER 2

Multiplier ideals and log canonical thresholds

In this chapter we discuss some invariants of singularities that play an important role in birational geometry. We follow rather closely the presentation in [**Laz04**], to which we refer to for a more in-depth discussion.

2.1. Divisorial valuations

We begin with a discussion of certain valuations which play an important role in the study of singularities in birational geometry.

Definition 2.1. A *discrete valuation* on a field K is a function $v\colon K \to \mathbf{Z} \cup \{\infty\}$ that satisfies the following conditions:

i) $v(a) = \infty$ if and only if $a = 0$.
ii) $v(ab) = v(a) + v(b)$ for all $a, b \in K$.
iii) $v(a + b) \geq \min\{v(a), v(b)\}$ for all $a, b \in K$.
iv) There is $a \in K$ such that $v(a) \neq 0, \infty$.

All valuations that we will encounter in this course arise as follows. Let k be a fixed algebraically closed field and Y a normal variety over k (by *variety over* k we mean a separated, integral scheme of finite type over k; moreover, unless explicitly mentioned otherwise, a *point* of Y is a closed point). If E is a prime divisor on Y (that is, E is a closed irreducible subset of Y of codimension 1 in Y), then the local ring $\mathcal{O}_{Y,E}$ is a DVR. If a is a generator of its maximal ideal, then every element φ of the function field $k(Y)$ can be uniquely written as $\varphi = ua^m$ for some $m \in \mathbf{Z}$ and $u \in \mathcal{O}_{Y,E}$ invertible. If we put $\mathrm{ord}_E(\varphi) = m$, then ord_E is a discrete valuation of $k(Y)$, the *order of vanishing along* E.

If X is an arbitrary variety over k, a *divisor over* X is given by a prime divisor E on a normal variety Y that has a birational morphism to X (note that in this case $k(Y) = k(X)$). We identify two such divisors if the corresponding valuations of $k(X)$ agree. A valuation of $k(X)$ that arises in this way is a *divisorial valuation* of X. We also say that (Y, E) gives a *model* for the valuation ord_E.

Remark 2.2. If E is a prime divisor on a normal variety Y and V is an open subset of Y with $E \cap V \neq \emptyset$, it is clear that E and $E \cap V$ (considered as a divisor on V) give the same valuation of $k(Y)$. Going in the opposite direction, when considering a divisor over X, we can always realize it as a prime divisor E on Y, where the birational morphism $Y \to X$ is proper. Indeed, it follows from a theorem of Nagata and Deligne [**Con07**] that there is a variety Z with a proper morphism $Z \to X$ and a morphism $Y \hookrightarrow Z$ over X which is an open immersion. After possibly replacing Z by its normalization, we may assume that Z is normal, and in this case we may replace any prime divisor E on Y by its closure in Z; as we have seen, the corresponding valuation does not change.

REMARK 2.3. If $f\colon Z \to Y$ is a proper birational morphism of normal varieties, let U be the domain of the rational map f^{-1}. This is the largest open subset of Y with the property that the induced morphism $f^{-1}(U) \to U$ is an isomorphism. Since Y is normal, it follows from the valuative criterion for properness that $\operatorname{codim}_Y(Y \smallsetminus U) \geq 2$. The closed subset $\operatorname{Exc}(f) := Z \smallsetminus f^{-1}(U)$ is the *exceptional locus* of f and a prime divisor E on Z is *exceptional* (or f*-exceptional* if f is not clear from the context) if it is contained in the exceptional locus. Note that E is exceptional if and only if $\dim\big(f(E)\big) < \dim(E)$.

If F is a prime divisor on Y, then $F \cap U \neq \emptyset$; the *strict transform* of F is $\widetilde{F} := \overline{f^{-1}(F \cap U)}$. Note that $\operatorname{ord}_F = \operatorname{ord}_{\widetilde{F}}$: this follows by restricting over U. It is clear that if E is a prime divisor on Z that is not f-exceptional, then $f(E)$ is a prime divisor on Y and $E = \widetilde{f(E)}$.

EXERCISE 2.4. Let $g\colon Z \to Y$ and $f\colon Y \to X$ be two proper birational morphisms between normal varieties.

i) Show that a prime divisor E on Z is $(f \circ g)$-exceptional if and only if either E is g-exceptional or it is the strict transform of an f-exceptional prime divisor on Y.
ii) Show that $\operatorname{Exc}(f \circ g) = \operatorname{Exc}(g) \cup g^{-1}\big(\operatorname{Exc}(f)\big)$.

REMARK 2.5. i) If Y is a normal variety and E_1 and E_2 are two prime divisors on Y, then $\operatorname{ord}_{E_1} = \operatorname{ord}_{E_2}$ if and only if $E_1 = E_2$. Indeed, this follows from the fact that $E_1 = E_2$ if and only if $\mathcal{O}_{Y,E_1} = \mathcal{O}_{Y,E_2}$ and

$$\mathcal{O}_{Y,E_i} = \{\varphi \in k(Y) \mid \operatorname{ord}_{E_i}(\varphi) \geq 0\}.$$

ii) Suppose that we have two divisors over X, realized as the prime divisors E_i on the normal varieties Y_i, where we have proper birational morphisms $f_i\colon Y_i \to X$ (for $i = 1, 2$). In this case there is a commutative diagram

$$\begin{tikzcd} W \arrow[r, "g_2"] \arrow[d, "g_1"'] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & X, \end{tikzcd}$$

with W a normal variety and g_1 and g_2 proper and birational. For example, we can take W to be the normalization of the unique irreducible component of $Y_1 \times_X Y_2$ that dominates X. In this case, it follows from part i) and Remark 2.3 that $\operatorname{ord}_{E_1} = \operatorname{ord}_{E_2}$ if and only if $\widetilde{E_1} = \widetilde{E_2}$ as prime divisors on W.

REMARK 2.6. If v is a divisorial valuation of X and we have a birational morphism $g\colon Y \to X$, it is clear that every divisorial valuation of Y is also a divisorial valuation of X. Conversely, if g is also proper and v is a divisorial valuation of X, then v is also a divisorial valuation of Y. Indeed, if $v = \operatorname{ord}_E$, where E is a prime divisor on the normal variety Z, that has a proper birational morphism $f\colon Z \to X$, then we have a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Z \arrow[d, "f"] \\ Y \arrow[r, "g"'] & X, \end{tikzcd}$$

with W normal and all morphisms proper and birational. If $\widetilde{E}$ is the strict transform of E on W, then $v = \mathrm{ord}_{\widetilde{E}}$, which gives our assertion.

DEFINITION 2.7. Let X be a variety and v a divisorial valuation of X. The *center* $c_X(v)$ is defined as follows: choose a birational morphism $f\colon Y \to X$, with Y normal, and a prime divisor E on Y such that $v = \mathrm{ord}_E$. We then put $c_X(v) = c_X(E) := \overline{f(E)}$. Note that this indeed only depends on v: this follows easily from assertion ii) in Remark 2.5.

REMARK 2.8. If v is a divisorial valuation of X and U is an open subset of X, then v is a divisorial valuation of U if and only if $c_X(v) \cap U \neq \emptyset$.

DEFINITION 2.9. Given a variety X and a nonzero ideal[1] $\mathfrak{a}$ of $\mathcal{O}_X$, for every divisorial valuation v, we define the *order of vanishing* $v(\mathfrak{a})$ as follows. Let $\pi\colon Y \to X$ be a proper birational morphism, with Y normal, such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for an effective Cartier divisor F on Y and $v = \mathrm{ord}_E$ for a prime divisor E on Y. Note that we can find such Y by first choosing a model (Y, E) with $v = \mathrm{ord}_E$ and then replacing (Y, E) by $(\widetilde{Y}, \widetilde{E})$, where $\widetilde{Y} \to Y$ is the normalization of the blow-up of Y along $\mathfrak{a} \cdot \mathcal{O}_Y$ and $\widetilde{E}$ is the strict transform of E. We then take $v(\mathfrak{a})$ to be the coefficient of E in (the Weil divisor associated to) F.

REMARK 2.10. In order to see that the above definition is independent of the choice of (Y, E), note the following alternative description: if U is an open subset of X with $U \cap c_X(v) \neq \emptyset$ and such that $\mathfrak{a} \cdot \mathcal{O}_U$ is generated by $f_1, \ldots, f_r \in \mathcal{O}_X(U)$, then

$$v(\mathfrak{a}) = \min\{v(f_i) \mid 1 \leq i \leq r\}.$$

If v is a divisorial valuation of X and D is an effective Cartier divisor on X, we also write $v(D)$ for $v\big(\mathcal{O}_X(-D)\big)$.

The assertions in the next lemma follow directly from the definition:

LEMMA 2.11. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be nonzero ideal sheaves on the variety X and v a divisorial valuation on X.*

i) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $v(\mathfrak{a}) \geq v(\mathfrak{b})$.*
ii) *For every positive integer m, we have $v(\mathfrak{a}^m) = m \cdot v(\mathfrak{a})$.*

Given a variety X and a divisorial valuation v of X, for every $m \in \mathbf{Z}_{\geq 0}$ we get an ideal $\mathfrak{a}_m(v) \subseteq \mathcal{O}_X$, as follows. For every open subset $U \subseteq X$, we take

$$\Gamma\big(U, \mathfrak{a}_m(v)\big) = \begin{cases} \{f \in \mathcal{O}_X(U) \mid v(f) \geq 0\}, & \text{if } U \cap c_X(v) \neq \emptyset; \\ \mathcal{O}_X(U), & \text{otherwise.} \end{cases}$$

LEMMA 2.12. *If $\pi\colon Y \to X$ is a proper birational morphism of normal varieties and E is a prime divisor on Y, then*

$$\mathfrak{a}_m(\mathrm{ord}_E) = \pi_*\mathcal{O}_Y(-mE) \quad \textit{for all} \quad m \in \mathbf{Z}_{\geq 0}.$$

In particular, the sheaf $\mathfrak{a}_m(\mathrm{ord}_E)$ is coherent.

PROOF. Recall that $\mathcal{O}_Y(-mE)$ is the sheaf of rational functions on Y whose sections over an open subset V of Y consist of 0, together with those φ such that

[1]Unless explicitly mentioned otherwise, all ideal sheaves are assumed to be coherent.

the restriction of $\mathrm{div}_Y(\varphi) - mE$ to V is effective. Note that we have an injection $\mathcal{O}_Y(-mE) \hookrightarrow \mathcal{O}_Y$ which induces an injection

$$\pi_*\mathcal{O}_Y(-mE) \hookrightarrow \pi_*\mathcal{O}_Y = \mathcal{O}_X,$$

where the equality follows since π is proper and birational and X is normal (Zariski's Main Theorem). It is now straightforward to see that this injection identifies $\pi_*\mathcal{O}_Y(-mE)$ with $\mathfrak{a}_m(\mathrm{ord}_E)$. $\square$

Example 2.13. Suppose that X is a smooth variety and Z is a smooth subvariety, defined by the ideal $\mathcal{I}_Z$. Let $f\colon \widetilde{X} \to X$ be the blow-up of X along Z, with exceptional divisor E. Note that E is a projective bundle over Z, hence it is smooth and irreducible, and thus $\widetilde{X}$ is smooth as well (and irreducible). We put $\mathrm{ord}_Z := \mathrm{ord}_E$.

Lemma 2.14. *With the notation in the above example, for every $q \geq 0$, we have*

$$f_*\mathcal{O}_{\widetilde{X}}(-qE) = \mathcal{I}_Z^q.$$

Proof. It is clear from the definition that we have an inclusion $\mathcal{I}_Z^q \subseteq f_*\mathcal{O}_{\widetilde{X}}(-qE)$ for every q. Furthermore, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_Z^{q+1} \arrow[r] \arrow[d] & \mathcal{I}_Z^q \arrow[r] \arrow[d] & \mathcal{I}_Z^q/\mathcal{I}_Z^{q+1} \arrow[r] \arrow[d, "\varphi_q"'] & 0 \\
0 \arrow[r] & f_*\mathcal{O}_{\widetilde{X}}\big(-(q+1)E\big) \arrow[r] & f_*\mathcal{O}_{\widetilde{X}}(-qE) \arrow[r] & f_*\mathcal{O}_E(-qE). &
\end{tikzcd}$$

Recall that $E = \mathbf{P}(\mathcal{I}_Z/\mathcal{I}_Z^2)$ and $\mathcal{O}_E(1) \simeq \mathcal{O}_E(-E)$, hence the vertical map φ_q is an isomorphism. The assertion in the lemma thus follows by induction on q starting with the trivial case $q = 0$. $\square$

Remark 2.15. With the notation in Lemma 2.14, by taking the stalk at the generic point of Z, we see that for $a \in \mathcal{O}_{X,Z} \smallsetminus \{0\}$ we have

$$\mathrm{ord}_Z(a) = \max\{j \in \mathbf{Z}_{\geq 0} \mid a \in \mathfrak{m}^j\},$$

where $\mathfrak{m} \subset \mathcal{O}_{X,Z}$ is the maximal ideal.

Example 2.16. More generally, given any irreducible subset Z of a variety X, such that Z intersects the smooth locus X_{sm} of X, we have a divisorial valuation of X, denoted by ord_Z, defined as follows: choose an open subset W of X_{sm}, with $W \cap Z$ smooth and nonempty. Then take ord_Z to be the valuation associated to $W \cap Z \subset W$. It is clear from definition that $c_X(\mathrm{ord}_Z) = Z$. Note that if $\mathrm{codim}_X(Z) = 1$, then this definition of ord_Z agrees with our previous definition in the case of prime divisors.

Exercise 2.17. Let X be a smooth variety and let Z be a smooth subvariety of X. Let $f\colon Y \to X$ be the blow-up along Z, with exceptional divisor E. Show that if D is an effective divisor on X and $\mathrm{ord}_Z(D) = q$, then $f^*(D) = \widetilde{D} + qE$, where if $D = \sum_i a_i D_i$, we put $\widetilde{D} = \sum_i a_i \widetilde{D_i}$.

We next introduce the relative canonical divisor of a proper birational morphism between smooth varieties. Consider, more generally, a birational morphism $f\colon Y \to$

X, with X and Y smooth varieties. Let $d = \dim(X) = \dim(Y)$. We have a canonical morphism $f^*\Omega_X \to \Omega_Y$ that induces the morphism

$$f^*\omega_X = f^*\Omega_X^d \to \Omega_Y^d = \omega_Y.$$

This is injective, since it is a morphism of line bundles which is an isomorphism in the open subset $f^{-1}(U)$, where $U \subseteq X$ is such that $f^{-1}(U) \to U$ is an isomorphism. Therefore it is defined by a nonzero section of the line bundle $\omega_Y \otimes_{\mathcal{O}_Y} f^*\omega_X^{-1}$. The *relative canonical divisor* $K_{Y/X}$ is the corresponding effective divisor on Y, so that $\mathcal{O}_Y(K_{Y/X}) \simeq \omega_Y \otimes_{\mathcal{O}_Y} f^*\omega_X^{-1}$.

PROPOSITION 2.18. If $f\colon Y \to X$ is a proper birational morphism between normal varieties, then the largest open subset over which f is an isomorphism is equal to the set U consisting of those $x \in X$ with $\dim f^{-1}(x) = 0$. If we assume that X and Y are smooth, then $\mathrm{Supp}(K_{Y/X}) = \mathrm{Exc}(f)$; in particular, $\mathrm{Exc}(f)$ has pure codimension 1.

PROOF. By Zariski's Main Theorem, we know that for every $x \in X$, the fiber $f^{-1}(x)$ is connected. In particular, if an irreducible component of $f^{-1}(x)$ is 0-dimensional, then $f^{-1}(x)$ consists of only one point.

Note now that U is open in X by semicontinuity of fiber dimension. As we have seen, the induced map $g\colon f^{-1}(U) \to U$ is injective. Moreover, it is surjective (since f is proper and dominant), hence it is bijective; therefore g is a homeomorphism, being continuous and closed. Furthermore, Zariski's Main Theorem gives $\mathcal{O}_U \simeq g_*\mathcal{O}_{f^{-1}(U)}$, hence g is an isomorphism.

Since the fibers of f over $X \smallsetminus U$ have dimension ≥ 1, it is then clear that U is the largest open subset over which f is an isomorphism. We thus have $\mathrm{Exc}(f) = f^{-1}(X \smallsetminus U)$.

Suppose now that X and Y are smooth. It follows from the definition of $K_{Y/X}$ that $Z := \mathrm{Supp}(K_{Y/X})$ is the set of those $y \in Y$ such that f is not étale at y. Since f is an isomorphism over U, we clearly have $Z \subseteq f^{-1}(X \smallsetminus U) = \mathrm{Exc}(f)$; the opposite inclusion follows since for $x \notin f^{-1}(U)$, there is an irreducible component of $f^{-1}\big(f(x)\big)$ containing x, hence f can't be étale at x. $\square$

EXAMPLE 2.19. Let X be a smooth variety and Z a smooth subvariety of codimension r. If $f\colon Y \to X$ is the blow-up of X along Z, with exceptional divisor E, then $K_{Y/X} = (r-1)E$. Indeed, note first that by Proposition 2.18, we know that $K_{Y/X} = mE$, for some $m \in \mathbf{Z}_{>0}$, hence we just need to show that $m = r-1$. Let U be an affine open subset of X and $x_1, \ldots, x_n$ an algebraic system of coordinates[2] on U such that $Z \cap U$ is defined in U by $(x_1, \ldots, x_r)$. Consider inside $\mathrm{Bl}_{U\cap Z}(U) \subseteq Y$ the affine open subset V with algebraic coordinates $y_1, \ldots, y_n$ that satisfy

$$f^*(x_1) = y_1, \quad f^*(x_i) = y_1 y_i \quad \text{for} \quad 2 \leq i \leq r, \quad f^*(x_j) = y_j \quad \text{for} \quad r < j \leq n.$$

It is then clear that

$$f^*(dx_1) = dy_1, \quad f^*(dx_i) = y_1 dy_i + y_i dy_1 \quad \text{for} \quad 2 \leq i \leq r, \quad \text{and}$$
$$f^*(dx_j) = dy_j \quad \text{for} \quad r < j \leq n,$$

hence

$$f^*(dx_1 \wedge \ldots \wedge dx_n) = y_1^{r-1} dy_1 \wedge \ldots \wedge dy_n,$$

which gives $m = r-1$.

[2]This means that the 1-forms $dx_1, \ldots, dx_n$ trivialize Ω_U.

The following lemma describes the relative canonical divisor for a composition of morphisms:

LEMMA 2.20. *If $f\colon Y \to X$ and $g\colon Z \to Y$ are birational morphisms between smooth varieties, then*

$$K_{Z/X} = K_{Z/Y} + g^*(K_{Y/X}).$$

PROOF. The assertion follows from the fact that the morphism $\omega_Z \to (f\circ g)^*\omega_X$ is the composition

$$\omega_Z \to g^*\omega_Y \to g^*(f^*\omega_X).$$

□

We can now define the log discrepancy of divisorial valuations of smooth varieties. Let X be a smooth variety and v a divisorial valuation of X. Suppose that $v = \mathrm{ord}_E$, where E is a prime divisor on the normal variety Y, with a birational morphism $f\colon Y \to X$. Since Y is smooth in codimension 1, we may replace it by the smooth locus Y_{sm} and E by $E \cap Y_{\mathrm{sm}}$, to assume that Y is smooth. The log discrepancy of v is $A_X(v) := 1 + k_E$, where k_E is the coefficient of E in $K_{Y/X}$. Note that this is independent on the chosen model (Y, E). Indeed, it follows from assertion ii) in Remark 2.5 that it is enough to show that if we have a proper birational morphism $g\colon Y' \to Y$, with Y' normal and $E' = \widetilde{E}$, then E' and E lead to the same invariant. Applying Lemma 2.20 to the composition $Y'_{\mathrm{sm}} \to Y \to X$, we see that the coefficient $k_{E'}$ of $E' \cap Y'_{\mathrm{sm}}$ in $K_{Y'_{\mathrm{sm}}/X}$ is equal to k_E: note that since E' is not g-exceptional, its coefficient in $K_{Y'_{\mathrm{sm}}/Y}$ is 0 (we can't apply Proposition 2.18 here since the morphism $Y'_{\mathrm{sm}} \to Y$ might not be proper, but the assertion follows directly from the definition of $K_{Y'_{\mathrm{sm}}/Y}$).

REMARK 2.21. It follows from Lemma 2.20 that if $f\colon Y \to X$ is a birational morphism between smooth varieties, then for every divisorial valuation v of Y, we have

$$A_X(v) = A_Y(v) + v(K_{Y/X}).$$

We end this section by recalling the notions of resolutions of singularities that we will need. From now on, we assume that the ground field has characteristic 0. Given a variety X, a *resolution of singularities* of X is a proper, birational morphism $f\colon Y \to X$, with Y smooth. Resolutions exist in characteristic 0 by a fundamental result of Hironaka [**Hir64**]. In fact, it is known that one can take f to be projective and an isomorphism over the smooth locus of X.

In this course, we will need a version adapted to the case of a pair $(X, \mathfrak{a})$, where $\mathfrak{a}$ is a nonzero ideal on X. We begin with the following key

DEFINITION 2.22. Suppose that X is a smooth variety and D is a divisor on X. We say that D has *simple normal crossings* (SNC, for short) if every point of X has an open neighborhood U with an algebraic system of coordinates $x_1, \ldots, x_n$ such that $D|_U = \mathrm{div}_U(x_1^{a_1} \cdots x_n^{a_n})$ for some $a_1, \ldots, a_n \in \mathbf{Z}$. Note that this is only on condition on the support of D.

Given a normal variety X and a nonzero ideal $\mathfrak{a}$ on X, a *log resolution* of the pair $(X, \mathfrak{a})$ is a proper birational morphism $f\colon Y \to X$, with Y smooth, such that the following conditions hold:

i) We have $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for an effective divisor F on Y.
ii) $\mathrm{Exc}(f)$ has pure codimension 1.

iii) If E is the sum of the f-exceptional divisors, then $F+E$ has SNC.

Note that by Proposition 2.18, condition ii) above is automatically satisfied if X is smooth. Log resolutions are known to exist in characteristic 0 by Hironaka's result [**Hir64**]. Moreover, by this result, we may assume that f is projective. In fact, one can assume in addition that f is an isomorphism over $U \subseteq X$ if the open subset U is smooth and $\mathfrak{a}|_U$ is the ideal defining an effective SNC divisor on U, see [**Kol07**]. If Y is a proper closed subscheme of X (for example, an effective Cartier divisor) defined by the ideal $\mathfrak{a}_Y$, then a log resolution of $(X, \mathfrak{a}_Y)$ will also be called a log resolution of (X, Y) (this is also called an *embedded resolution of singularities* of Y when X is smooth and Y is a hypersurface in X).

Remark 2.23. If we have several nonzero ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ on the normal variety X, we may consider a log resolution of $(X, \mathfrak{a})$, where $\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_r$. If this is given by $f\colon Y \to X$, then this has the property that we can write $\mathfrak{a}_i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_i)$ for some effective divisors F_i on Y and if E is the sum of the f-exceptional divisors, then $E + \sum_{i=1}^r F_i$ has SNC.

Remark 2.24. Let $f\colon Y \to X$ be a proper birational morphism, with X normal. If $\mathfrak{a}$ is a nonzero ideal on X, then we can find a log resolution $h\colon Z \to X$ of $(X, \mathfrak{a})$ that dominates Y (that is, which factors as $f \circ g$, for some $g\colon Z \to Y$). Indeed, after possibly replacing Y by its normalization, we may assume that it is normal. If $\mathfrak{b}$ is the ideal defining $\mathrm{Exc}(f)$ (with the reduced scheme structure) and if $g\colon Z \to Y$ is a log resolution of $\big(Y, (\mathfrak{a} \cdot \mathcal{O}_Y) \cdot \mathfrak{b}\big)$, then $h = f \circ g$ satisfies the desired condition: note that by assertion ii) in Exercise 2.4, $\mathrm{Exc}(h) = \mathrm{Exc}(g) \cup g^{-1}\big(\mathrm{Exc}(f)\big)$.

Remark 2.25. Given a nonzero ideal $\mathfrak{a}$ on the normal variety X and two log resolutions $f_1\colon Y_1 \to X$ and $f_2\colon Y_2 \to X$ of $(X, \mathfrak{a})$, we can find another such log resolution $f\colon Y \to X$ that dominates both of them. Indeed, arguing as in Remark 2.5 we see that we have a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & X, \end{tikzcd}$$

with all maps proper and birational. We can then find the log resolution $Y \to X$ that dominates $W \to X$ using the previous remark.

Exercise 2.26. Let X be a smooth variety, E a prime divisor on X, and $D = \sum_{i=1}^r a_i D_i$ a divisor on X such that $E \neq D_i$ for all i.

i) Show that D has simple normal crossings if and only if for every $J \subseteq \{1, \ldots, r\}$, the intersection $\bigcap_{i \in J} D_i$ is either empty or smooth, of codimension $|J|$ in X.

ii) Deduce that if the ground field has characteristic 0, the divisor D has simple normal crossings, and H is a general element of a base-point free linear system on X, then $D + H$ has simple normal crossings.

iii) Show that if $D + E$ has simple normal crossings, then E is smooth and $D|_E$ has simple normal crossings. Moreover, the divisors $D_i|_E$ are smooth (possibly disconnected), without common components.

iv) If E is smooth, the $D_i|_E$ have no common components, and $D|_E$ has simple normal crossings, then there is an open neighborhood U of E_1 such that $(D+E)|_U$ has simple normal crossings.

Exercise 2.27. Let X be a smooth variety over a ground field k, with $\mathrm{char}(k) = 0$, let Z be a smooth subvariety of X of codimension r, and let $\mathfrak{a}$ be an ideal in $\mathcal{O}_X$.

i) Suppose that $x_1, \ldots, x_n$ are algebraic coordinates on an open subset U of X such that $Z \cap U$ is nonempty and defined in U by $(x_1, \ldots, x_r)$. Let $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \in \mathrm{Der}_k\big(\mathcal{O}_X(U)\big)$ be the dual basis of $dx_1, \ldots, dx_n$, and for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n_{\geq 0}$, we consider $\frac{\partial^{|\alpha|}}{\partial x^\alpha} \colon \mathcal{O}_X(U) \to \mathcal{O}_X(U)$. Show that for $f \in \mathcal{O}_X(U)$ we have $\mathrm{ord}_Z(f) \geq m$ if and only if $\frac{\partial^{|\alpha|} f}{\partial x^\alpha}$ vanishes on X for all $\alpha = (\alpha_1, \ldots, \alpha_r, 0, \ldots, 0)$, with $|\alpha| = \alpha_1 + \ldots + \alpha_r \leq m-1$.

ii) Show that $\mathrm{ord}_Z(\mathfrak{a}) \geq m$ if and only if $\mathrm{ord}_x(\mathfrak{a}) \geq m$ for all $x \in Z$.

iii) Show that for every m, the set $\{x \in X \mid \mathrm{ord}_x(\mathfrak{a}) \geq m\}$ is closed in X.

2.2. The definition of multiplier ideals

Unless explicitly mentioned otherwise, from now on we work over a fixed algebraically closed field k, with $\mathrm{char}(k) = 0$. Let X be a smooth variety.

Definition 2.28. An $\mathbf{R}$-*divisor* on X is a finite linear combination of prime divisors on X, with $\mathbf{R}$-coefficients. Given such a divisor $D = \sum_{i=1}^N a_i D_i$ (whenever we write this we assume that the D_i are mutually distinct), we put

$$\lfloor D \rfloor = \sum_{i=1}^N \lfloor a_i \rfloor D_i \quad \text{and} \quad \lceil D \rceil = \sum_{i=1}^N \lceil a_i \rceil D_i,$$

where for $a \in \mathbf{R}$ we denote by $\lfloor a \rfloor$ (resp. $\lceil a \rceil$) the largest integer $\leq a$ (resp. the smallest integer $\geq a$).

We can now give the definition of the main object we will be concerned with in this chapter.

Definition 2.29. Let X be a smooth variety and $\mathfrak{a}$ a nonzero sheaf of ideals. For every $\lambda \in \mathbf{R}_{\geq 0}$, the *multiplier ideal* $\mathcal{J}(\mathfrak{a}^\lambda)$ is defined as follows: if $f \colon Y \to X$ is a log resolution of $(X, \mathfrak{a})$, with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(F)$, then

$$\mathcal{J}(\mathfrak{a}^\lambda) := f_*\mathcal{O}_Y\big(K_{Y/X} - \lfloor \lambda F \rfloor\big).$$

It is a bit hard to justify at this point why make this definition. However, this notion becomes more natural when considered in the context of vanishing theorems, as we will see later.

Remark 2.30. With the notation in the above definition, note that $\lfloor \lambda F \rfloor$ is an effective divisor, hence

$$\mathcal{O}_Y\big(K_{Y/X} - \lfloor \lambda F \rfloor\big) \subseteq \mathcal{O}_Y(K_{Y/X}) \quad \text{and thus} \quad \mathcal{J}(\mathfrak{a}^\lambda) \subseteq f_*\mathcal{O}_Y(K_{Y/X}).$$

Since $K_{Y/X}$ is an f-exceptional effective divisor, it follows from the lemma below that the inclusion $\mathcal{O}_Y \subseteq \mathcal{O}_Y(K_{Y/X})$ induces an equality

$$\mathcal{O}_X = f_*\mathcal{O}_Y = f_*\mathcal{O}_Y(K_{Y/X}).$$

Therefore $\mathcal{J}(\mathfrak{a}^\lambda)$ is an ideal of $\mathcal{O}_X$.

LEMMA 2.31. *Let $f\colon Y \to X$ be a proper birational morphism between normal varieties. If D is an effective f-exceptional*[3] *Weil divisor on Y, then the inclusion $\mathcal{O}_Y \hookrightarrow \mathcal{O}_Y(D)$ induces equalities*

$$\mathcal{O}_X = f_*\mathcal{O}_Y = f_*\mathcal{O}_Y(D).$$

PROOF. The first equality is a consequence of Zariski's Main Theorem but it will also follow from the argument below. In order to prove that the inclusions $\mathcal{O}_X \hookrightarrow f_*\mathcal{O}_Y \hookrightarrow f_*\mathcal{O}_Y(D)$ are equalities, we may assume that X is affine. We need to show that if $\varphi \in k(X) \smallsetminus \{0\}$ is such that $\mathrm{div}_Y(\varphi) + D$ is effective, then $\varphi \in \mathcal{O}_X(U)$. Indeed, if $\varphi \notin \mathcal{O}_X(U)$, then there is a prime divisor E on X with $\mathrm{ord}_E(\varphi) < 0$. Since $\widetilde{E}$ does not appear in D, it follows that the coefficient of E in $\mathrm{div}_Y(\varphi) + D$ is negative, a contradiction. $\square$

REMARK 2.32. With the notation in Definition 2.29, if U is open in X, we know that $\varphi \in \mathcal{O}_X(U)$ lies in $\Gamma\big(U, \mathcal{J}(\mathfrak{a}^\lambda)\big)$ if and only if for every prime divisor E on Y with $c_X(E) \cap U \neq \emptyset$, we have

$$\mathrm{ord}_E(K_{Y/X}) + \mathrm{ord}_E(\varphi) \geq \lfloor \lambda \cdot \mathrm{ord}_E(F) \rfloor = \lfloor \lambda \cdot \mathrm{ord}_E(\mathfrak{a}) \rfloor,$$

or equivalently

$$\mathrm{ord}_E(\varphi) + A_X(\mathrm{ord}_E) > \lambda \cdot \mathrm{ord}_E(\mathfrak{a}).$$

The following is the main result of this section:

THEOREM 2.33. *The definition of $\mathcal{J}(\mathfrak{a}^\lambda)$ does not depend on the choice of resolution.*

PROOF. Suppose that U is an open subset of X and $\varphi \in \mathcal{O}_X(U)$. If $f\colon Y \to X$ is a log resolution and F as in Definition 2.29, then we saw in Remark 2.32 that φ is a section of $\mathcal{J}(\mathfrak{a}^\lambda)$ if and only if for every prime divisor E on Y with $c_X(E) \cap U \neq \emptyset$, we have

$$\mathrm{ord}_E(\varphi) + A_X(\mathrm{ord}_E) > \lambda \cdot \mathrm{ord}_E(\mathfrak{a}). \tag{2.1}$$

We need to show that if this is the case, then the same inequality remains true for every divisor E' over X with $c_X(E') \cap U \neq \emptyset$.

Without any loss of generality, we may assume that $U = X$. Given a divisor E' over X, we want to show that

$$\mathrm{ord}_{E'}(\varphi) + A_X(\mathrm{ord}_{E'}) > \lambda \cdot \mathrm{ord}_{E'}(\mathfrak{a}). \tag{2.2}$$

Consider the divisors $E_1, \ldots, E_r$ on X that contain $c_Y(\mathrm{ord}_{E'})$ and that are contained in $\mathrm{Supp}(F) \cup \mathrm{Exc}(f)$. Let $q_i = \mathrm{ord}_{E'}(E_i)$ for $1 \leq i \leq r$. By assumption, $E_1 + \ldots + E_r$ is an SNC divisor. Note that since $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, we have

$$\mathrm{ord}_{E'}(\mathfrak{a}) = \sum_{i=1}^r q_i \cdot \mathrm{ord}_{E_i}(\mathfrak{a}). \tag{2.3}$$

Since the inequality (2.2) is trivial if $c_Y(\mathrm{ord}_{E'}) \not\subseteq \mathrm{Supp}(F) \cup \mathrm{Exc}(f)$, we may and will assume that $r \geq 1$ (for those interested also in the case when X is singular: in that setting, in order to run the same argument, we need here the condition that $\mathrm{Exc}(f)$ is of pure codimension 1). It is clear that we have

$$\mathrm{ord}_{E'}(\varphi) \geq \sum_{i=1}^r q_i \cdot \mathrm{ord}_{E_i}(\varphi),$$

[3]This means that all prime divisors in D are f-exceptional.

hence using (2.1) for $E = E_1, \ldots, E_r$ and (2.3), we get

$$\mathrm{ord}_{E'}(\varphi) > \sum_{i=1}^{r} q_i\big(\lambda \cdot \mathrm{ord}_{E_i}(\mathfrak{a}) - A_X(\mathrm{ord}_{E_i})\big) = \lambda \cdot \mathrm{ord}_{E'}(\mathfrak{a}) - \sum_{i=1}^{r} q_i \cdot A_X(\mathrm{ord}_{E_i}).$$

Therefore (2.2) follows if we prove that

$$A_X(\mathrm{ord}_{E'}) \geq \sum_{i=1}^{r} q_i \cdot A_X(\mathrm{ord}_{E_i}). \tag{2.4}$$

Lemma 2.34 below gives $A_Y(\mathrm{ord}_{E'}) \geq \sum_{i=1}^{r} q_i$. If $k_i = A_X(\mathrm{ord}_{E_i}) - 1$, then $K_{Y/X} = \sum_i k_i E_i$ and using Remark 2.21 we get

$$A_X(\mathrm{ord}_{E'}) = A_Y(\mathrm{ord}_{E'}) + \mathrm{ord}_{E'}(K_{Y/X}) \geq \sum_{i=1}^{r} q_i + \sum_{i=1}^{r} k_i q_i = \sum_{i=1}^{r} q_i \cdot A_X(\mathrm{ord}_{E_i}).$$

Therefore (2.4) holds and this completes the proof. □

LEMMA 2.34. *Let Y be a smooth variety and $E = E_1 + \ldots + E_r$ an SNC divisor on Y. If E' is a divisor over Y with $q_i := \mathrm{ord}_{E'}(E_i) \geq 1$ for all i and $s = \mathrm{codim}_Y\big(c_Y(\mathrm{ord}_{E'})\big)$, then*

$$A_Y(\mathrm{ord}_{E'}) \geq (s - r) + \sum_{i=1}^{r} q_i.$$

PROOF. Suppose that E' is a prime divisor on the normal variety W, that has a proper birational morphism $g\colon W \to Y$. Let $P \in E'$ be a general point, so that both W and E' are smooth at P. Furthermore, we may and will assume that $g(P)$ is a smooth point of $c_Y(\mathrm{ord}_{E'})$. Choose a regular system of parameters $y_1, \ldots, y_n$ in $\mathcal{O}_{Y,g(P)}$ such that E_i is defined at $g(P)$ by (y_i) for $1 \leq i \leq r$ and $c_Y(\mathrm{ord}_{E'})$ is defined at P by $(y_1, \ldots, y_s)$. Consider also a regular system of parameters $x_1, \ldots, x_n$ in $\mathcal{O}_{W,P}$, such that E' is defined at P by (x_1). By assumption, we can write

$$g^*(y_i) = x_1^{q_i} u_i \quad \text{for} \quad 1 \leq i \leq s$$

for some $u_i \in \mathcal{O}_{W,P}$, where we put $q_i = 1$ for $r < i \leq s$. We thus have

$$g^*(dy_i) = q_i x_1^{q_i - 1} u_i dx_1 + x_1^{q_i} du_i \quad \text{for} \quad 1 \leq i \leq s,$$

which implies

$$g^*(dy_1 \wedge \ldots \wedge dy_n) \in x_1^{q_1 + \ldots + q_s - 1} \omega_{W,P},$$

hence

$$A_Y(\mathrm{ord}_{E'}) \geq \sum_{i=1}^{s} q_i = (s - r) + \sum_{i=1}^{r} q_i.$$

□

Independence on resolution implies the following Change of Variable formula for multiplier ideals.

COROLLARY 2.35. Let $f\colon Y \to X$ be a proper birational morphism between smooth varieties. If $\mathfrak{a}$ is a nonzero ideal on X and $\mathfrak{b} = \mathfrak{a} \cdot \mathcal{O}_Y$, then for every $\lambda \in \mathbf{R}_{\geq 0}$, we have

$$\mathcal{J}(\mathfrak{a}^\lambda) = f_*\big(\mathcal{J}(\mathfrak{b}^\lambda) \cdot \mathcal{O}_Y(K_{Y/X})\big).$$

PROOF. We choose a log resolution $h\colon Z \to X$ for $(X, \mathfrak{a})$ that factors through f, that is, we have $h = f \circ g$ for a proper birational morphism $g\colon Z \to Y$ (see Remark 2.24). In this case g is a log resolution of $(Y, \mathfrak{b})$. Let F be the divisor on Z such that $\mathfrak{a} \cdot \mathcal{O}_Z = \mathcal{O}_Z(-F) = \mathfrak{b} \cdot \mathcal{O}_Z$. We thus have

$$\mathcal{J}(\mathfrak{a}^\lambda) = h_* \mathcal{O}_Z(K_{Z/X} - \lfloor \lambda F \rfloor) \quad \text{and}$$

$$\mathcal{J}(\mathfrak{b}^\lambda) = g_* \mathcal{O}_Z(K_{Z/Y} - \lfloor \lambda F \rfloor).$$

Recall that by Lemma 2.20 we have $K_{Z/X} = K_{Z/Y} + g^*(K_{Y/X})$, hence the projection formula gives

$$f_*\big(\mathcal{J}(\mathfrak{b}^\lambda) \cdot \mathcal{O}_Y(K_{Y/X})\big) = f_*\big(g_* \mathcal{O}_Z(K_{Z/Y} - \lfloor \lambda F \rfloor) \cdot \mathcal{O}_Y(K_{Y/X})\big)$$

$$= h_* \mathcal{O}_Z(K_{Z/Y} + g^*(K_{Y/X}) - \lfloor \lambda F \rfloor) = \mathcal{J}(\mathfrak{a}^\lambda).$$

□

REMARK 2.36. Note that if m is a positive integer, a log resolution $f\colon Y \to X$ of $(X, \mathfrak{a})$ is also a log resolution of $(X, \mathfrak{a}^m)$ and if $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, then $\mathfrak{a}^m \cdot \mathcal{O}_Y = \mathcal{O}_Y(-mF)$. It then follows from the definition that $\mathcal{J}\big((\mathfrak{a}^m)^\lambda\big) = \mathcal{J}(\mathfrak{a}^{m\lambda})$ for every $\lambda \in \mathbf{R}_{\geq 0}$.

If $\lambda = 1$, then we simply write $\mathcal{J}(\mathfrak{a})$ instead of $\mathcal{J}(\mathfrak{a}^1)$. Also, if D is an effective $\mathbf{Q}$-divisor, then we write $\mathcal{J}(D)$ for $\mathcal{J}(\mathfrak{a}^{1/m})$, where m is a positive integer such that mD has integer coefficients and $\mathfrak{a} = \mathcal{O}_X(-mD)$ (note that $\mathcal{J}(D)$ does not depend on the choice of m by Remark 2.36).

REMARK 2.37. Note that for every nonzero ideal $\mathfrak{a}$ and every $\lambda \in \mathbf{R}_{\geq 0}$, we have

$$\mathfrak{a} \subseteq \mathrm{rad}\big(\mathcal{J}(\mathfrak{a}^\lambda)\big).$$

In fact, if m is an integer such that $m \geq \lambda$, then it follows directly from the definition of the multiplier ideal that

$$\mathfrak{a}^m \subseteq \mathcal{J}(\mathfrak{a}^\lambda).$$

2.2.1. Jumping numbers and the log canonical threshold. The following proposition follows directly from the definition of multiplier ideals and the properties of the $\lfloor - \rfloor$ function.

PROPOSITION 2.38. If $\mathfrak{a}$ is a nonzero ideal on the smooth variety X, then the following hold:

i) If $\lambda, \mu \in \mathbf{R}_{\geq 0}$ and $\lambda \geq \mu$, then $\mathcal{J}(\mathfrak{a}^\lambda) \subseteq \mathcal{J}(\mathfrak{a}^\mu)$.
ii) For every $\lambda \in \mathbf{R}_{\geq 0}$, there is $\epsilon > 0$ such that

$$\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^\mu) \quad \text{for all } \mu \text{ with } \lambda \leq \mu \leq \lambda + \epsilon.$$

iii) $\mathcal{J}(\mathfrak{a}^0) = \mathcal{O}_X$.
iv) There is $m \in \mathbf{Z}_{>0}$ such that for all $\lambda \in \mathbf{R}_{>0} \smallsetminus \frac{1}{m}\mathbf{Z}$, there is $\epsilon > 0$ such that

$$\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^\mu) \quad \text{for all } \mu \text{ with } \lambda - \epsilon \leq \mu \leq \lambda.$$

DEFINITION 2.39. A positive λ is a *jumping number* of $\mathfrak{a}$ if $\mathcal{J}(\mathfrak{a}^\lambda) \subsetneq \mathcal{J}(\mathfrak{a}^\mu)$ for all $\mu < \lambda$.

It follows from Proposition 2.38 that there is a positive integer m such that all jumping numbers of $\mathfrak{a}$ lie in $\frac{1}{m}\mathbf{Z}_{>0}$. The jumping numbers of $\mathfrak{a}$ were introduced and studied systematically in [**ELSV04**]. The most important jumping number is the first one: this is the *log canonical threshold* $\mathrm{lct}(\mathfrak{a})$. By Proposition 2.38, we have

$$\mathrm{lct}(\mathfrak{a}) := \sup\{\lambda > 0 \mid \mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{O}_X\} = \min\{\lambda > 0 \mid \mathcal{J}(\mathfrak{a}^\lambda) \neq \mathcal{O}_X\}$$

(with the convention that $\mathrm{lct}(\mathfrak{a}) = \infty$ if $\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{O}_X$ for all λ). If $\mathfrak{a} = \mathcal{O}_X(-D)$, where D is an effective divisor on X, it is common to also write $\mathrm{lct}(X, D)$ (or $\mathrm{lct}(D)$ if X is understood) instead of $\mathrm{lct}(\mathfrak{a})$.

REMARK 2.40. Let $\mathfrak{a}$ be a nonzero ideal on the smooth variety X. If $f\colon Y \to X$ is a log resolution of $(X, \mathfrak{a})$, let F be the divisor on Y such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ and let us write

$$F = \sum_{i=1}^r a_i E_i \quad \text{and} \quad K_{Y/X} = \sum_{i=1}^r k_i E_i.$$

It follows from the definition of $\mathcal{J}(\mathfrak{a}^\lambda)$ that $1 \in \mathcal{J}(\mathfrak{a}^\lambda)$ if and only if $k_i + 1 > \lambda a_i$ for all i. We thus conclude that

$$\mathrm{lct}(\mathfrak{a}) = \min_i \frac{k_i + 1}{a_i}.$$

Since we know that this is independent of the resolution, we can also write

$$\mathrm{lct}(\mathfrak{a}) = \min_v \frac{A_X(v)}{v(\mathfrak{a})},$$

where the minimum is over all divisorial valuations of X. We note that $\mathrm{lct}(\mathfrak{a}) = \infty$ if and only if $\mathfrak{a} = \mathcal{O}_X$.

It is convenient to also have a local version of the log canonical threshold.

DEFINITION 2.41. If $\mathfrak{a}$ is a nonzero ideal on the smooth variety X and $P \in X$, then

$$\mathrm{lct}_P(\mathfrak{a}) := \sup\{\lambda > 0 \mid \mathcal{J}(\mathfrak{a}^\lambda)_P = \mathcal{O}_{X,P}\} = \min\{\lambda > 0 \mid \mathcal{J}(\mathfrak{a}^\lambda)_P \neq \mathcal{O}_{X,P}\}$$

(with the convention that $\mathrm{lct}_P(\mathfrak{a}) = \infty$ if $\mathcal{J}(\mathfrak{a}^\lambda)_P = \mathcal{O}_{X,P}$ for all λ). With the notation in Remark 2.40, we have

$$\mathrm{lct}(\mathfrak{a}) = \min_{i; P \in f(E_i)} \frac{k_i + 1}{a_i} = \min_{v; P \in c_X(v)} \frac{A_X(v)}{v(\mathfrak{a})}.$$

In particular, we have $\mathrm{lct}_P(\mathfrak{a}) < \infty$ if and only if $P \in V(\mathfrak{a})$.

REMARK 2.42. It follows from definition that $\mathrm{lct}(\mathfrak{a}) = \min_{P \in X} \mathrm{lct}_P(\mathfrak{a})$ and $\mathrm{lct}_P(\mathfrak{a}) = \max_{U \ni P} \mathrm{lct}(\mathfrak{a}|_U)$, where the maximum is over all open neighborhoods U of P.

REMARK 2.43. It follows from the definition of the log canonical threshold and Remark 2.36 that for every positive integer m and every nonzero ideal $\mathfrak{a}$, we have

$$\mathrm{lct}(\mathfrak{a}^m) = \frac{\mathrm{lct}(\mathfrak{a})}{m} \quad \text{and} \quad \mathrm{lct}_P(\mathfrak{a}^m) = \frac{\mathrm{lct}_P(\mathfrak{a})}{m}.$$

EXERCISE 2.44. Show that if $\mathfrak{a}$ and $\mathfrak{b}$ are nonzero ideals on the smooth variety X, then we have

$$\frac{1}{\mathrm{lct}(\mathfrak{a}\mathfrak{b})} \leq \frac{1}{\mathrm{lct}(\mathfrak{a})} + \frac{1}{\mathrm{lct}(\mathfrak{b})}$$

and for every $P \in X$, a similar inequality involving the log canonical thresholds at P.

EXERCISE 2.45. Let $\mathfrak{a}$ be a nonzero ideal on the smooth variety X. Show that for every $\lambda \in \mathbf{R}_{\geq 0}$, we have

$$\frac{1}{\operatorname{lct}\big(\mathcal{J}(\mathfrak{a}^\lambda)\big)} \geq \frac{\lambda}{\operatorname{lct}(\mathfrak{a})} - 1,$$

with the convention that the quotient is 0 if the log canonical threshold at the denominator is infinite.

2.2.2. Analytic description. When the ground field is $\mathbf{C}$, there is an analytic description of multiplier ideals that we now discuss. While we will not make use of this in what follows, it gives more intuition about this notion than then algebraic definition.

Suppose now that X is a smooth complex algebraic variety and $\mathfrak{a} = (f_1, \ldots, f_r)$, where $f_1, \ldots, f_r \in \mathcal{O}_X$. Let $g = \sum_{i=1}^r |f_i|^2$.

DEFINITION 2.46. For every $\lambda \in \mathbf{R}_{\geq 0}$ and every open subset U of X, we put

$$\Gamma\big(U, \mathcal{J}^{\mathrm{an}}(\mathfrak{a}^\lambda)\big) := \left\{ h \in \mathcal{O}_X(U) \mid \frac{|h|^2}{g^\lambda} \text{ is locally integrable on } U \right\}.$$

We note that the condition means that for every $P \in U$, there is an analytic open neighborhood $V \subseteq U$ of P and analytic coordinates $z_1, \ldots, z_n$ on V such that $\int_V |h|^2/|g|^\lambda dz d\overline{z} < \infty$.

It is easy to see from the definition that $\mathcal{J}^{\mathrm{an}}(\mathfrak{a}^\lambda)$ is a sheaf of ideals in $\mathcal{O}_X$, though coherence is not clear at this point (the independence of the choice of generators of $\mathfrak{a}$ is not obvious either). We have

THEOREM 2.47. *With the above notation, for every $\lambda \in \mathbf{R}_{\geq 0}$, we have*

$$\mathcal{J}^{\mathrm{an}}(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda).$$

We first prove the version of the Change of Variable formula for analytic multiplier ideals. Let $\pi \colon X' \to X$ be a proper birational morphism between smooth complex algebraic varieties. Suppose that $\mathfrak{a} = (f_1, \ldots, f_r)$, where $f_1, \ldots, f_r \in \mathcal{O}_X$ and let $f_i' = f_i \circ \pi$, so $\mathfrak{b} := \mathfrak{a} \cdot \mathcal{O}_Y = (f_1', \ldots, f_r')$. In the next lemma, we use these generators to compute the analytic multiplier ideals.

LEMMA 2.48. *With the above notation, we have*

$$\mathcal{J}^{\mathrm{an}}(\mathfrak{a}^\lambda) = f_*\big(\mathcal{J}^{\mathrm{an}}(\mathfrak{b}^\lambda) \cdot \mathcal{O}_Y(K_{Y/X})\big).$$

PROOF. This follows easily from the properness of π and the Change of Variable formula for Lebesgue integrals by noting that π gives a diffeomorphism between the complements of measure 0 subsets. $\square$

We now deduce the equality of the algebraic and analytic multiplier ideals.

PROOF. We need to check that the two sheaves have the same sections over any open subset U of X. After replacing X by U, it is enough to consider $h \in \mathcal{O}_X(X)$. Let $\pi \colon Y \to X$ be a log resolution of $h \cdot \mathfrak{a}$ and let $\mathfrak{b} = \mathfrak{a} \cdot \mathcal{O}_Y$. Because both algebraic and analytic multiplier ideals satisfy the Change of Variable formula (see Corollary 2.35 and Lemma 2.48), we see that it is enough to show that the following holds. Suppose that $P \in Y$ and we have coordinates $x_1, \ldots, x_n$ in a neighborhood of P, with $x_i(P) = 0$ for all i, such that $\mathfrak{b} = (w)$, with $w = x_1^{a_1} \cdots x_n^{a_n}$, $K_{Y/X}$ is defined by $v = x_1^{k_1} \cdots x_n^{k_n}$, and $h = ux_1^{b_1} \cdots x_n^{b_n}$, with u invertible; we then need to check that hv is a section of $\mathcal{J}^{\mathrm{an}}(\mathfrak{a}^\lambda)$ around P if and only if it is a section of $\mathcal{J}(\mathfrak{a}^\lambda)$ around P (where the former ideal is computed with respect to a given set of generators). Note that around P, we have

$$\mathcal{J}(\mathfrak{a}^\lambda) = (x_1^{\lfloor a_1\lambda \rfloor} \cdots x_n^{\lfloor a_n\lambda\rfloor})$$

and hv lies in this ideal in a neighborhood of P if and only if $b_i + k_i \geq \lfloor a_i\lambda \rfloor$ for all i.

On the other hand, suppose that the analytic multiplier ideal is associated to a system of generators $f_1, \ldots, f_r$. Note that in this case we can write $f_i = wq_i$ and $(q_1, \ldots, q_r) = 1$. Since $g := \sum_{i=1}^r |f_i|^2 = |f^2| \cdot \sum_{i=1}^r |q_i|^2$ and $|u|$ and $\sum_{i=1}^r |q_i|^2$ are locally bounded and bounded away from 0, we conclude that $|h|^2/g^\lambda$ is locally integrable around P if and only if $\prod_{i=1}^n |x_i|^{2b_i+2k_i} / \prod_{i=1}^n |x_i|^{2a_i\lambda}$ is integrable in a neighborhood of P. In turn, by Fubini's Theorem this is the case if and only if each function $|x|^{2b_i+2k_i-2a_i\lambda}$ is integrable in a neighborhood of 0. It is easy to see (and well-known) that this is the case if and only if $2b_i + 2k_i - 2a_i\lambda > -2$ for all i; equivalently, $b_i + k_i \geq \lfloor a_i\lambda \rfloor$ for all i. This completes the proof. $\square$

Of course, on a complex manifold it is more natural to associate multiplier ideals to sheaves of holomorphic functions. In fact, one can associate such multiplier ideals to *plurisubharmonic functions* and these ideals give a powerful tool in complex geometry, see [**Dem01**].

2.2.3. Mixed multiplier ideals. One can define similarly a *mixed* version of multiplier ideals, as follows. If $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are nonzero ideal sheaves on the smooth variety X, consider a log resolution $f\colon Y \to X$ for the pair $(X, \mathfrak{a}_1 \cdots \mathfrak{a}_r)$. If we write $\mathfrak{a}^i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_i)$ for all i, then for all $\lambda_1, \ldots, \lambda_r \in \mathbf{R}_{\geq 0}$, we define the *mixed multiplier ideal*

$$\mathcal{J}(\mathfrak{a}_1^{\lambda_1} \ldots \mathfrak{a}_r^{\lambda_r}) := f_*\mathcal{O}_Y\big(K_{Y/X} - \lfloor \lambda_1 F_1 + \ldots + \lambda_r F_r \rfloor\big).$$

Most of the time, we will not work in this level of generality in order to simplify the notation. However, all results easily extend to this setting. For example, the proof of Theorem 2.33 extends in a straightforward way to show that the definition of mixed test ideals is independent on the resolution. We also get an obvious extension of the Change of Variable formula in Corollary 2.35.

Remark 2.49. Mixed multiplier ideals can be described by usual multiplier ideals, as follows. First, it follows from the definition and the basic properties of the round-down function that given $\lambda_1, \ldots, \lambda_r \in \mathbf{R}_{\geq 0}$, we can find $\lambda_1', \ldots, \lambda_r' \in \mathbf{Q}_{\geq 0}$, with $0 < \lambda_i' - \lambda_i \ll 1$ such that

$$\mathcal{J}(\mathfrak{a}_1^{\lambda_1} \ldots \mathfrak{a}_r^{\lambda_r}) = \mathcal{J}(\mathfrak{a}_1^{\lambda_1'} \ldots \mathfrak{a}_r^{\lambda_r'}).$$

On the other hand, if $\lambda_1, \ldots, \lambda_r \in \mathbf{Q}_{\geq 0}$, then we can choose a positive integer m such that $m\lambda_i \in \mathbf{Z}$ for all i; if $\mathfrak{a} = \prod_{i=1}^r \mathfrak{a}_i^{m\lambda_i}$, then it follows from the definition that

$$\mathcal{J}(\mathfrak{a}_1^{\lambda_1} \ldots \mathfrak{a}_r^{\lambda_r}) = \mathcal{J}(\mathfrak{a}^{1/m}).$$

If we consider the constancy regions for mixed multiplier ideals, then the structure is more interesting than in the case of one ideal. Indeed, it follows easily from the definition of mixed multiplier ideals that there are linear functions

$L_1,\ldots,L_N\colon \mathbf{R}^r\to\mathbf{R}$ of the form $L_j=\sum_{i=1}^r a_{i,j}x_i$, with $a_{i,j}\in\mathbf{Z}_{\geq 0}$ for all i and j such that for all $\lambda=(\lambda_1,\ldots,\lambda_r)$ in a connected component of

$$\mathbf{R}^r_{\geq 0}\smallsetminus\bigcup_{1\leq i\leq N, m\in\mathbf{Z}_{\geq 0}}\{\lambda\mid L_i(\lambda)=m\}$$

the mixed multiplier ideal $\mathcal{J}(\mathfrak{a}_1^{\lambda_1}\cdots\mathfrak{a}_r^{\lambda_r})$ takes a constant value.

2.2.4. Multiplier ideals in positive characteristic. Suppose now that the ground field has positive characteristic. We can still define multiplier ideals, but the fact that it is not known whether log resolutions exist, makes it very hard to handle even the simplest questions.

Suppose that X is a smooth variety and $\mathfrak{a}$ is a nonzero ideal sheaf on X. We define

$$\mathcal{J}(\mathfrak{a}^\lambda)=\bigcap_v\mathfrak{a}_{m(v)}(v),$$

where the intersection is over all divisorial valuations v on X and $m(v)=\lfloor\lambda\cdot v(\mathfrak{a})\rfloor-A_X(v)+1$. However, since we have an infinite intersection in the definition, it is not clear whether this is a coherent ideal. Similarly, it seems very hard to say anything about the way these ideals vary with λ.

We can also define the log canonical threshold:

$$\mathrm{lct}(\mathfrak{a})=\inf\frac{A_X(v)}{v(\mathfrak{a})},$$

where the infimum is over all divisorial valuations v on X. In this case, it is not clear whether the infimum is achieved (in fact, it is not known whether $\mathrm{lct}(\mathfrak{a})$ is a rational number). Note also that while the definition implies that $\mathcal{J}(\mathfrak{a}^\lambda)=\mathcal{O}_X$ for $\lambda<\mathrm{lct}(\mathfrak{a})$, it is not clear whether $\mathcal{J}(\mathfrak{a}^{\mathrm{lct}(\mathfrak{a})})\neq\mathcal{O}_X$.

However, when we have a log resolution of $(X,\mathfrak{a})$, the proof of Theorem 2.33 goes through and we can describe $\mathcal{J}(\mathfrak{a}^\lambda)$ is terms of this resolution as in characteristic 0. In this case, of course, we have a good control on the jumping numbers. In particular, the minimum in the definition of the log canonical threshold is achieved by some divisorial valuation associated to a divisor on the log resolution.

2.3. Multiplier ideals: examples and first properties

In this section we always assume that the ambient variety X is smooth, over an algebraically closed field of characteristic 0. We begin with some easy examples.

Example 2.50. If $\mathfrak{a}$ is the ideal defining a smooth codimension r subvariety of X, then

$$\mathcal{J}(\mathfrak{a}^\lambda)=\mathfrak{a}^{\lfloor\lambda\rfloor-r+1}, \tag{2.5}$$

with the convention that $\mathfrak{a}^m=\mathcal{O}_X$ for $m\leq 0$. In particular, we have $\mathrm{lct}(\mathfrak{a})=r$.

Indeed, note that the blow-up $f\colon Y\to X$ of X along Z is a log resolution of $(X,\mathfrak{a})$. If E is the exceptional divisor, then we clearly have $\mathfrak{a}\cdot\mathcal{O}_Y=\mathcal{O}_Y(-E)$ and we have seen in Example 2.19 that $K_{Y/X}=(r-1)E$. We thus have

$$\mathcal{J}(\mathfrak{a}^\lambda)=f_*\mathcal{O}_Y\big((r-1-\lfloor\lambda\rfloor)E\big)$$

and using Lemma 2.14 we get the formula (2.5).

EXAMPLE 2.51. For every effective divisor D and every $\lambda \in \mathbf{R}_+$, we have

$$\mathcal{J}(\lambda D) = \mathcal{O}_X(-\lfloor \lambda \rfloor D) \cdot \mathcal{J}(tD), \quad \text{where} \quad t = \lambda - \lfloor \lambda \rfloor.$$

This follows directly from the definition of multiplier ideals and the projection formula. Hence the interesting multiplier ideals in this case are the $\mathcal{J}(\lambda D)$ for $\lambda < 1$. Moreover, we see that if D is nonzero, then $\mathrm{lct}(D) \leq 1$.

EXAMPLE 2.52. Suppose that X is a smooth surface and D is a divisor on X that has at most nodes as singularities. In this case $\mathcal{J}(X, tD) = \mathcal{O}_X$ for all $t < 1$. In particular, we have $\mathrm{lct}(D) = 1$.

Indeed, arguing locally we may assume that D has a unique singular point P which is a node. In this case if $f\colon Y = \mathrm{Bl}_P(X) \to X$ has exceptional divisor E, we have that $E + \widetilde{D}$ is a simple normal crossing divisor, hence f is a log resolution of (X, D). Moreover, we have $f^*(D) = \widetilde{D} + 2E$ and $K_{Y/X} = E$ by Example 2.19, hence for $t < 1/2$ we have

$$\mathcal{J}(tD) = f_*\mathcal{O}_Y(E) = \mathcal{O}_X$$

while for $1/2 \leq t < 1$, we have

$$\mathcal{J}(tD) = f_*\mathcal{O}_Y = \mathcal{O}_X.$$

EXAMPLE 2.53. The first nontrivial example is the case of a curve with a simple cusp: let D be the curve in $X = \mathbf{A}^2 = \operatorname{Spec} k[x_1, x_2]$ defined by $f = x_1^2 + x_2^3$. We need to construct a log resolution of (X, D). Let $\pi_1\colon X_1 \to X$ be the blow-up of X at 0, with exceptional divisor E_1. In order to simplify the notation, we will denote by the same letter a divisor and its strict transform, paying attention to what variety we are on. Note that

$$\pi_1^*(D) = D + 2E_1 \quad \text{and} \quad K_{X_1/X} = E_1.$$

Note that $D \cap E_1$ consists of a point P. In order to describe $D + E_1$ at P, let us consider the chart U on X_1 with coordinates y_1 and y_2 such that

$$\pi_1^*(x_1) = y_1 y_2 \quad \text{and} \quad \pi_1^*(x_2) = y_2.$$

In this case E_1 is defined by the equation y_2 and $\pi_1^*(D)$ is defined by the equation $y_2^2(y_1^2 + y_2)$. We thus see that D on X_1 is smooth, but $D + E$ does not have simple normal crossings.

Let $\pi_2\colon X_2 \to X_1$ be the blow-up of X_1 at P, with exceptional divisor E_2. Note that we have

$$\pi_2^*(D) = D + E_2, \quad \pi_2^*(E_1) = E_1 + E_2, \quad K_{X_2/X_1} = E_2,$$

hence

$$(\pi_1 \circ \pi_2)^*(D) = \pi_2^*(D + 2E_1) = D + 2E_1 + 3E_2 \quad \text{and} \quad K_{X_2/X} = E_2 + \pi_2^*(E_1) = E_1 + 2E_2.$$

Consider on X_2 the chart with coordinates z_1 and z_2, such that

$$\pi_2^*(y_1) = z_1 \quad \text{and} \quad \pi_2^*(y_2) = z_1 z_2.$$

In this chart E_2 is defined by the equation z_1, E_1 is defined by the equation z_2, and $\pi_2^*(D)$ is defined by $z_1(z_1 + z_2)$. We thus have 3 smooth curves on X_2 all of them passing through a point Q (it is easy to see that there are no other intersection points on X_2).

Consider now $\pi_3\colon X_3 \to X_2$ be the blow-up of X_2 at Q, with exceptional divisor E_3. This divisor intersects each of D, E_1, E_2 in distinct points and these are the only intersection points. The divisor $D + E_1 + E_2 + E_3$ on X_3 has simple normal crossings, hence $\pi = \pi_1 \circ \pi_2 \circ \pi_3$ is a log resolution of (X, D). Since

$$\pi_3^*(D) = D + E_3, \quad \pi_3^*(E_1) = E_1 + E_3, \quad \text{and} \quad \pi_3^*(E_2) = E_2 + E_3,$$

we have

$$\pi^*(D) = \pi_3^*(D + 2E_1 + 3E_2) = D + 2E_1 + 3E_2 + 6E_3 \quad \text{and}$$
$$K_{X_3/X} = E_3 + \pi_3^*(K_{X_2/X}) = E_3 + \pi_3^*(E_1 + 2E_2) = E_1 + 2E_2 + 4E_3.$$

We thus conclude that for $0 \leq t < 1$, we have

$$\mathcal{J}(tD) = \pi_* \mathcal{O}_{X_3}\big(-(\lfloor 2t \rfloor - 1)E_1 - (\lfloor 3t \rfloor - 2)E_2 - (\lfloor 6t \rfloor - 5)E_3\big).$$

It is clear that if $t < 5/6$, then we push forward the line bundle associated to an effective exceptional divisor, hence $\mathcal{J}(tD) = \mathcal{O}_X$ by Lemma 2.31. On the other hand, if $5/6 \leq t < 1$, we have

$$\mathcal{J}(tD) = \pi_* \mathcal{O}_{X_3}(-E_3) = (x, y).$$

The last equality follows from the fact that $c_X(E_3) = \{(0,0)\}$.

In particular, we see that $\mathrm{lct}(D) = 5/6$.

Example 2.54. Suppose that $\dim(X) = n$ and D is an effective divisor on X that has an *ordinary singular point* P of multiplicity d: this means that the projectivized tangent cone of D at P is a smooth hypersurface in $\mathbf{P}^{n-1}$ of degree d (for example, this is the case for a hypersurface in $\mathbf{A}^n$ defined by a homogeneous polynomial of degree d with an isolated singularity at 0). Let $\pi \colon Y \to X$ be the blow-up of X at P, with exceptional divisor $E \simeq \mathbf{P}^{n-1}$. The hypothesis says that the intersection $\widetilde{D} \cap E$ is smooth, of degree d in E. This implies that $\widetilde{D} + E$ has SNC in a neighborhood of E, hence there is an open neighborhood U of P such that π gives a log resolution of $(U, D|_U)$. Moreover, we have

$$\pi^*(D) = \widetilde{D} + dE \quad \text{and} \quad K_{Y/X} = (n-1)E$$

hence for $0 \leq t < 1$ we have

$$\mathcal{J}(tD) = \pi_* \mathcal{O}_Y\big((n - 1 - \lfloor td \rfloor)E\big).$$

Using Lemma 2.14, we see that

$$\mathcal{J}(tD) = \mathfrak{m}_P^{\lfloor td \rfloor - n + 1} \quad \text{for all} \quad t < 1,$$

where $\mathfrak{m}_P$ is the ideal defining P and we make the convention that $\mathfrak{m}_P^j = \mathcal{O}_X$ for $j \leq 0$. In particular, we see that $\mathrm{lct}_P(D) = \min\{1, n/d\}$.

We next record some easy properties of multiplier ideals:

Proposition 2.55. If $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ are nonzero ideals on X, then

$$\mathcal{J}(\mathfrak{a}_1^\lambda) \subseteq \mathcal{J}(\mathfrak{a}_2^\lambda) \quad \text{for all} \quad \lambda \in \mathbf{R}_{\geq 0}.$$

In particular, we have $\mathrm{lct}(\mathfrak{a}_1) \leq \mathrm{lct}(\mathfrak{a}_2)$ (and $\mathrm{lct}_P(\mathfrak{a}_1) \leq \mathrm{lct}_P(\mathfrak{a}_2)$ for all P).

Proof. Let $f \colon Y \to X$ be a log resolution of $(X, \mathfrak{a}_1 \cdot \mathfrak{a}_2)$. If we write $\mathfrak{a}_1 \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_1)$ and $\mathfrak{a}_2 \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_2)$, then the hypothesis implies $\mathcal{O}_Y(-F_1) \subseteq \mathcal{O}_Y(-F_2)$, hence

$$\mathcal{J}(\mathfrak{a}_1^\lambda) = f_* \mathcal{O}_Y(K_{Y/X} - \lfloor \lambda F_1 \rfloor) \subseteq f_* \mathcal{O}_Y(K_{Y/X} - \lfloor \lambda F_2 \rfloor) = \mathcal{J}(\mathfrak{a}_2^\lambda).$$

The last assertion is an immediate consequence. $\square$

Proposition 2.56. If $\dim(X) = n$, $P \in X$, and $\mathfrak{a}$ is an ideal on X with $\mathrm{ord}_P(\mathfrak{a}) = d$, then $\mathrm{lct}_P(\mathfrak{a}) \leq \frac{n}{d}$.

Proof. By hypothesis, if $\mathfrak{m}$ is the ideal defining P, we have $\mathfrak{a} \subseteq \mathfrak{m}^d$, hence using Proposition 2.55, Remark 2.43, and Example 2.50 we get

$$\mathrm{lct}_P(\mathfrak{a}) \leq \mathrm{lct}_P(\mathfrak{m}^d) = \frac{\mathrm{lct}_P(\mathfrak{m})}{d} = \frac{n}{d}.$$

$\square$

Exercise 2.57. Let X be a smooth variety. Show that if $\mathfrak{a}$ is a nonzero ideal on X and $\mathrm{lct}(\mathfrak{a}) = c$, then for every positive integer m, the locus

$$\{x \in X \mid \mathrm{ord}_x(\mathfrak{a}) \geq m\}$$

has codimension $\geq \lceil cm \rceil$.

Exercise 2.58. Show that if $\mathfrak{a}$ is a nonzero ideal on X, then $\mathfrak{a} \subseteq \mathcal{J}(\mathfrak{a})$. More generally, if $\mathfrak{b}$ is another nonzero ideal on X, then for every $\lambda, \mu \in \mathbf{R}_{\geq 0}$ we have

$$\mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^\lambda \mathfrak{b}^\mu) \subseteq \mathcal{J}(\mathfrak{a}^{\lambda+1}\mathfrak{b}^\mu).$$

The next proposition shows that at least locally, we can reduce computing multiplier ideals of arbitrary ideals to computing multiplier ideals of hypersurfaces.

Proposition 2.59. Let $\mathfrak{a}$ be a nonzero ideal on X generated by $f_1, \ldots, f_q \in \mathcal{O}_X(X)$. For every positive integer r, if we take $g_i = \sum_{j=1}^q a_{i,j} f_j$ for $1 \leq i \leq r$, where the tuple $(a_{i,j}) \in \mathbf{C}^{rq}$ is general, and D is the divisor defined by $\prod_{i=1}^r g_i$, then

$$\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}\left(\tfrac{\lambda}{r} D\right) \quad \text{for} \quad \lambda < r.$$

In particular, if $\mathfrak{a} \neq \mathcal{O}_X$, then $\mathrm{lct}(D) = \min\left\{\frac{\mathrm{lct}(\mathfrak{a})}{r}, 1\right\}$.

Proof. For every i, let $D_i = \mathrm{div}_X(g_i)$, so $D = D_1 + \ldots + D_r$. Let $\pi \colon Y \to X$ be a log resolution of $(X, \mathfrak{a})$, with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. If $|V|$ is the linear system generated by $f_1, \ldots, f_r$ on X, then $\pi^*|V| = F + |W|$, with $|W|$ base-point free. If we write $\pi^*(D_i) = F + E_i$, it follows that $E_1, \ldots, E_r$ are general elements of $|W|$. It is then a consequence of the Kleiman-Bertini theorem that since $K_{Y/X} + F$ has SNC and $|W|$ is base-point free, each E_i is smooth (possibly disconnected), without any common components with the components of $K_{Y/X} + F$ or with each other, and $K_{Y/X} + F + \sum_{i=1}^r E_i$ has SNC (see Exercise 2.26). We thus conclude that π is a log resolution of (X, D).

Note now that since $\lambda < r$, we have

$$\lfloor \pi^*\left(\tfrac{\lambda}{r} D\right) \rfloor = \lfloor \lambda F \rfloor + \sum_{i=1}^r \lfloor \tfrac{\lambda}{r} E_i \rfloor = \lfloor \lambda F \rfloor,$$

hence the equality of multiplier ideals follows directly from the definition. The last assertion follows from the definition of the log canonical threshold and the fact that $\mathfrak{a} \neq \mathcal{O}_X$ implies that $D \neq 0$, hence $\mathrm{lct}(D) \leq 1$. $\square$

We next discuss an important class of examples, that of monomial ideals. In this case the formula for the multiplier ideals is due to Howald [**How01**]. Suppose that $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ is a *monomial* ideal, that is, it can be generated by monomials.

In what follows, if $u=(u_1,\ldots,u_n)\in\mathbf{Z}^n$, we write x^u for the Laurent monomial $x_1^{u_1}\cdots x_n^{u_n}$. The *Newton polyhedron* of $\mathfrak{a}$ is the convex hull $P(\mathfrak{a})$ of the set

$$\{u\in\mathbf{Z}^n_{\geq 0}\mid x^u\in\mathfrak{a}\}.$$

For $u,v\in\mathbf{Z}^n$, we put $\langle u,v\rangle=\sum_{i=1}^n u_iv_i$, and we denote by $\mathbf{1}$ the vector $(1,\ldots,1)\in\mathbf{Z}^n$.

THEOREM 2.60. *If $\mathfrak{a}\subseteq k[x_1,\ldots,x_n]$ is a nonzero monomial ideal, then for every $\lambda\in\mathbf{R}_{\geq 0}$, we have*

$$\mathcal{J}(\mathfrak{a}^{\lambda})=\big(x^u\mid u+\mathbf{1}\in\operatorname{Int}\big(\lambda\cdot P(\mathfrak{a})\big)\big)\,.$$

PROOF. One can give a two-line argument using some basic facts about toric varieties. However, we do not assume familiarity with toric varieties, so we develop everything that we need from scratch.

We consider on $X=\mathbf{A}^n$ the standard action of the torus $T=(k^*)^n$ given by

$$(t_1,\ldots,t_n)\cdot(a_1,\ldots,a_n)=(t_1a_1,\ldots,t_na_n).$$

It is clear that every monomial ideal is preserved by this action (that is, by the corresponding action of T on $R=k[x_1,\ldots,x_n]$); moreover, it is straightforward to check that the converse also holds since k is an infinite field.

It is a general fact that if an algebraic group G acts algebraically on a variety X and a nonzero ideal sheaf $\mathfrak{a}$ is preserved by this action, then there is a log resolution $f\colon Y\to X$ of $(X,\mathfrak{a})$ such that G acts algebraically on Y and f is an equivariant morphism (this does not follow immediately from the results in [**Hir64**], but it is a consequence of the more recent work on resolution of singularities, see for example [**Kol07**]). In this case, it is an immediate consequence of the definition of multiplier ideals that $\mathcal{J}(\mathfrak{a}^{\lambda})$ is preserved by the G-action (note that if $\mathfrak{a}\cdot\mathcal{O}_Y=\mathcal{O}_Y(-F)$, then both F and $K_{Y/X}$ are preserved by the G-action. Moreover, if G is connected, then all f-exceptional divisors, as well as those that appear in F, are preserved by the G-action.

We thus conclude that in our setting the ideal $\mathcal{J}(\mathfrak{a}^{\lambda})$ is a monomial ideal. Moreover, in order to check whether a certain $h\in R$ is in $\mathcal{J}(\mathfrak{a}^{\lambda})$, we only need to check that $v(h)>\lambda\cdot v(\mathfrak{a})-A_X(v)$ for divisorial valuations v that are T-equivariant. Thus means that for every $g\in R$, we have $v(t\cdot g)=v(g)$ for all $t\in T$. It is easy to see that this means the following: if $v(x_i)=v_i$ and $g=\sum_u c_ux^u$, then

$$v(g)=\min_{u,c_u\neq 0}\langle u,v\rangle. \tag{2.6}$$

Note that since the image if v is $\mathbf{Z}$, we have $\gcd(v_1,\ldots,v_n)=1$.

We next show that conversely, if $\gcd(v_1,\ldots,v_n)=1$ and we define v as above, then indeed v is a divisorial valuation on X and we compute $A_X(v)$. Note first that there is a matrix $A=(a_{i,j})\in M_n(\mathbf{Z})$ with $a_{1,i}=v_i$ for $1\leq i\leq n$, and $\det(A)=\pm 1$. Indeed, the hypothesis on $v=(v_1,\ldots,v_n)$ implies that there are vectors $(a_{i,1},\ldots,a_{i,n})$ for $2\leq i\leq n$ that together with v give a basis of $\mathbf{Z}^n$.

We define a k-algebra homomorphism

$$\varphi\colon R=k[x_1,\ldots,x_n]\to k[y_1,y_2^{\pm 1}\ldots,y_n^{\pm 1}]=S,\quad \varphi(x_j)=\prod_{i=1}^n y_i^{a_{i,j}}.$$

Since $\det(A)=\pm 1$, it follows that φ induces an isomorphism between the corresponding Laurent polynomial rings; therefore the morphism $\operatorname{Spec}(S)\to X$ induced by π birational. If E is the divisor on $\mathrm{Spec}(S)$ defined by (y_1), we see that the corresponding valuation ord_E is the valuation v defined by (2.6).

Let's compute now $A_X(v)$. It follows from the definition of π that

$$\pi^*\big(\mathrm{dlog}(x_j)\big) = \sum_{i=1}^n a_{i,j}\mathrm{dlog}(y_i) \quad \text{for} \quad 1 \leq j \leq n.$$

Here for a regular function g, we write $\mathrm{dlog}(g) = \frac{dg}{g}$. Note that $\mathrm{dlog}(gh) = \mathrm{dlog}(g) + \mathrm{dlog}(h)$.

Therefore we have

$$\pi^*\big(\mathrm{dlog}(x_1)\wedge\ldots\wedge\mathrm{dlog}(x_n)\big) = \det(A)\cdot\mathrm{dlog}(y_1)\wedge\ldots\wedge\mathrm{dlog}(y_n) = \pm\mathrm{dlog}(y_1)\wedge\ldots\wedge\mathrm{dlog}(y_n).$$

We thus conclude that

$$\pi^*(dx_1 \wedge \ldots \wedge dx_n) = \prod_{i=1}^n {y_i}^{a_{i,1}+\ldots+a_{i,n}-1} dy_1 \wedge \ldots \wedge dy_n,$$

hence $A_X(v) = v_1 + \ldots + v_n = \langle \mathbf{1}, v\rangle$. Note also that

$$v(\mathfrak{a}) = \min_{x^w \in \mathfrak{a}} \langle w, v\rangle = \min_{w \in P(\mathfrak{a})} \langle w, v\rangle.$$

We thus conclude that a monomial x^u lies in $\mathcal{J}(\mathfrak{a}^\lambda)$ if and only if for every $v = (v_1, \ldots, v_n) \in \mathbf{Z}^n_{\geq 0}$ with $\gcd(v_1, \ldots, v_n) = 1$, we have

$$\langle u, v\rangle > \lambda \cdot \min_{w\in P(\mathfrak{a})} \langle w, v\rangle - \langle \mathbf{1}, v\rangle.$$

Of course, this holds if and only if we have the same inequality for all nonzero $v \in \mathbf{Z}^n_{\geq 0}$. Since $P(\mathfrak{a})$ is the convex hull of a subset of $\mathbf{Z}^n_{\geq 0}$, there are nonzero $v^{(1)}, \ldots, v^{(N)} \in \mathbf{Z}^n_{\geq 0}$ such that if $b_i = \min_{u\in P(\mathfrak{a})}\langle u, v^{(i)}\rangle$ for $1 \leq i \leq N$, then

$$P(\mathfrak{a}) = \bigcap_{i=1}^N \big\{u \in \mathbf{R}^n_{\geq 0} \mid \langle u, v^{(i)}\rangle \geq b_i\big\} \quad \text{and}$$

$$\mathrm{Int}\big(P(\mathfrak{a})\big) = \bigcap_{i=1}^N \big\{u \in \mathbf{R}^n_{>0} \mid \langle u, v^{(i)}\rangle > b_i\big\}.$$

It is now easy to see that $x^u \in \mathcal{J}(\mathfrak{a}^\lambda)$ if and only if $u + \mathbf{1} \in \mathrm{Int}\big(\lambda \cdot P(\mathfrak{a})\big)$. This completes the proof. □

COROLLARY 2.61. If $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ is a nonzero proper monomial ideal, then

$$\mathrm{lct}(\mathfrak{a}) = \mathrm{lct}_0(\mathfrak{a}) = \max\{\lambda > 0 \mid \mathbf{1} \in \lambda \cdot P(\mathfrak{a})\}.$$

EXAMPLE 2.62. Let $\mathfrak{a} = (x_1^{a_1}, \ldots, x_r^{a_r}) \subseteq k[x_1, \ldots, x_n]$, with $a_1, \ldots, a_r \in \mathbf{Z}_{>0}$. In this case

$$P(\mathfrak{a}) = \left\{u = (u_1, \ldots, u_n) \in \mathbf{R}^n_{\geq 0} \mid \frac{u_1}{a_1} + \ldots + \frac{u_r}{a_r} \geq 1\right\}.$$

By Theorem 2.60, we have

$$\mathcal{J}(\mathfrak{a}^\lambda) = \left(x^u \mid \frac{u_1+1}{a_1} + \ldots + \frac{u_r+1}{a_r} > \lambda\right).$$

In particular, we have $\mathrm{lct}(\mathfrak{a}) = \sum_{i=1}^r \frac{1}{a_i}$.

Suppose now that $f = \sum_{i=1}^r x_i^{a_i}$. It follows from Proposition 2.59 that if $g = \alpha_1 x_1^{a_1} + \ldots + \alpha_r x_r^{a_r}$, with $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbf{C}^r$ general, then

$$\mathcal{J}(g^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda) = \left(x^u \mid \frac{u_1+1}{a_1} + \ldots + \frac{u_r+1}{a_r} > \lambda\right) \quad \text{for} \quad \lambda < 1.$$

Of course, we may assume that $\alpha_i \neq 0$ for every i. If β_i are such that $\beta_i^{a_i} = \alpha_i$, the automorphism φ of $k[x_1, \ldots, x_r]$ given by $\varphi(x_i) = \beta_i x_i$ maps f to g. Since φ clearly preserves monomial ideals, we conclude that

$$\mathcal{J}(f^\lambda) = \left(x^u \mid \frac{u_1+1}{a_1} + \ldots + \frac{u_r+1}{a_r} > \lambda\right) \quad \text{for} \quad \lambda < 1.$$

In particular, we have

$$\mathrm{lct}(f) = \min\left\{1, \sum_{i=1}^r \frac{1}{a_i}\right\}.$$

2.4. Multiplier ideals and vanishing theorems

The relevance of multiplier ideals in birational geometry comes from their role in vanishing theorems. Let us begin by recalling the basic vanishing result in algebraic geometry. Recall that we work over a ground field of characteristic 0.

THEOREM 2.63 (Kodaira). *If X is a smooth projective variety and $\mathcal{L}$ is an ample line bundle on X, then*

$$H^i(X, \omega_X \otimes \mathcal{L}) = 0 \quad \textit{for all} \quad i > 0.$$

REMARK 2.64. Via Serre duality, the assertion in the above theorem is equivalent to the fact that if $\mathcal{L}$ is ample, then $H^i(X, \mathcal{L}^{-1}) = 0$ for $i < \dim(X)$.

The proofs of this result either rely on analytic tools, such as Hodge theory (see for example [**Laz04**, Chapter 4.2]) or go by reduction to positive characteristic (see [**DI87**]). We will discuss the latter approach in Chapter 4.3 below. However, it is important to note that there are counterexamples in positive characteristic (see [**Ray78**]).

The result that is actually used most in higher-dimensional birational geometry is a powerful generalization due independently to Kawamata and Viehweg. The generalization goes in two directions: first, ampleness is replaced by the more flexible condition *big and nef*. Second, it is not the line bundle that is required to satisfy this condition, but a small perturbation. Let us begin by introducing the relevant terminology.

DEFINITION 2.65. Let X be an n-dimensional projective variety. A line bundle $\mathcal{L}$ on X is *nef* if for every curve C on X (assumed to be irreducible and reduced), we have $\deg(\mathcal{L}|_C) \geq 0$. We say that $\mathcal{L}$ is *big* if there is $m > 0$ such that $\mathcal{L}^m$ defines a rational map $\varphi_m = \varphi_{\mathcal{L}^m} \colon X \dashrightarrow \mathbf{P}\big(H^0(X, \mathcal{L}^m)\big)$ whose image has dimension n.

We do not discuss in detail these important notions, but only point out some basic facts. Clearly, an ample line bundle is big and nef. In fact, if a line bundle $\mathcal{L}$ is *semiample* (that is, some positive multiple of $\mathcal{L}$ is globally generated), then it is nef. Properties iv) and v) in the proposition below show that being big and nef is a more flexible property than ampleness.

PROPOSITION 2.66. Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on the projective variety X.

i) If d is a positive integer, then $\mathcal{L}$ is nef (big) if and only if $\mathcal{L}^d$ is nef (big).
ii) If $\mathcal{L}$ and $\mathcal{M}$ are nef, then $\mathcal{L} \otimes \mathcal{M}$ is nef.
iii) If $\mathcal{L}$ is big and $H^0(X, \mathcal{M}^m) \neq 0$ for some $m \in \mathbf{Z}_{>0}$, then $\mathcal{L} \otimes \mathcal{M}$ is big.
iv) If $f \colon Y \to X$ is a projective, birational morphism, then $\mathcal{L}$ is nef if and only if $f^*(\mathcal{L})$ is nef.
v) If f is as in iv) and X is normal[4], then $\mathcal{L}$ is big if and only if $f^*(\mathcal{L})$ is big.

Proof. The assertion in i) is trivial for the nef condition. For the big condition it is enough to show that if $\mathcal{L}^m$ defines a rational map whose image has dimension $n = \dim(X)$, then any multiple $\mathcal{L}^{mq}$ satisfies the same property. Note that we have an obvious linear map

$$\mathrm{Sym}^q H^0(X, \mathcal{L}^m) \to H^0(X, \mathcal{L}^{mq}),$$

which induces the bottom rational map in the commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi_m"] \arrow[d, dashed, "\varphi_{mq}"'] & \mathbf{P}\big(H^0(X, \mathcal{L}^m)\big) \arrow[d, hook, "j"] \\ \mathbf{P}\big(H^0(X, \mathcal{L}^{mq})\big) \arrow[r, dashed] & \mathbf{P}\big(\mathrm{Sym}^q H^0(X, \mathcal{L}^m)\big), \end{tikzcd}$$

in which j is the Veronese embedding. Since the image of φ_m has dimension n, it follows that also the image of φ_{mq} has dimension n.

The assertion in ii) is trivial. Suppose now $\mathcal{L}$ and $\mathcal{M}$ are as in iii). Clearly, we have $H^0(X, \mathcal{M}^{mq}) \neq 0$ for all $q \geq 1$. Therefore, by the argument in the proof of i), we may assume that there is m such that $H^0(X, \mathcal{M}^m) \neq 0$ (hence the rational map $\varphi_{\mathcal{M}^m}$ is defined and also that the rational map $\varphi_{\mathcal{L}^m}$ has n-dimensional image.

Note we have a multiplication map

$$\alpha \colon H^0(X, \mathcal{L}^m) \otimes_k H^0(X, \mathcal{M}^m) \to H^0(X, \mathcal{N}^m),$$

where $\mathcal{N} = \mathcal{L} \otimes \mathcal{M}$, and a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi_{\mathcal{N}^m}"] \arrow[d, dashed, "(\varphi_{\mathcal{L}^m}, \varphi_{\mathcal{M}^m})"'] & \mathbf{P}\big(H^0(X, \mathcal{N}^m)\big) \arrow[d, dashed, "g"] \\ \mathbf{P}\big(H^0(X, \mathcal{L}^m)\big) \times \mathbf{P}\big(H^0(X, \mathcal{M}^m)\big) \arrow[r] & \mathbf{P}\big(H^0(X, \mathcal{L}^m) \otimes H^0(X, \mathcal{M}^m)\big), \end{tikzcd}$$

where the bottom map is the Segre embedding and g is the rational map induced by α. Since the image of $\varphi_{\mathcal{L}^m}$ is n-dimensional, also the image of $(\varphi_{\mathcal{L}^m}, \varphi_{\mathcal{M}^m})$ is n-dimensional, and the commutative diagram implies that the image of $\varphi_{\mathcal{N}^m}$ is n-dimensional. Therefore $\mathcal{N}$ is big.

We next prove iv). If $\mathcal{L}$ is nef, then it is clear that $f^*(\mathcal{L})$ is nef: if C is a curve on Y and $C' = f(C)$ is a curve on X, then the projection formula gives

$$\deg\big(f^*(\mathcal{L})|_C\big) = \deg(C/C') \cdot \deg(\mathcal{L}|_{C'})$$

(on the other hand, if $f(C)$ is a point, then we clearly have $\deg\big(f^*(\mathcal{L})|_C\big) = 0$). The converse follows in the same way if we show that for every curve C' on X, there is a curve C on Y with $f(C) = C'$. This follows by choosing an irreducible component

[4]The same property holds without assuming X normal, but we will not need the more general statement.

T of $f^{-1}(C)$ that dominates C and by cutting T with r general hyperplane sections (with respect to a suitable projective embedding of Y), where $\dim(T) = r+1$.

Suppose now that X is normal and let us show now that $\mathcal{L}$ is big if and only if $f^*(\mathcal{L})$ is big. Since X is normal, we have $f_*(\mathcal{O}_Y) = \mathcal{O}_X$, and using the projection formula we get

$$H^0\big(Y, f^*(\mathcal{L}^m)\big) \simeq H^0\big(X, f_*f^*(\mathcal{L}^m)\big) \simeq H^0\big(X, \mathcal{L}^m \otimes f_*(\mathcal{O}_Y)\big) \simeq H^0(X, \mathcal{L}^m).$$

We thus have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[d, "f"'] & \mathbf{P}\big(H^0(Y, f^*(\mathcal{L}^m))\big) \arrow[d, "g"] \\ X \arrow[r, dashed] & \mathbf{P}\big(H^0(X, \mathcal{L}^m)\big), \end{tikzcd}$$

in which g is an isomorphism. It is thus clear that the image of the top horizontal map has dimension n if and only if the image of the bottom horizontal map has dimension n. This gives the assertion in iv). □

In light of property i) in the above proposition, it makes sense to extend the definition to $\mathbf{Q}$-divisors, as follows.

Definition 2.67. Let X be a smooth, projective variety. A $\mathbf{Q}$-divisor D on X is *nef* (*big*) if $\mathcal{O}_X(mD)$ has the corresponding property when m is a positive integer such that mD is a divisor.

Theorem 2.68 (Kawamata-Viehweg). *Let X be a smooth projective variety. If $\mathcal{L} = \mathcal{O}_X\big(\lceil D\rceil\big)$, where D is a big and nef $\mathbf{Q}$-divisor on X such that $\lceil D\rceil - D$ is an SNC divisor, then*

$$H^i(X, \omega_X \otimes \mathcal{L}^{-1}) = 0 \quad \textit{for} \quad i > 0.$$

The proof of this result is reduced to the assertion in Kodaira's theorem via a clever use of cyclic covers (see for example [**Laz04**, Chapter 9.1.C]).

We next discuss some applications of this theorem in the context of multiplier ideals. The first result shows that the line bundle whose direct image gives a multiplier ideal has vanishing higher direct images.

Theorem 2.69 (Relative vanishing). *Let X be a smooth variety and $\mathfrak{a}$ a nonzero ideal on X. If $f\colon Y \to X$ is a projective log resolution of $(X, \mathfrak{a})$, with $\mathfrak{a} \cdot \mathcal{O}_X = \mathcal{O}_X(-F)$, then for every $\lambda \in \mathbf{R}_{\geq 0}$, we have*

$$R^i f_* \mathcal{O}_Y\big(K_{Y/X} - \lfloor \lambda F\rfloor\big) = 0 \quad \textit{for} \quad i > 0.$$

Before giving the proof of the theorem, we need the following lemma that allows us to deduce relative vanishing from absolute vanishing.

Lemma 2.70. *Let $f\colon Y \to X$ be a morphism of projective varieties, $\mathcal{F}$ a coherent sheaf on Y, and $\mathcal{L}$ an ample line bundle on X. If $m \in \mathbf{Z}$ is such that*

$$H^i\big(Y, \mathcal{F} \otimes f^*(\mathcal{L}^j)\big) = 0 \quad \textit{for} \quad i > 0,\ j \geq m,$$

then the following hold:

i) *$R^i f_*(\mathcal{F}) = 0$ for all $i > 0$.*
ii) *$H^i\big(X, f_*(\mathcal{F}) \otimes \mathcal{L}^j\big) = 0$ for all $j \geq m$.*

Proof. For every j, consider the Leray spectral sequence associated to the sheaf $\mathcal{F} \otimes f^*(\mathcal{L}^j)$:

$$E_2^{p,q} = H^p\big(X, R^q f_*(\mathcal{F} \otimes f^*(\mathcal{L}^j))\big) \Rightarrow H^{p+q}\big(Y, \mathcal{F} \otimes f^*(\mathcal{L}^j)\big).$$

Note that by the projection formula, we have

$$E_2^{p,q} \simeq H^p\big(X, R^q f_*(\mathcal{F}) \otimes \mathcal{L}^j\big).$$

Since $\mathcal{L}$ is ample, it follows that there is m' such that for all q we have

a) $E_2^{p,q} = 0$ for all $p \geq 1$ and $j \geq m'$, and
b) $R^q f_*(\mathcal{F}) \otimes \mathcal{L}^j$ is globally generated for all $j \geq m'$.

Condition a) implies that for $j \geq m'$, the spectral sequence gives

$$H^0\big(X, R^q f_*(\mathcal{F}) \otimes \mathcal{L}^j\big) \simeq H^q\big(Y, \mathcal{F} \otimes f^*(\mathcal{L}^j)\big).$$

By assumption, the right-hand side vanishes for $j \geq \max\{m, m'\}$ and $q \geq 1$, which implies $H^0\big(X, R^q f_*(\mathcal{F}) \otimes \mathcal{L}^j\big) = 0$. Condition b) then implies $R^q f_*(\mathcal{F}) \otimes \mathcal{L}^j = 0$, hence $R^q f_*(\mathcal{F}) = 0$ for $q \geq 1$, which is assertion i) in the lemma.

Once we know this, the spectral sequence gives

$$E_2^{p,0} = H^p\big(X, f_*(\mathcal{F}) \otimes \mathcal{L}^j\big) \simeq H^p\big(Y, \mathcal{F} \otimes f^*(\mathcal{L}^j)\big) \quad \text{for all} \quad j \in \mathbf{Z},$$

hence assertion ii) in the theorem follows from the hypothesis. □

We can now give the proof of the Relative Vanishing theorem.

Proof of Theorem 2.69. We may replace λ by $\lambda' > \lambda$, with $\lambda' \in \mathbf{Q}_{\geq 0}$, such that $\lfloor \lambda F \rfloor = \lfloor \lambda' F \rfloor$. Hence we may and will assume that $\lambda \in \mathbf{Q}_{\geq 0}$. We first treat the case when X is a projective variety. Note that in this case, since f is projective, Y is a projective variety too.

Let D be an ample divisor on X. Using the projection formula, we see that

$$R^i f_* \mathcal{O}_Y\big(K_{Y/X} - \lfloor \lambda F \rfloor\big) \simeq R^i f_* \omega_Y(-\lceil \lambda F \rceil) \otimes \omega_X^{-1}.$$

By Lemma 2.70, it is thus enough to show that

$$H^i\big(Y, \omega_Y(-\lfloor \lambda F \rfloor + f^*(jD))\big) = 0 \quad \text{for} \quad i > 0 \text{ and } j \gg 0. \tag{2.7}$$

Note that $-\lfloor \lambda F \rfloor + f^*(jD) = \lceil -\lambda F + f^*(jD) \rceil$. Since the divisor $\lceil -\lambda F \rceil + \lambda F$ has simple normal crossings, it follows from Theorem 2.68 that it is enough to show that for $j \gg 0$, the divisor $f^*(jD) - \lambda F$ is big and nef.

Since $\mathfrak{a} \cdot \mathcal{O}_Y$ is locally principal, the morphism f factors as the composition

$$Y \xrightarrow{h} B = \mathrm{Bl}_{\mathfrak{a}}(X) \xrightarrow{g} X,$$

where g is the blow-up along $\mathfrak{a}$, with exceptional divisor E. Note that we have $F = h^*(E)$. We know that on B we have a g-ample line bundle $\mathcal{O}_B(1)$ such that $\mathcal{O}_B(1) \simeq \mathcal{O}_B(-E)$. Therefore there is $j_1 > 0$ such that $g^*(j_1 D) - E$ is an ample divisor on B. It thus follows from Proposition 2.66 that

$$f^*(jD) - \lambda F = \lambda h^*\big(g^*(j_1 D) - E\big) + (j - j_1\lambda) f^*(D)$$

is big and nef for all $j > j_1 \lambda$. This completes the proof in the case when X is projective.

The reduction to the projective case is a standard argument. First, the assertion we want to prove is local on X, hence we may assume that X is affine. Consider an open immersion $j \colon X \hookrightarrow \overline{X}$, where $\overline{X}$ is projective. We may and will assume that $\overline{X}$ is smooth: indeed, there is a projective morphism $\widetilde{X} \to \overline{X}$ that is an isomorphism over X and we replace $X \hookrightarrow \overline{X}$ by $X \hookrightarrow \widetilde{X}$. Let $\overline{\mathfrak{a}}$ be an ideal on $\overline{X}$ such that $\overline{\mathfrak{a}}|_X = \mathfrak{a}$. We similarly consider an open immersion $Y \hookrightarrow \overline{Y}$, with $\overline{Y}$ smooth and projective (note that Y is projective over the affine variety X, hence it is quasi-projective). After taking a suitable projective morphism $\widetilde{Y} \to \overline{Y}$ that is an isomorphism over Y, with $\widetilde{Y}$ smooth, and replacing $\overline{Y}$ by $\widetilde{Y}$, we may and will assume that the rational map $g \colon \overline{Y} \dashrightarrow \overline{X}$ is a morphism. We clearly have $Y \subseteq g^{-1}(X)$ and since f is proper, we have in fact $Y = g^{-1}(X)$. If G is the sum of the g-exceptional divisors, consider a projective log resolution $W \to \overline{Y}$ of $\mathcal{O}_{\overline{Y}}(-G) \cdot (\overline{\mathfrak{a}} \cdot \mathcal{O}_W)$ which is an isomorphism over Y. After replacing $\overline{Y} \to \overline{X}$ by the composition $W \to \overline{Y} \to \overline{X}$, we may assume that g is a log resolution of $(\overline{X}, \overline{\mathfrak{a}})$. It is then clear that applying the conclusion of the theorem for g, we also obtain it for f. This completes the proof. □

We next use relative vanishing to give the following generalization of Kawamata-Viehweg vanishing.

THEOREM 2.71 (Nadel). *Let X be a smooth projective variety and D an effective $\mathbf{Q}$-divisor on X. If A is a divisor on X such that $A - D$ is big and nef, then*

$$H^i\big(X, \omega_X \otimes \mathcal{O}_X(A) \otimes \mathcal{J}(D)\big) = 0 \quad \textit{for all} \quad i > 0.$$

REMARK 2.72. Suppose that E is a big and nef $\mathbf{Q}$-divisor on the smooth projective variety X such that $D := \lceil E \rceil - E$ has simple normal crossings. Note that in this case $\mathcal{J}(D) = \mathcal{O}_X$ and we can apply Theorem 2.71 with $A = \lceil E \rceil$ to conclude that

$$H^i\big(X, \omega_X \otimes \mathcal{O}_X(\lceil E \rceil)\big) = 0 \quad \text{for} \quad i > 0.$$

Hence the statement of the Kawamata-Viehweg vanishing theorem is a special case of Theorem 2.71.

In fact, we prove the following more general statement.

THEOREM 2.73 (Nadel). *Let $\mathfrak{a}$ be a nonzero ideal sheaf on the smooth projective variety X. If M is a divisor on X such that $\mathfrak{a} \otimes \mathcal{O}_X(M)$ is globally generated, $\lambda \in \mathbf{Q}_{\geq 0}$, and A is a divisor on X such that $A - \lambda M$ is big and nef, then*

$$H^i\big(X, \omega_X \otimes \mathcal{O}_X(A) \otimes \mathcal{J}(\mathfrak{a}^\lambda)\big) = 0 \quad \textit{for} \quad i > 0.$$

REMARK 2.74. With the notation in Theorem 2.71, let m be a positive integer such that mD has integer coefficients and let $\mathfrak{a} = \mathcal{O}_X(-mD)$. If we take $M = mD$ and $\lambda = \frac{1}{m}$, then we may apply the vanishing in Theorem 2.73 to deduce the one in Theorem 2.71. Hence it is enough to prove Theorem 2.71.

PROOF OF THEOREM 2.73. Let $f \colon Y \to X$ be a projective resolution of $(X, \mathfrak{a})$. Let us write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. Note that by the projection formula, we can write

$$\omega_X \otimes \mathcal{O}_X(A) \otimes \mathcal{J}(\mathfrak{a}^\lambda) = \omega_X \otimes \mathcal{O}_X(A) \otimes f_*\mathcal{O}_Y(K_{Y/X} - \lfloor \lambda F \rfloor) = f_*\omega_Y\big(f^*(A) - \lfloor \lambda F \rfloor\big),$$

and using also Theorem 2.69 we have

$$R^q f_*\omega_Y\big(f^*(A) - \lfloor \lambda F \rfloor\big) \simeq \omega_X \otimes \mathcal{O}_X(A) \otimes R^q f_*\mathcal{O}_Y(K_{Y/X} - \lfloor \lambda F \rfloor) = 0 \quad \text{for} \quad q > 0.$$

The Leray spectral sequence for f and $\omega_Y\big(f^*(A) - \lfloor \lambda F \rfloor\big)$ then gives an isomorphism

$$H^i\big(X, \omega_X \otimes \mathcal{O}_X(A) \otimes \mathcal{J}(\mathfrak{a}^\lambda)\big) \simeq H^i\big(Y, \omega_Y(f^*(A) - \lfloor \lambda F \rfloor)\big). \tag{2.8}$$

By assumption, we have a surjection

$$\mathcal{O}_X(-M)^{\oplus d} \to \mathfrak{a}$$

for a suitable d, which induces on Y the surjection

$$\mathcal{O}_Y\big(-f^*(M)\big)^{\oplus d} \to \mathcal{O}_Y(-F).$$

Therefore $\mathcal{O}_Y\big(f^*(M) - F\big)$ is globally generated; in particular, $f^*(M) - F$ is nef.

On the other hand, since $A - \lambda M$ is big and nef, it follows from properties iv) and v) in Proposition 2.66 that $f^*(A - \lambda M)$ is big and nef on Y. Using now properties i), ii), and iii) in the same proposition, we conclude that

$$G := f^*(A) - \lambda F = f^*(A - \lambda M) + \lambda\big(f^*(M) - F\big) \quad \text{is big and nef.}$$

Note that $\lceil G \rceil - G$ is supported on $\mathrm{Supp}(F)$, hence it is an SNC divisor. By the Kawamata-Viehweg vanishing theorem, we have

$$H^i\big(Y, \omega_Y(\lceil G \rceil)\big) = 0 \quad \text{for} \quad i > 0. \tag{2.9}$$

Since $\lceil G \rceil = f^*(A) - \lfloor \lambda F \rfloor$, the conclusion of the theorem follows from (2.8) and (2.9). $\square$

Remark 2.75. The most obvious application of vanishing theorems is to lifting of sections: given a short exact sequence of sheaves on X:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

if $H^1(X, \mathcal{F}') = 0$, then it follows from the long exact sequence in cohomology that the map

$$H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'')$$

is surjective.

Another possible application of vanishing results is to existence of sections. Suppose for example that A is an ample divisor on the n-dimensional projective variety X and $\mathcal{F}$ is a nonzero coherent sheaf on X such that $H^i\big(X, \mathcal{F}(jA)\big) = 0$ for all $j \geq j_0$. In this case there is j with $j_0 \leq j \leq j_0 + n$ such that $H^0\big(X, \mathcal{F}(jA)\big) \neq 0$. Indeed, recall that there is a polynomial $P \in \mathbf{Q}[x]$ of degree $\leq n$ such that $P(j) = \chi\big(\mathcal{F}(j)\big)$ for all $j \in \mathbf{Z}$. Our assumption implies that $P(j) = h^0\big(X, \mathcal{F}(j)\big)$ for $j \geq j_0$. If $H^0\big(X, \mathcal{F}(jA)\big) = 0$ for $j_0 \leq j \leq j_0 + n$, it follows that P has $n+1$ roots, hence $P = 0$. Since $\mathcal{F}(jA)$ is globally generated for $j \gg 0$, this implies that $\mathcal{F} = 0$, a contradiction.

If $\mathcal{O}_X(A)$ is also globally generated, then we can say more: $\mathcal{F}(jA)$ is globally generated for all $j \geq j_0 + n$. Indeed, the hypothesis gives, in particular, that $H^i\big(X, \mathcal{F}((j-i)A)\big) = 0$ for all $i > 0$, hence $\mathcal{F}(jA)$ is 0-regular in the sense of Castelnuovo-Mumford regularity; this implies that it is globally generated (for the basic facts about Castelnuovo-Mumford regularity, see for example [**Laz04**, Chapter 1.8]).

Example 2.76. We give an application of Theorem 2.71 to bounding the number of singular points on a plane curve with simple cusps. Suppose that $C \subseteq \mathbf{P}^2$ is a plane curve of degree d that has r simple cusps[5]. Note that if $\Gamma \subseteq C$ is the (reduced) set of singular points, then it follows from Example 2.53 that $\mathcal{J}\left(\frac{5}{6}C\right)$ is the ideal $\mathcal{I}_\Gamma$ defining Γ in $\mathbf{P}^2$ (it is easy to see that for every simple cusp, a log

[5]A point $P \in C$ is a *simple cusp* if $\widehat{\mathcal{O}_{C,P}} \simeq k[\![x,y]\!]/(x^2 - y^3)$.

resolution in a neighborhood of this point is obtained as in Example 2.53). On the other hand, since $\omega_{\mathbf{P}^2} \simeq \mathcal{O}_{\mathbf{P}^2}(-3)$, it follows from Theorem 2.71 that

$$H^1\big(\mathbf{P}^2, \mathcal{I}_\Gamma \otimes \mathcal{O}_{\mathbf{P}^2}(m-3)\big) = 0 \quad \text{for} \quad m > 5d/6.$$

By tensoring the short exact sequence

$$0 \to \mathcal{I}_\Gamma \to \mathcal{O}_{\mathbf{P}^2} \to \mathcal{O}_\Gamma \to 0$$

with $\mathcal{O}_{\mathbf{P}^2}(m-3)$, we conclude that the map

$$H^0\big(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(m-3)\big) \to H^0\big(\Gamma, \mathcal{O}_\Gamma(m-3)\big)$$

is surjective as long as $m > 5d/6$. Since the dimension of the left-hand side is $\binom{m-1}{2}$, by taking $m = \lfloor 5d/6 \rfloor + 1$, we conclude that $r \leq \binom{\lfloor 5d/6 \rfloor}{2}$. For example, if C has degree 4, then the number of cusps is ≤ 3.

Example 2.77. An early application of vanishing theorems has been towards finding projective hypersurfaces of small degree passing through a set of points (see [**EV83**] for a discussion of such results). Given a subset $S \subseteq \mathbf{P}^n$ and a positive integer m, let $\alpha_m(S)$ be the smallest degree of a hypersurface H in $\mathbf{P}^n$ such that $\mathrm{ord}_P(H) \geq m$ for all $P \in S$. It was conjectured by Chudnovsky that the following inequality holds:

$$\frac{\alpha_m(S)}{m} \geq \frac{\alpha_1(S) + n - 1}{n}.$$

We show using Nadel vanishing that the following weaker version holds:

$$\frac{\alpha_m(S)}{m} \geq \frac{\alpha_1(S)}{n}. \tag{2.10}$$

Indeed, suppose that D is an effective divisor of degree d in $\mathbf{P}^n$ such that $\mathrm{ord}_P(D) \geq m$ for all $P \in S$. Let Z be the closed subscheme of $\mathbf{P}^n$ defined by $\mathcal{I}_Z = \mathcal{J}\big(\frac{n}{m}D\big)$. For every $P \in S$, it follows from Proposition 2.56 that $\mathrm{lct}_P(D) \leq \frac{n}{m}$, hence $S \subseteq Z$. On the other hand, since $\omega_{\mathbf{P}^n} \simeq \mathcal{O}_{\mathbf{P}^n}(-n-1)$, it follows from Theorem 2.71 that if $j + n + 1 > \frac{dn}{m}$, then

$$H^i\big(\mathbf{P}^n, \mathcal{I}_Z \otimes \mathcal{O}_{\mathbf{P}^n}(j)\big) = 0 \quad \text{for all} \quad i \geq 1.$$

We thus deduce that $H^0\big(\mathbf{P}^n, \mathcal{I}_Z \otimes \mathcal{O}_{\mathbf{P}^n}(j)\big) \neq 0$ if $j + 1 > \frac{dn}{m}$ (see Remark 2.75). This implies that Z (hence also S) is contained in a hypersurface of degree $\lfloor dn/m \rfloor$, so we get the inequality (2.10).

Example 2.78 (Kollár). Suppose that A is an Abelian variety and Θ is a principal polarization on A (recall that this means that Θ is an ample effective divisor on A, with $h^0\big(A, \mathcal{O}_A(\Theta)\big) = 1$). We claim that $\mathrm{lct}(\Theta) = 1$. In particular, using Example 2.56, we deduce that $\mathrm{ord}_x(\Theta) \leq \dim(A)$ for every $x \in \Theta$.

In order to show that $\mathrm{lct}(\Theta) = 1$, we need to show that for every $\lambda \in [0,1) \cap \mathbf{Q}$, the closed subscheme Z defined by $\mathcal{I}_Z = \mathcal{J}(\lambda\Theta)$ is empty. Since A is an Abelian variety, we have $\omega_A \simeq \mathcal{O}_A$, hence Theorem 2.71 gives

$$H^1\big(A, \mathcal{I}_Z \otimes \mathcal{O}_A(\Theta)\big) = 0. \tag{2.11}$$

By tensoring with $\mathcal{O}_A(\Theta)$ the short exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_A \to \mathcal{O}_Z \to 0$$

and taking cohomology, we deduce using (2.11) that the restriction map

$$H^0\big(A, \mathcal{O}_A(\Theta)\big) \to H^0(A, \mathcal{O}_A(\Theta)|_Z)$$

is surjective. By assumption, the left-hand side is generated by a section defining Θ; since Z is clearly contained in Θ, it follows that the map is in fact 0. Therefore $H^0(A, \mathcal{O}_A(\Theta)|_Z) = 0$.

On the other hand, if $Z \neq \emptyset$, then for $a \in A$ general, the divisor $\Theta_a = a + \Theta$ does not contain Z, hence $H^0\big(A, \mathcal{O}_A(\Theta_a)|_Z)\big) \neq 0$. Since this holds for general $a \in A$, the semicontinuity theorem implies $H^0(A, \mathcal{O}_A(\Theta)|_Z) \neq 0$, a contradiction. We thus conclude that $Z = \emptyset$. Since this holds for every $\lambda < 1$, it follows that $\mathrm{lct}(\Theta) = 1$.

2.5. Main properties of multiplier ideals

In this section we discuss the main properties of multiplier ideals. The key result concerns the behavior under restriction to a smooth hypersurface. The proof of this result makes use of the Relative Vanishing theorem.

2.5.1. The restriction theorem. In order to state the main result, it is convenient to introduce a variant of multiplier ideals, the *adjoint ideal*.

DEFINITION 2.79. Let X be a smooth variety and H a prime divisor in X. If $\mathfrak{a}$ is an ideal on X with $\mathfrak{a} \cdot \mathcal{O}_H \neq 0$ and $\lambda \in \mathbf{R}_{\geq 0}$, then the adjoint ideal $\mathrm{adj}_H(\mathfrak{a}^\lambda)$ is defined as follows: consider a log resolution $f\colon Y \to X$ of $\big(X, \mathfrak{a} \cdot \mathcal{O}_X(-H)\big)$ and write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ and $f^*H = G$; then

$$\mathrm{adj}_H(\mathfrak{a}) := f_*\mathcal{O}_Y(K_{Y/X} - G - \lfloor \lambda F \rfloor + \widetilde{H}).$$

REMARK 2.80. Note that our assumption on $\mathfrak{a}$ implies that $\mathrm{ord}_{\widetilde{H}}(F) = 0$, hence

$$\mathrm{ord}_{\widetilde{H}}(K_{Y/X} - G - \lfloor \lambda F \rfloor + \widetilde{H}) = 0.$$

We thus have

$$f_*\mathcal{O}_Y(K_{Y/X} - G - \lfloor \lambda F \rfloor + \widetilde{H}) \subseteq f_*\mathcal{O}_Y(K_{Y/X}) = \mathcal{O}_X,$$

where the last equality follows from Lemma 2.31. Therefore $\mathrm{adj}_H(\mathfrak{a})$ is an ideal of $\mathcal{O}_X$.

REMARK 2.81. Since the divisors $\widetilde{H}$ and $G - \widetilde{H}$ are effective, it follows from the definition of the adjoint ideal that we have the inclusions

$$\mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_X(-H) = \mathcal{J}\big(\mathfrak{a}^\lambda \cdot \mathcal{O}_X(-H)\big) \subseteq \mathrm{adj}_H(\mathfrak{a}^\lambda) \subseteq \mathcal{J}(\mathfrak{a}^\lambda),$$

where the first equality is a consequence of the projection formula.

As in the case of multiplier ideals, we need to show that the definition is independent of the resolution.

PROPOSITION 2.82. The ideal $\mathrm{adj}_H(\mathfrak{a})$ is independent of the log resolution f.

PROOF. The proof follows along the same lines as the proof of Theorem 2.33, with some care due to the special role of the divisor $\widetilde{H}$. Note that by definition, if $U \subseteq X$ is an open subset, $\varphi \in \mathcal{O}_X(U)$ lies in $\Gamma\big(U, \mathrm{adj}_H(\mathfrak{a})\big)$ if and only if for every divisor E on Y different from $\widetilde{H}$ and with $c_X(\mathrm{ord}_E) \cap U \neq \emptyset$ we have

$$\mathrm{ord}_E(\varphi) > \lambda \cdot \mathrm{ord}_E(\mathfrak{a}) + \mathrm{ord}_E(H) - A_Y(\mathrm{ord}_E). \tag{2.12}$$

Note that if $E = \widetilde{H}$, then the right-hand side of (2.12) is 0, hence we have the weak inequality in this case, too. We need to show that (2.12) holds if we replace

ord_E by any $\mathrm{ord}_{E'} \neq \mathrm{ord}_H$, with $c_X(\mathrm{ord}_{E'}) \cap U \neq \emptyset$. We may and will assume that $U = X$.

Suppose that $E_1, \ldots, E_r$ are the divisors on Y containing $c_Y(\mathrm{ord}_{E'})$ and which are contained in $\mathrm{Exc}(f) \cup \mathrm{Supp}(F+G)$. Let $q_i = \mathrm{ord}_{E'}(E_i)$ for $1 \leq i \leq r$. By assumption, we know that for all i we have

$$\mathrm{ord}_{E_i}(\varphi) \geq \lambda \cdot \mathrm{ord}_{E_i}(\mathfrak{a}) + \mathrm{ord}_{E_i}(H) - A_Y(\mathrm{ord}_{E_i})$$

and the equality is strict unless $E_i = \widetilde{H}$. Recall that by Lemma 2.34, we have $A_Y(\mathrm{ord}_{E'}) \geq \sum_{i=1}^r q_i$ and the inequality is strict is $c_Y(\mathrm{ord}_{E'})$ has codimension $> r$. Arguing as in the proof of Theorem 2.33, we see that

$$A_X(\mathrm{ord}_{E'}) \geq \sum_{i=1}^r q_i \cdot A_X(\mathrm{ord}_{E'}) \tag{2.13}$$

and this inequality is strict if $c_Y(\mathrm{ord}_{E'})$ has codimension $> r$. Furthermore, we have

$$\mathrm{ord}_{E'}(\varphi) \geq \sum_{i=1}^r q_i \cdot \mathrm{ord}_{E_i}(\varphi) \geq \sum_{i=1}^r q_i \big(\lambda \cdot \mathrm{ord}_{E_i}(\mathfrak{a}) + \mathrm{ord}_{E_i}(H) - A_X(\mathrm{ord}_{E_i})\big). \tag{2.14}$$

Moreover, the second inequality is strict unless $r = 1$ and $E_1 = \widetilde{H}$. Using (2.13), we conclude that

$$\sum_{i=1}^r q_i \big(\lambda \cdot \mathrm{ord}_{E_i}(\mathfrak{a}) + \mathrm{ord}_{E_i}(H) - A_Y(\mathrm{ord}_{E_i})\big) = \lambda \cdot \mathrm{ord}_{E'}(\mathfrak{a}) + \mathrm{ord}_{E'}(H) - \sum_{i=1}^r q_i \cdot A_X(\mathrm{ord}_{E_i})$$

$$\geq \lambda \cdot \mathrm{ord}_{E'}(\mathfrak{a}) + \mathrm{ord}_{E'}(H) - A_X(\mathrm{ord}_{E'}).$$

Therefore we are done, unless $r = 1$ and $E_1 = \widetilde{H}$. However, in this case since $\mathrm{ord}_{E'} \neq \mathrm{ord}_H$, the codimension of $c_Y(\mathrm{ord}_{E'})$ is > 1, hence

$$A_X(\mathrm{ord}_{E'}) \geq q_1 \cdot A_X(\mathrm{ord}_{\widetilde{H}}) + 1 = q_1 + 1.$$

We thus have

$$\lambda \cdot \mathrm{ord}_{E'}(\mathfrak{a}) + \mathrm{ord}_{E'}(H) - A_X(\mathrm{ord}_{E'}) = q_1 - A_X(\mathrm{ord}_{E'}) \leq -1 < \mathrm{ord}_{E'}(\varphi).$$

This completes the proof. $\square$

It is common to denote the ideal $\mathrm{adj}_H(\mathcal{O}_X)$ by $\mathrm{adj}(H)$.

The following result describes the restriction of a multiplier ideal to a smooth hypersurface.

THEOREM 2.83 (Restriction Theorem). *Let X be a smooth variety and H a smooth irreducible hypersurface in X. If $\mathfrak{a}$ is an ideal on X with $\mathfrak{b} := \mathfrak{a} \cdot \mathcal{O}_H \neq 0$, then for every $\lambda \in \mathbf{R}_{\geq 0}$, we have an exact sequence*

$$0 \longrightarrow \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_X(-H) \xrightarrow{i} \mathrm{adj}_H(\mathfrak{a}) \xrightarrow{p} \mathcal{J}(\mathfrak{b}^\lambda) \longrightarrow 0,$$

with i is the natural inclusion of ideals and p is induced by the projection $\mathcal{O}_X \to \mathcal{O}_H$.

PROOF. Let $f \colon Y \to X$ be a log resolution of $\big(X, \mathfrak{a} \cdot \mathcal{O}_X(-H)\big)$. We write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ and $f^*(H) = \widetilde{H} + R$. Let $\mathcal{L} = \mathcal{O}_Y\big(K_{Y/X} - \lfloor \lambda F \rfloor - f^*(H)\big)$ and consider the short exact sequence on Y:

$$0 \to \mathcal{L} \to \mathcal{L}(\widetilde{H}) \to \mathcal{L}(\widetilde{H})|_{\widetilde{H}} \to 0. \tag{2.15}$$

Using the projection formula and Theorem 2.69], we get

$$f_*\mathcal{L} \simeq f_*\mathcal{O}_Y(K_{Y/X} - \lfloor \lambda F \rfloor) \otimes \mathcal{O}_X(-H) = \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_X(-H) \quad \text{and}$$
$$R^1 f_*\mathcal{L} \simeq R^1 f_*\mathcal{O}_Y(K_{Y/X} - \lfloor \lambda F \rfloor) \otimes \mathcal{O}_X(-H) = 0.$$

Since we also have

$$f_*\mathcal{L}(\widetilde{H}) = \mathrm{adj}_H(\mathfrak{a}^\lambda),$$

we get from the exact sequence (2.15) another exact sequence on X:

$$0 \longrightarrow \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_X(-H) \xrightarrow{i} \mathrm{adj}_H(\mathfrak{a}) \xrightarrow{p} f_*\big(\mathcal{L}(\widetilde{H})|_{\widetilde{H}}\big) \longrightarrow 0. \tag{2.16}$$

Note now that the morphism $g\colon \widetilde{H} \to H$ is a log resolution of $(H, \mathfrak{b})$. Indeed, this follows from the following observations:

i) If E is the sum of the f-exceptional divisors, then $\widetilde{H}$ does not appear in $E + F$ and $\widetilde{H} + E + F$ has simple normal crossings (see assertion iii) in Exercise 2.26). Therefore $\widetilde{H}$ is smooth and $(E + F)|_{\widetilde{H}}$ has simple normal crossings.
ii) $\mathfrak{b} \cdot \mathcal{O}_{\widetilde{H}} = \mathcal{O}_{\widetilde{H}}(-F|_{\widetilde{H}})$.
iii) $\mathrm{Exc}(g) \subseteq \mathrm{Supp}(E|_{\widetilde{H}})$.

Finally, note that $\widetilde{H}$ does not appear in $K_{Y/X} - R$ (this is clear) and $K_{\widetilde{H}/H} = (K_{Y/X} - R)|_{\widetilde{H}}$: the fact that they are linearly equivalent follows from the adjunction formula, but the fact that we have indeed an equality is proved in Lemma 2.84 below. Moreover, since $H + F$ has simple normal crossings, we easily see that $\lfloor \lambda F \rfloor|_{\widetilde{H}} = \lfloor \lambda F|_{\widetilde{H}} \rfloor$. This implies that

$$f_*\big(\mathcal{L}(\widetilde{H})|_{\widetilde{H}}\big) = g_*\mathcal{O}_{\widetilde{H}}(K_{\widetilde{H}/H} - \lfloor \lambda F|_{\widetilde{H}} \rfloor) = \mathcal{J}(\mathfrak{b}^\lambda).$$

The exact sequence (2.16) is thus the exact sequence in the theorem. The fact that the maps are as described follows by restricting to the maximal open subset over which f is an isomorphism, where this fact follows easily. □

Lemma 2.84. *Let $f\colon Y \to X$ be a proper birational morphism between smooth varieties. If H is a smooth prime divisor on H such that $\widetilde{H}$ is smooth, and if we write $f^*(H) = \widetilde{H} + R$, then*

$$K_{\widetilde{H}/H} = (K_{Y/X} - R)|_{\widetilde{H}}.$$

Proof. Let $g\colon \widetilde{H} \to H$ be the restriction of f. We prove this equality at any point $P \in \widetilde{H}$. Let $x_1, \ldots, x_n$ be a regular system of parameters in $\mathcal{O}_{X,f(P)}$ such that H is defined at $f(P)$ by (x_1) and let $y_1, \ldots, y_n$ be a regular system of parameters in $\mathcal{O}_{Y,P}$ such that $\widetilde{H}$ is defined at P by (y_1). We then have regular systems of parameters in $\mathcal{O}_{H,f(P)} = \mathcal{O}_{X,P}/(x_1)$ and $\mathcal{O}_{\widetilde{H},P} = \mathcal{O}_{Y,P}/(y_1)$ given by $\overline{x_2}, \ldots, \overline{x_n}$ and $\overline{y_2}, \ldots, \overline{y_n}$, respectively.

We can write $f^*(x_1) = y_1 h$ and h defines R at P. Note that $f^*(dx_1) = y_1 dh + h dy_1$. If we write

$$f^*(dx_2 \wedge \ldots \wedge dx_n) = \sum_{i=1}^{n} \varphi_i dy_1 \wedge \ldots \wedge \widehat{dy_i} \wedge \ldots \wedge dy_n,$$

with $\varphi_1, \ldots, \varphi_n \in \mathcal{O}_{Y,P}$, then

$$g^*(d\overline{x_2} \wedge \ldots \wedge d\overline{x_n}) = \varphi_1|_{\widetilde{H}} d\overline{y_2} \wedge \ldots \wedge d\overline{y_n},$$

hence $K_{\widetilde{H}/H}$ is defined at P by $\varphi|_{\widetilde{H}}$. On the other hand, if we write

$$f^*(dx_1 \wedge \ldots \wedge dx_n) = \psi dy_1 \wedge \ldots \wedge dy_n = (y_1 dh + h dy_1) \wedge f^*(dx_2 \wedge \ldots \wedge dx_n),$$

for some $\psi \in \mathcal{O}_{Y,P}$, so that $K_{Y/X}$ is defined at P by (ψ), then we see that we can write

$$\psi = y_1 w + h\varphi_1 \quad \text{for some} \quad w \in \mathcal{O}_{Y,P}.$$

In this case $\psi|_{\widetilde{H}} = (h\varphi_1)|_{\widetilde{H}}$, which implies the equality in the lemma at P. $\square$

Corollary 2.85. With the notation in the theorem, we have

$$\mathcal{J}(\mathfrak{b}^\lambda) \subseteq \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_H \quad \text{for all} \quad \lambda \in \mathbf{R}_{\geq 0}.$$

In particular, there is an open neighborhood U of H such that $\mathrm{lct}(\mathfrak{a}|_U) \geq \mathrm{lct}(\mathfrak{b})$ and $\mathrm{lct}_P(\mathfrak{a}) \geq \mathrm{lct}_P(\mathfrak{b})$ for every $P \in H$.

Proof. Indeed, it follows from the theorem and Remark 2.81 that

$$\mathcal{J}(\mathfrak{b}^\lambda) = \mathrm{adj}_H(\mathfrak{a}) \cdot \mathcal{O}_H \subseteq \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_H.$$

The assertions about the log canonical thresholds follow from the definition of these invariants. $\square$

As a consequence, we obtain the following lower bound for the local threshold in terms of the order of vanishing at a point.

Corollary 2.86. Let $\mathfrak{a}$ be a nonzero ideal on the smooth variety X. If $P \in X$ is such that $\mathrm{ord}_P(\mathfrak{a}) = d \geq 1$, then $\mathrm{lct}_P(\mathfrak{a}) \geq \frac{1}{d}$.

Proof. We argue by induction on $n = \dim(X) \geq 1$. If $n = 1$, then $\mathfrak{a} = \mathcal{O}_X(-dP)$ in a neighborhood of P and we have $\mathrm{lct}_P(\mathfrak{a}) = \frac{1}{d}$. For the induction step, after possibly replacing X with a suitable open neighborhood of P, we may assume that X is affine and we have $x_1, \ldots, x_n \in \mathcal{O}_X(X)$ that give a regular system of parameters at P. Let H be the hypersurface in X defined by $h = \sum_{i=1}^n a_i x_i$, with $a_1, \ldots, a_n$ general and $\mathfrak{b} = \mathfrak{a} \cdot \mathcal{O}_H$. In this case $\mathrm{ord}_P(\mathfrak{b}) = d$. Indeed, note that we have

$$\widehat{\mathcal{O}_{X,P}} = k[\![x_1, \ldots, x_n]\!]$$

and by assumption $\mathfrak{a} \cdot \widehat{\mathcal{O}_{X,P}}$ contains an element of the form $\sum_{i \geq d} f_i(x)$, with each f_i homogeneous of degree i and $f_d \neq 0$. It follows that as long as h does not divide $f_d(x)$, we have $\mathrm{ord}_P(\mathfrak{b}) = d$ (the inequality $\mathrm{ord}_P(\mathfrak{b}) \geq d$ holds for all H). Using the inductive assumption and Corollary 2.85, we conclude that

$$\mathrm{lct}_P(\mathfrak{a}) \geq \mathrm{lct}_P(\mathfrak{a}|_H) \geq \frac{1}{d}.$$

$\square$

The result in Corollary 2.85 can be extended to the pull-back via an arbitrary morphism, as follows.

Theorem 2.87. *Let $f\colon W \to X$ be a morphism of smooth varieties. If $\mathfrak{a}$ is an ideal on X such that $\mathfrak{b} := \mathfrak{a} \cdot \mathcal{O}_W \neq 0$, then*

$$\mathcal{J}(\mathfrak{b}^\lambda) \subseteq \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_W.$$

In particular, there is an open neighborhood of $f(W)$ such that $\mathrm{lct}(\mathfrak{a}|_U) \geq \mathrm{lct}(\mathfrak{b})$ and for every $P \in W$, we have $\mathrm{lct}_{f(P)}(\mathfrak{a}) \geq \mathrm{lct}_P(\mathfrak{a})$.

We first prove a more precise result concerning the behavior of multipler ideals under pull-back via a smooth morphism.

PROPOSITION 2.88. If $p\colon W \to X$ is a smooth morphism of smooth varieties, $\mathfrak{a}$ is a nonzero ideal on X and $\mathfrak{b} = \mathfrak{a} \cdot \mathcal{O}_W$, then

$$\mathcal{J}(\mathfrak{b}^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_W \quad \text{for all} \quad \lambda \geq 0.$$

PROOF. Let $f\colon Y \to X$ be a log resolution of $(X, \mathfrak{a})$, with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. Consider the Cartesian diagram

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ W \arrow[r, "p"'] & X. \end{tikzcd}$$

Since p is smooth, it follows that q is smooth as well, hence Z is smooth. Since f is birational, it is an isomorphism over some open subset $U \subseteq X$, hence g is an isomorphism over $p^{-1}(U)$. Note that Z is irreducible (since $f_*\mathcal{O}_Y = \mathcal{O}_X$, it follows from flat base-change that $g_*\mathcal{O}_Z = \mathcal{O}_W$, hence g has connected fibers). Since $\Omega_{Z/W} \simeq q^*(\Omega_{Y/X})$, it follows easily that $K_{Z/W} = q^*(K_{Y/X})$. Moreover, since q is smooth, we deduce that $q^*(F + K_{Y/X})$ has simple normal crossings. Therefore g is a log resolution of $(W, \mathfrak{b})$, with $\mathfrak{b} \cdot \mathcal{O}_Z = \mathcal{O}_Z(-q^*F)$. Moreover, we have $\lfloor \lambda q^*F)\rfloor = q^*\lfloor \lambda F\rfloor$, and thus

$$\mathcal{J}(\mathfrak{b}^\lambda) = g_*q^*\mathcal{O}_Y(K_{Y/X} - \lfloor \lambda F\rfloor) \simeq p^*f_*\mathcal{O}_Y(K_{Y/X} - \lfloor \lambda F\rfloor) \simeq \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_W,$$

where the first isomorphism follows by flat base-change and the second one also follows from the flatness of p. The fact that the isomorphism commutes with the embedding in $\mathcal{O}_W$ follows by restricting to $p^{-1}(U)$. $\square$

PROOF OF THEOREM 2.87. Again, it is enough to prove the assertion about multiplier ideals, since the one about log canonical thresholds follows immediately. We factor f as the composition

$$W \overset{\iota}{\hookrightarrow} W \times X \xrightarrow{p} X, \quad \text{where} \quad \iota(x) = \big(x, f(x)\big),\ p(w, x) = x.$$

Clearly, it is enough to prove the inclusion in the theorem separately for ι and p. For p, this follows from Proposition 2.88, hence to complete the proof we may assume that f is a closed immersion. The assertion to prove is local on X. Since f is a closed immersion of smooth varieties, after restricting to a suitable cover of X, we may assume that if $r = \operatorname{codim}_X(W)$, we have closed immersions

$$W = Z_r \hookrightarrow Z_{r-1} \hookrightarrow \ldots \hookrightarrow Z_1 \hookrightarrow Z_0 = X,$$

identifying each Z_i with a smooth divisor in Z_{i-1}, for $1 \leq i \leq r$. Therefore it is enough to treat the case of codimension 1, which follows from Corollary 2.85. $\square$

We end this section by showing that in Corollary 2.85 we have equality if H is the general member of a base-point free linear system. More generally, we have the following:

COROLLARY 2.89. Let X be a smooth variety and $\mathfrak{a}$ a nonzero ideal on X. If $r \leq \dim(X) - 1$ and $Y = H_1 \cap \ldots \cap H_r$, where $H_1, \ldots, H_r$ are general elements of a base-point free linear system on X, then for $\mathfrak{b} = \mathfrak{a} \cdot \mathcal{O}_Y$, we have

$$\mathcal{J}(\mathfrak{b}^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_Y \quad \text{for all} \quad \lambda \in \mathbf{R}_{\geq 0}.$$

PROOF. We first consider the case $r = 1$. The equality follows from Theorem 2.83 if we show that if H is a general member of a base-point free linear system on H, then $\mathrm{adj}_H(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)$ for all $\lambda \in \mathbf{R}_{\geq 0}$. Let $f\colon Y \to X$ be a log resolution of $(X, \mathfrak{a})$, with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. By hypothesis, f^*H is a general element of a base-point free linear system on Y, hence by the Kleiman-Bertini theorem, we see that f^*H is smooth (possibly disconnected), without common components with $K_{Y/X}$ and F, and such that $f^*H + F + K_{Y/X}$ has simple normal crossings. In particular, we have $f^*H = \widetilde{H}$, and f is a log resolution on $\mathfrak{a} \cdot \mathcal{O}_X(-H)$. It is then clear from the definition of adjoint ideals that $\mathrm{adj}_H(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)$ for all $\lambda \in \mathbf{R}_{\geq 0}$.

For general r, we use the fact that if $H_1, \ldots, H_r$ are general, then we may apply the above argument to each $H_1 \cap \ldots \cap H_i \hookrightarrow H_1 \cap \ldots \cap H_{i-1}$ for $1 \leq i \leq r$. $\square$

EXERCISE 2.90. Let $X = M_{m,n}(\mathbf{C})$, with $m \leq n$, and let $\mathfrak{a} \subseteq \mathbf{C}[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n]$ be the ideal generated by the maximal minors of the $m \times n$ matrix of indeterminates $(x_{i,j})$. Show that $\mathrm{lct}(\mathfrak{a}) = n - m + 1$.

2.5.2. The Subadditivity Theorem. We now discuss a result of Demailly, Ein, and Lazarsfeld [**DEL00**], which gives a somewhat surprising behavior of multiplier ideals with respect to products. This has been the source of interesting applications in commutative algebra, such as [**ELS01**] and [**ELS03**]. This is one result that makes essential use of the fact that we work on smooth varieties.

THEOREM 2.91 (Subadditivity Theorem). *If $\mathfrak{a}$ and $\mathfrak{b}$ are nonzero ideals on the smooth variety X, then for every $\lambda, \mu \in \mathbf{R}_{\geq 0}$, we have*

$$\mathcal{J}(\mathfrak{a}^\lambda \mathfrak{b}^\mu) \subseteq \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{J}(\mathfrak{b}^\mu).$$

The idea is to first treat the case when we have two ideals on a product of varieties, each of them coming from one of the projections. In this case it is easy to see that the inclusion in the theorem is in fact an equality.

LEMMA 2.92. *Let X and Y be smooth varieties and $\mathfrak{a}$ and $\mathfrak{b}$ nonzero ideals on X and Y, respectively. If $\widetilde{\mathfrak{a}} = \mathfrak{a} \cdot \mathcal{O}_{X \times Y}$ and $\widetilde{\mathfrak{b}} = \mathfrak{b} \cdot \mathcal{O}_{X \times Y}$, then for every $\lambda, \mu \in \mathbf{R}_{\geq 0}$, we have*

$$\mathcal{J}(\widetilde{\mathfrak{a}}^\lambda \widetilde{\mathfrak{b}}^\mu) = \mathcal{J}(\widetilde{\mathfrak{a}}^\lambda) \cdot \mathcal{J}(\widetilde{\mathfrak{b}}^\mu) = \mathcal{J}(\mathfrak{a}^\lambda) \boxtimes \mathcal{J}(\mathfrak{b}^\mu). \tag{2.17}$$

PROOF. Let $f\colon X' \to X$ and $g\colon Y' \to Y$ be log resolutions of $(X, \mathfrak{a})$ and $(Y, \mathfrak{b})$, respectively. We write $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ and $\mathfrak{b} \cdot \mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-G)$. Let $p\colon X' \times Y' \to X'$ and $q'\colon X' \times Y' \to Y'$ be the projections and similarly for $p\colon X \times Y \to X$, $q\colon X \times Y \to Y$. Note that $h = (f, g)\colon X' \times Y' \to X \times Y$ is birational and

$$K_{X' \times Y'/X \times Y} = p'^*(K_{X'/X}) + q'^*(K_{Y'/Y}).$$

Moreover, we have

$$\widetilde{\mathfrak{a}} \cdot \mathcal{O}_{X' \times Y'} = \mathcal{O}_{X' \times Y'}(-p'^*F) \quad \text{and} \quad \widetilde{\mathfrak{b}} \cdot \mathcal{O}_{X' \times Y'} = \mathcal{O}_{X' \times Y'}(-q'^*G)$$

and clearly $p'^*(F)$ and q'^*G have no common components, hence

$$\lfloor \lambda p'^*F + \mu q'^*G \rfloor = \lfloor \lambda p'^*F \rfloor + \lfloor \mu q'^*G \rfloor = p'^*\lfloor \lambda F \rfloor + q'^*\lfloor \mu G \rfloor.$$

Note that if A and B are SNC divisors on X' and Y', respectively, then $p'^*(A) + q'^*(B)$ is SNC. We thus see that h is a log resolution of $(X \times Y, \widetilde{\mathfrak{a}} \cdot \widetilde{\mathfrak{b}})$ and using Künneth's formula we obtain

$$\mathcal{J}(\widetilde{\mathfrak{a}}^\lambda \widetilde{\mathfrak{b}}^\mu) = h_* \mathcal{O}_{X' \times Y'}(p'^*K_{X'/X} + q'^*K_{Y'/Y} - p'^*\lfloor \lambda F \rfloor - q'^*\lfloor \mu G \rfloor)$$

$$= h_*\big(\mathcal{O}_{X'}(K_{X'/X} - \lfloor \lambda F \rfloor) \boxtimes \mathcal{O}_{Y'}(K_{Y'/Y} - \lfloor \mu G \rfloor)\big)$$
$$= f_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor \lambda F \rfloor) \boxtimes g_*\mathcal{O}_{Y'}(K_{Y'/Y} - \lfloor \mu G \rfloor)\big) = \mathcal{J}(\mathfrak{a}^\lambda) \boxtimes \mathcal{J}(\mathfrak{b}^\mu).$$

A special case of this (or a consequence of Proposition 2.88) is that

$$\mathcal{J}(\widetilde{\mathfrak{a}}^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda) \boxtimes \mathcal{O}_Y \quad \text{and} \quad \mathcal{J}(\widetilde{\mathfrak{b}}^\mu) = \mathcal{O}_X \boxtimes \mathcal{J}(\mathfrak{b}^\mu),$$

hence the equality

$$\mathcal{J}(\mathfrak{a}^\lambda) \boxtimes \mathcal{J}(\mathfrak{b}^\mu) = \mathcal{J}(\widetilde{\mathfrak{a}}^\lambda) \cdot \mathcal{J}(\widetilde{\mathfrak{b}}^\mu)$$

is clear. This completes the proof of the lemma. □

Proof of Theorem 2.91. Let $\Delta\colon X \hookrightarrow X \times X$ be the diagonal embedding and let $p, q\colon X \times X \to X$ be the projections on the first and second components, respectively. We consider

$$\widetilde{\mathfrak{a}} = p^{-1}(\mathfrak{a}) \quad \text{and} \quad \widetilde{\mathfrak{b}} = q^{-1}(\mathfrak{b}).$$

Note that $\Delta^{-1}(\widetilde{\mathfrak{a}}) = \mathfrak{a}$ and $\Delta^{-1}(\widetilde{\mathfrak{b}}) = \mathfrak{b}$, so that it follows from Theorem 2.87 (in fact, we use the corresponding version for mixed multiplier ideals, which can be either proved in the same way or can be deduced from the version for usual multiplier ideals via Remark 2.49) that

$$\mathcal{J}(\mathfrak{a}^\lambda \mathfrak{b}^\mu) \subseteq \Delta^{-1}\big(\mathcal{J}(\widetilde{\mathfrak{a}}^\lambda \widetilde{\mathfrak{b}}^\mu)\big). \tag{2.18}$$

On the other hand, it follows from the lemma that

$$\Delta^{-1}\big(\mathcal{J}(\widetilde{\mathfrak{a}}^\lambda \widetilde{\mathfrak{b}}^\mu)\big) = \Delta^{-1}\big(p^{-1}\mathcal{J}(\mathfrak{a}^\lambda) \cdot q^{-1}\mathcal{J}(\mathfrak{b}^\mu)\big) = \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{J}(\mathfrak{b}^\mu). \tag{2.19}$$

The assertion in the theorem follows by combining (2.18) and (2.19). □

2.5.3. Behavior of multiplier ideals in families. In this section we consider a smooth morphism $\pi\colon X \to T$ of varieties and an ideal $\mathfrak{a}$ on X. For every $t \in T$, we denote by X_t the fiber $\pi^{-1}(t)$ and by $\mathfrak{a}_t$ the ideal $\mathfrak{a}|_{X_t}$. We assume that for any $t \in T$, the ideal $\mathfrak{a}_t$ is everywhere nonzero and we consider its multiplier ideals (since we don't assume that the fibers X_t are connected, the multiplier ideals are defined separately on each connected component).

Proposition 2.93. With the above notation, if T is smooth, then there is a nonempty open subset $T_0 \subseteq T$ such that

$$\mathcal{J}(\mathfrak{a}_t^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_{X_t} \quad \text{for all} \quad t \in T_0, \lambda \in \mathbf{R}_{\geq 0}.$$

Proof. Note that since T is smooth and π is a smooth morphism, X is smooth too. After possibly replacing T by an affine open subset, we may assume that T is affine. Let $|V|$ be a very ample linear system on T and $|W|$ its inverse image on X, which is base-point free. Let $r = \dim(Z)$. By Corollary 2.89, if $H_1, \ldots, H_r$ are general elements of $|W|$ and if $Y = H_1 \cap \ldots \cap H_r$ and $\mathfrak{a}_Y = \mathfrak{a} \cdot \mathcal{O}_Y$, then

$$\mathcal{J}(\mathfrak{a}_Y^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_Y \quad \text{for all} \quad \lambda \in \mathbf{R}_{\geq 0}.$$

Note that Y is a union of fibers of π. Moreover, there is an open subset $T_0 \subseteq T$ such that every fiber X_t, with $t \in T_0$, is a component of such Y. This gives the assertion in the proposition. □

The above proposition describes the generic behavior of multiplier ideals in families. The next theorem gives a semicontinuity result regarding the triviality of such ideals.

THEOREM 2.94. *If $\pi\colon X \to T$ is a smooth morphism and $\mathfrak{a}$ an ideal on X that for every $t \in T$, the ideal $\mathfrak{a}_t$ is everywhere nonzero, then for every $\lambda \in \mathbf{R}_{\geq 0}$, the set*

$$W_\lambda := \left\{x \in X \mid \mathcal{J}(\mathfrak{a}_{\pi(x)}^\lambda)_x \neq \mathcal{O}_{X_{\pi(x)},x}\right\}$$

is closed in X.

PROOF. We divide the proof in several steps.
Step 1. We may assume that T is smooth. Indeed, let $g\colon \widetilde{T} \to T$ be a resolution of singularities and consider the Cartesian diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "h"] \arrow[d, "\widetilde{\pi}"'] & X \arrow[d, "\pi"] \\ \widetilde{T} \arrow[r, "g"'] & T. \end{tikzcd}$$

If $\widetilde{\mathfrak{a}} = \mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}}$ and

$$\widetilde{W_\lambda} = \left\{x \in \widetilde{X} \mid \mathcal{J}(\widetilde{\mathfrak{a}}_{\widetilde{\pi}(x)}^\lambda)_x \neq \mathcal{O}_{\widetilde{X}_{\widetilde{\pi}(x)},x}\right\},$$

then $\widetilde{W_\lambda} = h^{-1}(W_\lambda)$. Since h is proper, hence closed, it is clear that W_λ is closed in X if and only if $\widetilde{W_\lambda}$ is closed in $\widetilde{X}$.
Step 2. Given any T, there is a nonempty open subset T_0 of T, such that the assertion in the theorem holds for the morphism $f^{-1}(T_0) \to T_0$. Indeed, in order to see this, after possibly restricting to the smooth locus of T and then to the open subset provided by Proposition 2.93, we may assume that $\mathcal{J}(\mathfrak{a}_t^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_{X_t}$ for every $t \in T$. In this case it follows from Nakayama's Lemma that

$$W_\lambda = \left\{x \in X \mid \mathcal{J}(\mathfrak{a}^\lambda)_x \neq \mathcal{O}_{X,x}\right\},$$

which is clearly closed in X.
Step 3. After repeatedly applying Step 2, we can write T as the disjoint union of finitely many locally closed subsets T_α such that $W_\lambda \cap \pi^{-1}(T_\alpha)$ is closed in $\pi^{-1}(T_\alpha)$ for every α. This implies that W_λ is constructible.
Step 4. Since W_λ is constructible, in order to show that it is closed, it is enough to show that if Z is a 1-dimensional locally closed subvariety of X and $x \in Z$ such that $Z \smallsetminus \{x\} \subseteq W_\lambda$, then $x \in W_\lambda$. Since we may replace T by $\overline{\pi(Z)}$, we may assume that Z dominates T. If T is a point, the assertion is clear, hence we may assume that $\dim(T) = 1$. Furthermore, by Step 1, we may assume that T, hence also X is smooth.

Arguing by contradiction, let us assume that $x \not\in W_\lambda$, hence $\mathcal{J}(\mathfrak{a}_{\pi(x)}^\lambda)_x = \mathcal{O}_{X_{\pi(x)},x}$. Applying Corollary 2.85 for $X_{\pi(x)} \hookrightarrow X$, we conclude that $\mathcal{J}(\mathfrak{a}^\lambda)_x = \mathcal{O}_{X,x}$. Therefore there is an open neighborhood V of x such that $\mathcal{J}(\mathfrak{a}^\lambda)_y = \mathcal{O}_{X,y}$ for every $y \in V$. On the other hand, it follows from Proposition 2.93 that there is a nonempty open subset U of T such that

$$\mathcal{J}(\mathfrak{a}_t^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_{X_t} \quad \text{for all} \quad t \in U.$$

Since $\dim(T) = 1$, it follows that $U' = U \cup \{\pi(x)\}$ is an open neighborhood of $\pi(x)$, hence $\pi^{-1}(U')$ is an open neighborhood of x. The intersection $Z \cap \pi^{-1}(U') \cap V$ is nonempty, hence 1-dimensional, and it is not contained in the fiber over $\pi(x)$. Therefore there is $y \in Z \cap \pi^{-1}(U) \cap V$, with $y \neq x$. Since

$$\mathcal{J}(\mathfrak{a}_{\pi(y)}^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_{X_{\pi(y)},y} = \mathcal{O}_{X_{\pi(y)},y},$$

this contradicts our assumption on $Z \smallsetminus \{x\}$. This completes the proof of the theorem. $\square$

COROLLARY 2.95. It $\pi\colon X \to T$ and $\mathfrak{a}$ are as in Theorem 2.94, then for every $\lambda \in \mathbf{R}_{\geq 0}$, the set

$$U_\lambda := \{x \in X \mid \mathrm{lct}_x(\mathfrak{a}_{\pi(x)}) \geq \lambda\}$$

is open in X. In particular, if $s\colon T \to X$ is such that $\pi \circ s = \mathrm{id}_T$, then the set

$$\{t \in T \mid \mathrm{lct}_{s(t)}(\mathfrak{a}_t) \geq \lambda\}$$

is open in T.

PROOF. Since $\mathcal{J}(\mathfrak{a}^\mu_{\pi(x)})_x \neq \mathcal{O}_{X_{\pi(x)},x}$ if and only if $\mathrm{lct}_x(\mathfrak{a}_{\pi(x)}) \leq \mu$, we deduce from Theorem 2.94 that for every μ, the set

$$A_\mu = \{x \in X \mid \mathrm{lct}_x(\mathfrak{a}_{\pi(x)}) > \mu\}$$

is open in X. If the set

$$L := \{\mathrm{lct}_x(\mathfrak{a}_{\pi(x)}) \mid x \in X\}$$

is finite, then the set U_λ is equal to A_μ, where μ is the largest element in L that is $< \lambda$.

Finiteness of L follows by induction on $\dim(T)$ using Proposition 2.93. Indeed, if we choose $T_0 \subseteq T_{\mathrm{sm}}$ as in the proposition, then it is clear that for $x \in \pi^{-1}(T_0)$, the log canonical threshold $\mathrm{lct}_x(\mathfrak{a}_{\pi(x)})$ lies in the finite set $\{\mathrm{lct}_x(\mathfrak{a}) \mid x \in \pi^{-1}(T_0)\}$. This completes the proof of the corollary. $\square$

2.6. Integral closure and the Briançon-Skoda Theorem

In this section we discuss an application of multiplier ideals to integral closure due to Ein and Lazarsfeld [**EL99**]. We begin with an introduction to integral closure for ideals. Roughly speaking, this records the relevant information about an ideal from the point of view of valuation theory.

2.6.1. Integral closure of ideals. In this section, unless explicitly mentioned otherwise, we work over a field of arbitrary characteristic. Let X be a normal variety and $\mathfrak{a}$ a nonzero ideal on X. Consider a proper birational morphism $f\colon Y \to X$, with Y normal, such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$ for an effective Cartier divisor E on Y (the latter condition is equivalent to the fact that f factors through the blow-up of X along $\mathfrak{a}$).

DEFINITION 2.96. The *integral closure* of $\mathfrak{a}$ is

$$\overline{\mathfrak{a}} := f_*\mathcal{O}_Y(-E).$$

The ideal $\mathfrak{a}$ is *integrally closed* if $\mathfrak{a} = \overline{\mathfrak{a}}$.

REMARK 2.97. Since X is normal and E is effective, we have $f_*\mathcal{O}_Y(-E) \hookrightarrow f_*(\mathcal{O}_Y) = \mathcal{O}_X$, hence $\overline{\mathfrak{a}}$ is an ideal of $\mathcal{O}_X$. It is also clear that we have an inclusion $\mathfrak{a} \subseteq \overline{\mathfrak{a}}$. Finally, note that the integral closure does not depend on the choice of f: since any two such morphisms are dominated by a third one, it is enough to show that if $g\colon Z \to Y$ is proper, birational, with Z normal, such that $\mathfrak{a} \cdot \mathcal{O}_Z = \mathcal{O}_Z(-F)$, and $h = f \circ g$, then $h_*\mathcal{O}_Z(-F) = f_*\mathcal{O}_Y(-E)$. Note that we have $F = g^*(E)$, hence

$$h_*\mathcal{O}_Z(-F) = f_*\big(g_*g^*\mathcal{O}_Y(-E)\big) = f_*\mathcal{O}_Y(-E),$$

where the second equality follows from the projection formula and the fact that $g_*\mathcal{O}_Z = \mathcal{O}_Y$.

REMARK 2.98. The independence of $\overline{\mathfrak{a}}$ on f means that for every open subset $U \subseteq X$ and nonzero $\varphi \in \mathcal{O}_X(U)$, we have $\varphi \in \Gamma(U, \overline{\mathfrak{a}})$ if and only if for every divisorial valuation $v = \mathrm{ord}_E$ with $c_X(v) \cap U \neq \emptyset$, we have $v(\varphi) \geq v(\mathfrak{a})$. Moreover, it is enough to consider those prime divisors E in the inverse image of $V(\mathfrak{a})$ on the normalization of the blow-up of X along $\mathfrak{a}$.

PROPOSITION 2.99. Let $\mathfrak{a}$ and $\mathfrak{b}$ be nonzero ideals on X.

i) For every $f\colon Y \to X$ proper, birational, with Y normal, such that $\mathfrak{a}\cdot\mathcal{O}_Y = \mathcal{O}_Y(-E)$ for an effective Cartier divisor E, we have $\overline{\mathfrak{a}} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$.
ii) The ideal $\overline{\mathfrak{a}}$ is integrally closed.
iii) If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\overline{\mathfrak{a}} \subseteq \overline{\mathfrak{b}}$. In particular, if $\mathfrak{b}$ is integrally closed, then $\overline{\mathfrak{a}} \subseteq \mathfrak{b}$.
iv) We have $\overline{\mathfrak{a}} \cdot \overline{\mathfrak{b}} \subseteq \overline{\mathfrak{a} \cdot \mathfrak{b}}$.

PROOF. Let $f\colon Y \to X$ be as in i). Note that on Y we have inclusions

$$\mathfrak{a} \cdot \mathcal{O}_Y \subseteq \overline{\mathfrak{a}} \cdot \mathcal{O}_Y \subseteq \mathcal{O}_Y(-E),$$

where the first one follows from $\mathfrak{a} \subseteq \overline{\mathfrak{a}}$ and the second one follows from the definition of $\overline{\mathfrak{a}}$. Since $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$, these are both equalities, giving the assertion in i). The assertion in ii) is an immediate consequence.

Given $\mathfrak{a}$ and $\mathfrak{b}$, we consider $f\colon Y \to X$ as above such that we also have $\mathfrak{b} \cdot \mathcal{O}_Y(-F)$, for an effective Cartier divisor F (simply take f to dominate both the blow-ups of X along $\mathfrak{a}$ and $\mathfrak{b}$). If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mathcal{O}_Y(-E) \subseteq \mathcal{O}_Y(-F)$, hence

$$\overline{\mathfrak{a}} = f_*\mathcal{O}_Y(-E) \subseteq f_*\mathcal{O}_Y(-F) = \overline{\mathfrak{b}},$$

giving the assertion in iii).

Finally, note that $(\mathfrak{a} \cdot \mathfrak{b}) \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E-F)$. The assertion in iv) follows then from the obvious inclusion

$$f_*\mathcal{O}_Y(-E) \cdot f_*\mathcal{O}_Y(-F) \subseteq f_*\mathcal{O}_Y(-E-F).$$

□

REMARK 2.100. Let $f\colon Y \to X$ be a proper, birational morphism, with X and Y normal, and let D be a Weil divisor on Y such that every prime divisor that appears in D with positive coefficient is f-exceptional. Let us write $D = A - B$, with A and B effective, without common components. We have

$$f_*\mathcal{O}_Y(D) \subseteq f_*\mathcal{O}_Y(A) = \mathcal{O}_X,$$

where the equality follows from Lemma 2.31. This implies that the ideal $\mathfrak{a} := f_*\mathcal{O}_Y(-B)$ is in fact equal to $f_*\mathcal{O}_Y(D)$.

Note that every such ideal $\mathfrak{a}$ is integrally closed. Indeed, after possibly replacing Y by Z, for a suitable $Z \to Y$ and $B = \sum_i b_i E_i$ by $\widetilde{B} = \sum_i b_i \widetilde{E_i}$, we may assume that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$ for some effective divisor E. Since $\mathfrak{a} \cdot \mathcal{O}_Y \subseteq \mathcal{O}_Y(-B)$, we have $\mathcal{O}_Y(-E) \subseteq \mathcal{O}_Y(-B)$, and thus

$$\overline{\mathfrak{a}} = f_*\mathcal{O}_Y(-E) \subseteq f_*\mathcal{O}_Y(-B) = \mathfrak{a},$$

hence $\mathfrak{a} = \overline{\mathfrak{a}}$.

In particular, we see that for every smooth variety X over a field of characteristic 0, every nonzero ideal $\mathfrak{b}$ on X, and every $\lambda \in \mathbf{R}_{\geq 0}$, the multiplier ideal $\mathcal{J}(\mathfrak{b}^\lambda)$ is integrally closed.

REMARK 2.101. For every nonzero ideal $\mathfrak{a}$ on the smooth variety X over a field of characteristic 0, we have

$$\overline{\mathfrak{a}} \subseteq \mathcal{J}(\mathfrak{a}).$$

Indeed, this follows from the inclusion $\mathfrak{a} \subseteq \mathcal{J}(\mathfrak{a})$ and the fact that $\mathcal{J}(\mathfrak{a})$ is integrally closed by assertion iii) in Proposition 2.99.

REMARK 2.102. If $\mathfrak{a}$ is a nonzero ideal on the smooth variety X defined over a ground field of characteristic 0, then

$$\mathcal{J}(\mathfrak{a}^{\lambda}) = \mathcal{J}(\overline{\mathfrak{a}}^{\lambda}) \quad \text{for all} \quad \lambda \in \mathbf{R}_{\geq 0}.$$

Indeed, if $f\colon Y \to X$ is a log resolution of $(X, \mathfrak{a})$ and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, then it follows from assertion i) in Proposition 2.99 that $\overline{\mathfrak{a}} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. Therefore f is a log resolution of $(X, \overline{\mathfrak{a}})$ and

$$\mathcal{J}(\overline{\mathfrak{a}}^{\lambda}) = f_*\mathcal{O}_Y(K_{Y/X} - \lfloor \lambda F \rfloor) = \mathcal{J}(\mathfrak{a}^{\lambda}) \quad \text{for all} \quad \lambda \in \mathbf{R}_{\geq 0}.$$

EXAMPLE 2.103. Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be a monomial ideal. Recall from the proof of Theorem 2.60 that we have an action of the torus $T = (k^*)^n$ on $X = \mathbf{A}^n$ that preserves the ideal $\mathfrak{a}$. This implies that if $f\colon Y \to X$ is the normalization of the blow-up of X along $\mathfrak{a}$, then T acts naturally on Y such that f is T-equivariant. This implies that if we write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, then all prime divisors that appear in F are preserved by the T-action. First, this implies that $\overline{\mathfrak{a}} = f_*\mathcal{O}_Y(-F)$ is preserved by the T-action, hence it is a monomial ideal. Second, using the description of valuations associated to torus-invariant divisors in the proof of Theorem 2.60 and Remark 2.98, we see that a monomial x^u lies in $\overline{\mathfrak{a}}$ if and only if for every $v \in \mathbf{Z}^n_{\geq 0} \smallsetminus \{0\}$, we have

$$\langle u, v\rangle \geq \min_{x^w \in \mathfrak{a}} \langle w, v\rangle = \min_{w \in P(\mathfrak{a})} \langle w, v\rangle.$$

We thus conclude that $x^u \in \overline{\mathfrak{a}}$ if and only if $u \in P(\mathfrak{a})$.

EXAMPLE 2.104. Every nonzero ideal $\mathfrak{a}$ on X that is locally principal is integrally closed: this follows directly from definition.

EXAMPLE 2.105. If $\mathfrak{a} = (x_1^d, \ldots, x_n^d) \subseteq k[x_1, \ldots, x_n]$, then it follows from Example 2.103 that $\overline{\mathfrak{a}} = (x_1, \ldots, x_n)^d$.

PROPOSITION 2.106. *If $\mathfrak{a}$ is a nonzero ideal on the normal variety X, then there is a positive integer m such that*

$$\overline{\mathfrak{a}^i} = \overline{\mathfrak{a}^m} \cdot \mathfrak{a}^{i-m} \quad \text{for all} \quad i \geq m.$$

PROOF. The inclusion "$\supseteq$" holds for every m: indeed we have

$$\overline{\mathfrak{a}^m} \cdot \mathfrak{a}^{i-m} \subseteq \overline{\mathfrak{a}^m} \cdot \overline{\mathfrak{a}^{i-m}} \subseteq \overline{\mathfrak{a}^i},$$

where the second inclusion follows from assertion iv) in Proposition 2.99. The interesting inclusion is the opposite one.

Let $g\colon \widetilde{X} \to X$ be the blow-up along $\mathfrak{a}$ and $h\colon Y \to \widetilde{X}$ the normalization of $\widetilde{X}$. We put $f = g \circ h$. Recall that $\widetilde{X} = \mathcal{P}roj(\mathcal{S})$, where $\mathcal{S} = \bigoplus_{j \geq 0} \mathfrak{a}^j$. We have an effective Cartier divisor F on $\widetilde{X}$ such that

$$\mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-F) \simeq \mathcal{O}_{\widetilde{X}}(1).$$

Let $E = h^*(F)$, so that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$.

By the standard properties of the $\mathcal{P}roj$ construction, the graded $\mathcal{S}$-module associated to the coherent sheaf $h_*(\mathcal{O}_Y)$ on $\widetilde{X}$:

$$\bigoplus_{j\geq 0} g_*\big(h_*(\mathcal{O}_Y)\otimes \mathcal{O}_{\widetilde{X}}(j)\big) \simeq \bigoplus_{j\geq 0} f_*\mathcal{O}_Y(-jE) = \bigoplus_{j\geq 0}\overline{\mathfrak{a}^j}$$

is locally finitely generated. If it is locally generated over $\mathcal{S}$ in degrees $\leq m$, it follows that for every $i\geq m$, we have

$$\overline{\mathfrak{a}^i}\subseteq \mathfrak{a}^{i-m}\cdot\overline{\mathfrak{a}^m}.$$

This completes the proof of the proposition. □

We can now give the connection with the usual description of integral closure of an ideal (which explains its name).

PROPOSITION 2.107. If $X=\operatorname{Spec}(R)$ is a normal affine variety, $\mathfrak{a}\subseteq R$ is a nonzero ideal, and $u\in R$, then the following are equivalent:

i) $u\in\overline{\mathfrak{a}}$.
ii) There is a nonzero ideal $\mathfrak{b}$ in R such that $u\cdot\mathfrak{b}\subseteq\mathfrak{a}\cdot\mathfrak{b}$.
iii) There is a positive integer r and $a_i\in\mathfrak{a}^i$ for $1\leq i\leq r$, such that

$$u^r+\sum_{i=1}^r a_iu^{r-i}=0.$$

iv) For every discrete valuation v of the fraction field of R, with $v(a)\geq 0$ for all $a\in R$, we have $v(u)\geq v(\mathfrak{a})$.
iv') For every divisorial valuation v of X, we have $v(u)\geq v(\mathfrak{a})$.
v) There is a nonzero $c\in R$ such that $cu^i\in\mathfrak{a}^i$ for all $i\geq 1$.

PROOF. If m is as given in Proposition 2.106 and $\mathfrak{b}=\overline{\mathfrak{a}^m}$, then

$$\overline{\mathfrak{a}}\cdot\mathfrak{b}\subseteq\overline{\mathfrak{a}^{m+1}}\subseteq\mathfrak{a}\cdot\mathfrak{b}.$$

This proves the implication i)⇒ii).

The implication ii)⇒iii) follows using the determinant trick: if $\mathfrak{b}$ is generated by $g_1,\ldots,g_r$, then we can write $ug_i=\sum_{j=1}^n a_{i,j}g_j$ for $1\leq i\leq r$, with $a_{i,j}\in\mathfrak{a}$. If $A=(a_{i,j})$, then $\det(uI_r-A)$ annihilates $\mathfrak{b}$, hence it is 0. Expending this determinant, we get the assertion in iii).

Given an equation as in iii) and a valuation v as in iv), the equality

$$u^m=\sum_{i=1}^r(-a_i)u^{r-i}$$

implies that there is i, with $1\leq i\leq r$ such that

$$m\cdot v(u)=v(u^m)\geq v(-a_iu^{r-i})=v(-a_i)+(r-i)\cdot v(u).$$

We thus conclude that $i\cdot v(u)\geq v(-a_i)\geq i\cdot v(\mathfrak{a})$, hence $v(u)\geq v(\mathfrak{a})$. This proves iii)⇒iv). The implication iv)⇒iv') is trivial, while iv')⇒i) follows from the definition of integral closure.

We have thus shown that i), ii), iii), iv), and iv') are equivalent. In order to complete the proof, we will show that i)⇒v)⇒iv). If m is chosen as in Proposition 2.106, then $\overline{\mathfrak{a}}^i\subseteq\overline{\mathfrak{a}^i}\subseteq\mathfrak{a}^{i-m}$ for all $i\geq m$. If we choose a nonzero $c\in\mathfrak{a}^m$, then $c\cdot\overline{\mathfrak{a}^i}\subseteq\mathfrak{a}^i$ for all i. This proves i)⇒v).

Finally, if $cu^i \in \mathfrak{a}^i$ for all i, with $c \neq 0$, and v is a discrete valuation as in iv), then
$$v(c) + i \cdot v(u) = v(cu^i) \geq v(\mathfrak{a}^i) = i \cdot v(\mathfrak{a}) \quad \text{for all} \quad i \geq 1.$$
Dividing by i and letting i go to infinity gives $v(u) \geq v(\mathfrak{a})$. Therefore v)⇒iv), completing the proof of the proposition. □

We end this section with a discussion of *reductions* of ideals. This notion will be important in the context of Skoda's theorem.

DEFINITION 2.108. If $\mathfrak{a}$ is a nonzero ideal sheaf on the normal variety X, then a *reduction* of $\mathfrak{a}$ is an ideal $\mathfrak{b} \subseteq \mathfrak{a}$ such that $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$.

PROPOSITION 2.109. If $\mathfrak{a}$ is a nonzero ideal sheaf on the n-dimensional variety X, then for every $P \in X$ there is an open neighborhood U of P and a reduction of $\mathfrak{a}|_U$ generated by n elements $a_1, \ldots, a_n \in \Gamma(U, \mathfrak{a})$.

PROOF. After possibly replacing X by an affine open neighborhood of P, we may and will assume that X is affine. Let $f\colon Y \to X$ be the normalization of the blow-up of X along $\mathfrak{a}$, and let us write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. If $\mathfrak{a}$ is generated by linearly independent elements $u_1, \ldots, u_r \in \Gamma(X, \mathfrak{a})$ and if $|V|$ is the linear system on X generated by these elements, then $f^*|V| = F + |M|$ for a basepoint-free linear system $|M|$ on Y. Since the fiber $f^{-1}(P)$ has dimension $\leq n-1$, it follows that if $M_1, \ldots, M_n \in |M|$ are general elements, then $f^{-1}(P) \cap M_1 \cap \ldots \cap M_n = \emptyset$. Since f is proper, there is an open neighborhood U of P such that $f^{-1}(U) \cap M_1 \cap \ldots \cap M_n = \emptyset$. If $a_i \in \Gamma(X, \mathfrak{a})$ defines the element of $|V|$ whose inverse image is $F + M_i$ and $\mathfrak{b} = (a_1, \ldots, a_n)$, it follows that $\mathfrak{b} \cdot \mathcal{O}_{f^{-1}(U)} = \mathcal{O}(-F)|_{f^{-1}(U)}$. Therefore $\mathfrak{b}|_U$ is a reduction of $\mathfrak{a}|_U$ generated by n elements. □

2.6.2. Skoda type theorem for multiplier ideals. In this section we assume that the ground field has characteristic 0. The following is Skoda's theorem for multiplier ideals [**EL99**].

THEOREM 2.110. *If $\mathfrak{a}$ is a nonzero ideal on the smooth variety X and $\mathfrak{a}$ is locally generated by q elements, then*
$$\mathcal{J}(\mathfrak{a}^\lambda) = \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{\lambda-1}) \quad \textit{for all} \quad \lambda \geq q.$$

PROOF. After taking a suitable affine open cover, we may assume that X is affine and $\mathfrak{a} = (a_1, \ldots, a_q)$. Let $f\colon Y \to X$ be a log resolution of $(X, \mathfrak{a})$ and let us write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. If $V = k^q$, then we have a surjective morphism
$$V \otimes_k \mathcal{O}_Y \to \mathcal{O}_Y(-F)$$
given by $\big(f^*(a_1), \ldots, f^*(a_n)\big)$. This induces an exact Koszul complex
$$0 \to \wedge^q V \otimes_k \mathcal{O}_Y(qF) \to \ldots \to \wedge^2 V \otimes_k \mathcal{O}_Y(2F) \to V \otimes_k \mathcal{O}_Y(F) \to \mathcal{O}_Y \to 0.$$
By tensoring with $\mathcal{O}_Y\big(K_{Y/X} - \lfloor \lambda F \rfloor\big)$, we obtain an exact complex
$$0 \to \mathcal{E}_q \to \ldots \to \mathcal{E}_1 \to \mathcal{E}_0 \to 0,$$
where
$$\mathcal{E}_j = \wedge^j V \otimes_k \mathcal{O}_Y\big(K_{Y/X} - \lfloor (\lambda - i)F \rfloor\big).$$
Note that it follows from Theorem 2.69 that if $\lambda \geq q$, then
$$R^i f_*(\mathcal{E}_j) = 0 \quad \text{for all} \quad i \geq 1. \tag{2.20}$$

If $\mathcal{F}_j = \ker(\mathcal{E}_j \to \mathcal{E}_{j-1})$ for $1 \leq j \leq q-1$, then it follows by descending induction on j, using the long exact sequence for higher direct images associated to

$$0 \to \mathcal{F}_j \to \mathcal{E}_j \to \mathcal{F}_{j-1} \to 0$$

that $R^i f_*(\mathcal{F}_j) = 0$ for all $i \geq 1$ and $j \geq 1$. In particular, since $R^1 f_*(\mathcal{F}_1) = 0$, we conclude that the induced map

$$V \otimes_k \mathcal{J}(\mathfrak{a}^{\lambda-1}) = V \otimes_k f_*(E_1) \to f_*(\mathcal{E}_0) = \mathcal{J}(\mathfrak{a}^{\lambda})$$

is surjective. Since this map is defined by $(a_1, \ldots, a_q)$, we conclude that

$$\mathcal{J}(\mathfrak{a}^{\lambda}) = \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{\lambda-1}).$$

□

COROLLARY 2.111. If $\mathfrak{a}$ is a nonzero ideal on the n-dimensional smooth variety X, then

$$\mathcal{J}(\mathfrak{a}^{\lambda}) = \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{\lambda-1}) \quad \text{for all} \quad \lambda \geq n.$$

In particular, we have

$$\overline{\mathfrak{a}^n} \subseteq \mathfrak{a}.$$

PROOF. In order to prove the first assertion, we may take a suitable open cover of X. By Proposition 2.109, we may thus assume that $\mathfrak{a}$ has a reduction $\mathfrak{b}$ that is generated by n global sections. Since $\overline{\mathfrak{b}} = \overline{\mathfrak{a}}$, we have $\mathcal{J}(\mathfrak{a}^{\mu}) = \mathcal{J}(\mathfrak{b}^{\mu})$ for all μ (see Remark 2.102), hence Theorem 2.110 gives

$$\mathcal{J}(\mathfrak{a}^{\lambda}) = \mathfrak{b} \cdot \mathcal{J}(\mathfrak{a}^{\lambda-1}) \subseteq \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{\lambda-1}) \quad \text{for all} \quad \lambda \geq n.$$

On the other hand, the inclusion

$$\mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{\lambda-1}) \subseteq \mathcal{J}(\mathfrak{a}^{\lambda})$$

is a general easy fact (see Exercise 2.58), hence we obtain the first assertion in the proposition. The second assertion follows from the first one and the fact that $\overline{\mathfrak{a}^n} \subseteq \mathcal{J}(\mathfrak{a}^n)$ (see Remark 2.101). □

EXERCISE 2.112. Show that, more generally, if $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are nonzero ideals on the smooth n-dimensional variety X, then

$$\mathcal{J}(\mathfrak{a}^{\lambda_1} \cdot \ldots \mathfrak{a}_r^{\lambda_r}) = \mathfrak{a}_1 \cdot \mathcal{J}(\mathfrak{a}_1^{\lambda_1 - 1} \cdots \mathfrak{a}_r^{\lambda_r}) \quad \text{for} \quad \lambda_1 \geq n.$$

EXERCISE 2.113. Let $\mathfrak{a}$ be a nonzero ideal on the normal variety X such that all associated subvarieties of $\mathfrak{a}$ have codimension $\leq r$ in X.

i) Show that around the generic point of each associated subvariety of $\mathfrak{a}$, there is a reduction of $\mathfrak{a}$ generated by r elements.
ii) Deduce that $\mathcal{J}(\mathfrak{a}^r) \subseteq \mathfrak{a}$.

The last assertion in Corollary 2.111 is due to Briançon and Skoda [**SB74**]. It implies an interesting relation between a regular function and its Jacobian ideal that we now discuss.

DEFINITION 2.114. If X is a smooth variety and $f \in \mathcal{O}_X(X)$, then the Jacobian ideal J_f of f is defined as follows: in an open subset U with algebraic coordinates $x_1, \ldots, x_n$, the ideal J_f is generated by $\frac{\partial f}{\partial x_1}, \ldots \frac{\partial f}{\partial x_1}$. It is easy to check that this is independent of the choice of coordinates and thus the local definitions glue to give the coherent ideal J_f.

PROPOSITION 2.115. If f is a nonconstant regular function on the smooth variety X, then $f \in \overline{J_f}$ in a neighborhood of the zero-locus of f.

PROOF. We apply Generic Smoothness for the morphism $f\colon X \to \mathbf{A}^1$ to conclude that after possibly replacing X by an open neighborhood of the zero-locus of f, we may assume that $f^{-1}(t)$ is smooth for all $t \neq 0$. We thus have $V(J_f) \subseteq V(f)$, or equivalently, $f \in \mathrm{rad}(J_f)$. The assertion to prove is local, hence we may assume that $X = \mathrm{Spec}(R)$ is affine and we have algebraic coordinates $x_1, \ldots, x_n$ defined globally on X.

We need to show that for every divisorial valuation ord_E of X with $c_X(\mathrm{ord}_E) \subseteq V(J_f)$, we have $\mathrm{ord}_E(f) \geq q := \mathrm{ord}_E(J_f)$. Suppose that E is a prime divisor on the normal variety Y, that has a birational morphism $\pi\colon Y \to X$. Let $Q \in E$ be a point where both Y and E are smooth and $P = \pi(Q)$. We choose a regular system of parameters $x_1, \ldots, x_n$ of $\mathcal{O}_{X,P}$ and a regular system of parameters $y_1, \ldots, y_n$ of $\mathcal{O}_{Y,Q}$ such that E is defined at Q by (y_1). Let

$$\varphi\colon \widehat{\mathcal{O}_{X,P}} = k[\![x_1, \ldots, x_n]\!] \to \widehat{\mathcal{O}_{Y,Q}} = k[\![y_1, \ldots, y_n]\!]$$

be the homomorphism induced by π. If $u_i = \varphi(x_i)$ for $1 \leq i \leq n$, then we know that

$$\frac{\partial f}{\partial x_i}(u_1, \ldots, u_n) \in (y_1^q) \quad \text{for all} \quad 1 \leq i \leq n.$$

Note that if $g = f(u_1, \ldots, u_n) = \varphi(f)$, then we have

$$\frac{\partial g}{\partial y_1} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(u_1, \ldots, u_n) \cdot \frac{\partial u_i}{\partial y_1} \in (y_1^q). \tag{2.21}$$

Since by assumption f vanishes on $\pi(E)$, we know that $g \in (y_1)$, and we thus conclude[6] using (2.21) that $g \in (y_1^{q+1})$. This completes the proof of the proposition. □

COROLLARY 2.116. If f is a nonconstant regular function on the smooth n-dimensional variety X, then $f^n \in J_f$ in a neighborhood of the zero-locus of f.

PROOF. The statement follows by combining the last assertion in Corollary 2.111 with Proposition 2.115. □

2.7. The Summation Theorem

In this section we give a description of the multiplier ideals of a sum of ideals. More generally, we prove the following:

THEOREM 2.117. *Let $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ be nonzero ideals on the smooth variety X. For every $\lambda, \gamma \in \mathbf{R}_{\geq 0}$, we have*

$$\mathcal{J}\big((\mathfrak{a}+\mathfrak{b})^{\lambda}\mathfrak{c}^{\gamma}\big) = \sum_{\alpha+\beta=\gamma} \mathcal{J}(\mathfrak{a}^{\alpha}\mathfrak{b}^{\beta}\mathfrak{c}^{\gamma}). \tag{2.22}$$

REMARK 2.118. Since the function $\lfloor - \rfloor$ only takes finitely many values on bounded intervals, it follows from the definition of mixed multiplier ideals that in the sum in (2.22) we have only finitely many distinct terms.

[6] Note that we use here that the ground field has characteristic 0.

REMARK 2.119. By combining the above result with the Subadditivity Theorem, we deduce that

$$\mathcal{J}\big((\mathfrak{a}+\mathfrak{b})^{\lambda}\big) \subseteq \sum_{\alpha+\beta=\lambda} \mathcal{J}(\mathfrak{a}^{\alpha}) \cdot \mathcal{J}(\mathfrak{b}^{\beta}).$$

This weaker version was proved in [**Mus02**], while the above stronger version was first proved in [**Tak06**] using positive characteristic methods and then it was reproved using the Relative Vanishing Theorem in [**JoMi08**]. Here we follow the latter approach.

COROLLARY 2.120. If $\mathfrak{a}$ and $\mathfrak{b}$ are nonzero ideals on the smooth variety X, then for every $P \in X$, we have

$$\mathrm{lct}_P(\mathfrak{a}+\mathfrak{b}) \leq \mathrm{lct}_P(\mathfrak{a}) + \mathrm{lct}_P(\mathfrak{b}).$$

PROOF. It follows from Theorem 2.117 that $\mathcal{J}\big((\mathfrak{a}+\mathfrak{b})^{\lambda}\big)_P = \mathcal{O}_{X,P}$ if and only if there are $\alpha, \beta \in \mathbf{R}_{\geq 0}$ with $\alpha+\beta=\lambda$ such that $\mathcal{J}(\mathfrak{a}^{\alpha}\mathfrak{b}^{\beta}) = \mathcal{O}_{X,P}$. If this is the case, since we have the inclusions

$$\mathcal{J}(\mathfrak{a}^{\alpha}\mathfrak{b}^{\beta}) \subseteq \mathcal{J}(\mathfrak{a}^{\alpha}) \quad \text{and} \quad \mathcal{J}(\mathfrak{a}^{\alpha}\mathfrak{b}^{\beta}) \subseteq \mathcal{J}(\mathfrak{b}^{\beta}),$$

it follows that $\alpha < \mathrm{lct}_P(\mathfrak{a})$ and $\beta < \mathrm{lct}(\mathfrak{b})$. Therefore we have $\lambda = \alpha+\beta < \mathrm{lct}_P(\mathfrak{a}) + \mathrm{lct}_P(\mathfrak{b})$. This gives the inequality in the corollary. □

COROLLARY 2.121. If P is a point on the smooth n-dimensional variety X defined by the ideal $\mathfrak{m}_P$ and if $\mathfrak{a}$ and $\mathfrak{b}$ are nonzero ideals on X such that $\mathfrak{a}+\mathfrak{m}_P^d = \mathfrak{b}+\mathfrak{m}_P^d$ for some positive integer d, then

$$|\,\mathrm{lct}_P(\mathfrak{a}) - \mathrm{lct}_P(\mathfrak{b})| \leq \frac{n}{d}.$$

PROOF. By assumption, we have $\mathfrak{a} \subseteq \mathfrak{b}+\mathfrak{m}_P^d$, hence Corollary 2.120 gives

$$\mathrm{lct}_P(\mathfrak{a}) \leq \mathrm{lct}_P(\mathfrak{b}+\mathfrak{m}_P^d) \leq \mathrm{lct}_P(\mathfrak{b}) + \mathrm{lct}_P(\mathfrak{m}_P^d).$$

Since $\mathrm{lct}_P(\mathfrak{m}_P^d) = \frac{n}{d}$ by Remark 2.43 and Example 2.50, we obtain

$$\mathrm{lct}_P(\mathfrak{a}) - \mathrm{lct}_P(\mathfrak{b}) \leq \frac{n}{d}$$

and the inequality

$$\mathrm{lct}_P(\mathfrak{b}) - \mathrm{lct}_P(\mathfrak{a}) \leq \frac{n}{d}$$

follows by symmetry. □

PROOF OF THEOREM 2.117. Let $f\colon Y \to X$ be a log resolution of the ideal $\mathfrak{a}\cdot\mathfrak{b}\cdot\mathfrak{c}\cdot(\mathfrak{a}+\mathfrak{b})$. Let us write

$$\mathfrak{a}\cdot\mathcal{O}_Y = \mathcal{O}_Y(-A), \quad \mathfrak{b}\cdot\mathcal{O}_Y = \mathcal{O}_Y(-B), \quad \mathfrak{c}\cdot\mathcal{O}_Y = \mathcal{O}_Y(-C).$$

By assumption, the ideal

$$(\mathfrak{a}+\mathfrak{b})\cdot\mathcal{O}_Y = \mathcal{O}_Y(-A) + \mathcal{O}_Y(-B)$$

is locally principal. In this case, for every $P \in Y$, there is an open neighborhood U of P such that either $A|_U \geq B|_U$ or $A|_U \leq B|_U$. Clearly, if $F = \min\{A, B\}$, then $(\mathfrak{a}+\mathfrak{b})\cdot\mathcal{O}_Y = \mathcal{O}_Y(-F)$.

For every $\alpha \in [0,\lambda]$, let us write $D(\alpha) = \lfloor \alpha A + (\lambda-\alpha)B + \gamma C \rfloor$. By the properties of the round-down function, we can choose

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r = \lambda$$

such that the divisor $D(\alpha)$ is independent of α for every $\alpha \in (\alpha_{i-1}, \alpha_i)$ and all $1 \leq i \leq r$. In fact, after possibly adding intermediate points in each interval (α_{i-1}, α_i), we may and will assume that for every $1 \leq i \leq r$, the divisor $D(\alpha)$ is constant either on the interval $(\alpha_{i-1}, \alpha_i]$ or on the interval $[\alpha_{i-1}, \alpha_i)$. In this case, if we put $D_i = D(\alpha_i)$ and $E_i = \max\{D_i, D_{i-1}\}$, then it follows from the elementary Lemma 2.122 below that for every i with $1 \leq i \leq r$, we have

$$E_i = D_i \quad \text{or} \quad E_i = D_{i-1}.$$

For every i with $0 \leq i \leq r$, we have

$$\alpha_i A + (\lambda - \alpha_i)B + \gamma C \geq \lambda F + \gamma C,$$

hence we have inclusions

$$u_i \colon \mathcal{O}_Y(-D_i) \hookrightarrow \mathcal{O}_Y\big(-\lfloor \lambda F + \gamma C \rfloor\big).$$

Similarly, for every i with $1 \leq i \leq r$, we have inclusions

$$v_i' \colon \mathcal{O}_Y(-E_i) \hookrightarrow \mathcal{O}_Y(-D_i) \quad \text{and} \quad v_i'' \colon \mathcal{O}_Y(-E_i) \hookrightarrow \mathcal{O}_Y(-D_{i-1}).$$

We claim that we have a short exact sequence

$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}_Y(-E_i) \xrightarrow{v} \bigoplus_{i=0}^{r} \mathcal{O}_Y(-D_i) \xrightarrow{u} \mathcal{O}_Y\big(-\lfloor \lambda F + \gamma C \rfloor\big) \to 0, \tag{2.23}$$

where

$$v(x_1, \ldots, x_r) = \big(v_1''(x_1), v_2''(x_2) - v_1'(x_1), \ldots, v_r''(x_r) - v_{r-1}'(x_{r-1}), -v_r'(x_r)\big)$$

$$\text{and} \quad u(y_0, \ldots, y_n) = \sum_{i=0}^{r} u_i(y_i).$$

It is clear that this is a complex. In order to check that it is a short exact sequence, we can argue locally and we may thus assume that we are in an open subset U on which $A \leq B$ or $B \leq A$. Without any loss of generality, let us suppose that $A \leq B$, in which case $D_0 \leq D_1 \leq \ldots \leq D_r$. Therefore in U the exact sequence is given by

$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}_Y(-D_i) \to \bigoplus_{i=0}^{r} \mathcal{O}_Y(-D_i) \to \mathcal{O}_Y(-D_0) \to 0. \tag{2.24}$$

If we define

$$\mathcal{O}_Y(-D_0) \to \mathcal{O}_Y(-D_0) \oplus \mathcal{O}_Y(-D_1) \oplus \ldots \oplus \mathcal{O}_Y(-D_r), \quad \text{given by} \quad x \to (x, 0, \ldots, 0) \quad \text{and}$$

$$\mathcal{O}_Y(-D_0) \oplus \mathcal{O}_Y(-D_1) \oplus \ldots \oplus \mathcal{O}_Y(-D_r) \to \mathcal{O}_Y(-D_1) \oplus \ldots \oplus \mathcal{O}_Y(-D_r), \quad \text{given by}$$

$$(y_0, \ldots, y_r) \to (-y_1 - \ldots - y_r, \ldots, -y_{r-1} - y_r, -y_r),$$

it is straightforward to check that we get a homotopy on (2.24) between the identity and the zero map. This proves the exactness of (2.23).

Recall that since E_i is equal to either D_i or D_{i-1}, we may apply Theorem 2.69 to conclude that

$$R^q f_* \mathcal{O}_Y(K_{Y/X} - E_i) = 0 \quad \text{for all} \quad q \geq 1,\ 1 \leq i \leq r.$$

After tensoring the exact sequence (2.23) with $\mathcal{O}_Y(K_{Y/X})$ and taking the long exact sequence for higher direct images, we obtain an exact sequence

$$0 \to \bigoplus_{i=1}^{r} f_* \mathcal{O}_Y(K_{Y/X} - E_i) \to \bigoplus_{i=0}^{r} \mathcal{J}(\mathfrak{a}^{\alpha_i} \mathfrak{b}^{\lambda - \alpha_i} \mathfrak{c}^{\gamma}) \to \mathcal{J}\big((\mathfrak{a} + \mathfrak{b})^{\lambda} \mathfrak{c}^{\gamma}\big) \to 0.$$

This proves that

$$\mathcal{J}\big((\mathfrak{a}+\mathfrak{b})^{\lambda}\mathfrak{c}^{\gamma}\big)=\sum_{i=0}^{r}\mathcal{J}(\mathfrak{a}^{\alpha_i}\mathfrak{b}^{\lambda-\alpha_i}\mathfrak{c}^{\gamma}).$$

Since given $\alpha\in[0,\lambda]$, the α_i can be chosen to include α as one of them, we obtain the assertion in the theorem. $\square$

LEMMA 2.122. *Let $\varphi\colon \mathbf{R}\to\mathbf{Z}$ be given by $\varphi(x)=\lfloor ax+b\rfloor$ for some $a,b\in\mathbf{R}$.*

i) *If φ is constant on an interval $[c,d)$, then $\varphi(c)\leq\varphi(d)$.*
ii) *If φ is constant on an interval $(c,d]$, then $\varphi(d)\leq\varphi(c)$.*

PROOF. We only prove the assertion in i), the proof of ii) being similar. If $a\geq 0$, then φ is nondecreasing, hence we clearly have $\varphi(c)\leq\varphi(d)$. Suppose now that $a<0$. In this case, the behavior of the round-down function implies that if $0<\epsilon\ll 1$, then $\varphi(d)=\varphi(d-\epsilon)$. Since φ is constant on $[c,d)$, it follows that $\varphi(c)=\varphi(d)$. $\square$

EXERCISE 2.123. Show that for every nonzero ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_r,\mathfrak{b}_1,\ldots,b_s$ on the smooth variety X, and every $\lambda,\mu_1,\ldots,\mu_s\in\mathbf{R}_{\geq 0}$, we have

$$\mathcal{J}\big((\mathfrak{a}_1+\ldots+\mathfrak{a}_r)^{\lambda}\mathfrak{b}_1^{\mu_1}\ldots\mathfrak{b}_s^{\mu_s}\big)=\sum_{\alpha_1+\ldots+\alpha_r=\lambda}\mathcal{J}(\mathfrak{a}_1^{\alpha_1}\ldots\mathfrak{a}_r^{\alpha_r}\mathfrak{b}_1^{\mu_1}\ldots\mathfrak{b}_s^{\mu_s}).$$

EXERCISE 2.124. Let P be a point on the smooth n-dimensional variety X defined by the ideal $\mathfrak{m}_P$. Show that if $\mathfrak{a}$ is a nonzero ideal on X with $\mathrm{ord}_P(\mathfrak{a})=d\geq 1$, then for every $r>d$, we have

$$\mathrm{lct}_P(\mathfrak{a}+\mathfrak{m}_P^r)\leq\frac{n+\mathrm{lct}_P(\mathfrak{a})\cdot(r-d)}{r}.$$

EXERCISE 2.125. Let $f\in\mathbf{C}[x_1,\ldots,x_n]$ and $g\in\mathbf{C}[y_1,\ldots,y_m]$ be nonzero, with $f(0)=0=g(0)$. Show that if $h(x,y)=f(x)+g(y)$, then we have the following Thom-Sebastiani formula:

$$\mathrm{lct}_0(h)=\min\{\mathrm{lct}_0(f)+\mathrm{lct}_0(g),1\}.$$

2.8. Multiplier ideals of graded sequences of ideals

Some of the most interesting applications of multiplier ideals involve an asymptotic construction associated to suitable sequences of ideals. This construction first appeared in the analytic setting in [**Siu98**] and then it was formalized in the algebraic framework that we discuss here in [**ELS01**]. We only give a brief introduction to this topic and present the application to symbolic powers from [**ELS01**].

DEFINITION 2.126. Let X be an arbitrary Noetherian scheme. A *graded sequence of ideals* $\mathfrak{a}_\bullet=(\mathfrak{a}_p)_{p\geq 0}$ on X is a sequence of ideals in $\mathcal{O}_X$ that satisfy the following properties:

i) $\mathfrak{a}_0=\mathcal{O}_X$.
ii) There is $p>0$ with $\mathfrak{a}_p\neq 0$.
iii) For every $p,q\in\mathbf{Z}_{\geq 0}$, we have $\mathfrak{a}_p\cdot\mathfrak{a}_q\subseteq\mathfrak{a}_{p+q}$.

EXAMPLE 2.127. If $\mathfrak{a}$ is a nonzero ideal in $\mathcal{O}_X$, then $(\mathfrak{a}^m)_{m\geq 0}$ is a graded sequence of ideals.

EXAMPLE 2.128. If $v=\mathrm{ord}_E$ is a divisorial valuation on the variety X, then $\mathfrak{a}_\bullet(v)=\big(\mathfrak{a}_m(v)\big)_{m\geq 0}$ is a graded sequence of ideals.

EXAMPLE 2.129. Let $\mathcal{L}$ be a line bundle on the projective variety X such that $H^0(X, \mathcal{L}^m) \neq 0$ for some $m \geq 1$. For every $p \geq 0$, let $\mathfrak{a}_p$ be the ideal defining the base locus of $|\mathcal{L}^p|$, that is, the image of the evaluation map

$$H^0(X, \mathcal{L}^p) \otimes \mathcal{O}_X \to \mathcal{L}^p$$

is $\mathfrak{a}_p \otimes \mathcal{L}^p$. Since for every p, $q \geq 0$, we have the multiplication map

$$H^0(X, \mathcal{L}^p) \otimes H^0(X, \mathcal{L}^q) \to H^0(X, \mathcal{L}^{p+q}),$$

we deduce that $\mathfrak{a}_p \cdot \mathfrak{a}_q \subseteq \mathfrak{a}_{p+q}$. Therefore $\mathfrak{a}_\bullet$ is a graded sequence of ideals. These graded sequences play an important role in birational geometry.

EXAMPLE 2.130. Recall that if $\mathfrak{q}$ is a prime ideal in a Noetherian ring R, then the m-th *symbolic power* of $\mathfrak{q}$ is

$$\mathfrak{q}^{(m)} = \mathfrak{q}^m R_{\mathfrak{q}} \cap R.$$

For $m \geq 1$ this is clearly a $\mathfrak{q}$-primary ideal. It is then clear that $\left(\mathfrak{q}^{(m)}\right)_{m \geq 0}$ is a graded sequence of ideals in R.

Note that if $X = \operatorname{Spec}(R)$ is a variety and the zero-locus of $\mathfrak{q}$ meets the smooth locus of X, then this exact sequence is equal to $\mathfrak{a}_\bullet(\operatorname{ord}_Z)$, hence this is a special case of the Example 2.128. If both X and Z are smooth, then it follows from Lemma 2.14 and Remark 2.15 that $\mathfrak{q}^{(m)} = \mathfrak{q}^m$ for all $m \geq 0$. However, when Z is singular, the behavior of symbolic powers is much more subtle.

REMARK 2.131. If $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X, then $\bigoplus_{p \geq 0} \mathfrak{a}_p$ is a quasi-coherent $\mathcal{O}_X$-algebra. If this is locally a finitely generated $\mathcal{O}_X$-algebra, then it is an elementary fact that there is $d > 0$ such that $\mathfrak{a}_{dp} = \mathfrak{a}_d^p$ for all $p \geq 1$. This is a "trivial" situation: in general, one is interested in the behavior of graded systems when this algebra is not finitely generated (or, at least, when this is not known *a priori*). The notion that we will introduce shortly provides a tool for handling such graded sequences.

From now on we assume that we work on a smooth variety X over an algebraically closed field of characteristic 0.

PROPOSITION 2.132. Let $\mathfrak{a}_\bullet$ be a graded sequence of ideals on X and let $\lambda \in \mathbf{R}_{\geq 0}$.

i) If p is a positive integer such that $\mathfrak{a}_p \neq 0$, then for every positive integer q, we have
$$\mathcal{J}(\mathfrak{a}_p^{\lambda/p}) \subseteq \mathcal{J}(\mathfrak{a}_{pq}^{\lambda/pq}).$$

ii) There is an ideal J such that $\mathcal{J}(\mathfrak{a}_p^{\lambda/p}) \subseteq J$ for all p with $\mathfrak{a}_p \neq 0$, with equality if p is divisible enough.

DEFINITION 2.133. The ideal J in part ii) of the above proposition is the *asymptotic multiplier ideal* $\mathcal{J}(\mathfrak{a}_\bullet^\lambda)$.

PROOF OF PROPOSITION 2.132. Since $\mathfrak{a}_p^q \subseteq \mathfrak{a}_{pq}$, we have

$$\mathcal{J}(\mathfrak{a}_p^\lambda) = \mathcal{J}\left((\mathfrak{a}_p^q)^{\lambda/pq}\right) \subseteq \mathcal{J}(\mathfrak{a}_{pq}^{\lambda/pq}),$$

giving the assertion in i).

Let's consider now the set of ideals $\{\mathcal{J}(\mathfrak{a}_p^{\lambda/p}) \mid \mathfrak{a}_p \neq 0\}$. Since X is Noetherian, we can choose an element $J = \mathcal{J}(\mathfrak{a}_p^{\lambda/p})$ of this set that is not properly contained in any ideal in this set. If q is such that $\mathfrak{a}_q \neq 0$, we deduce from i) that

$$J = \mathcal{J}(\mathfrak{a}_p^{\lambda/p}) \subseteq \mathcal{J}(\mathfrak{a}_{pq}^{\lambda/pq}) \quad \text{and} \quad \mathcal{J}(\mathfrak{a}_q^{\lambda/q}) \subseteq \mathcal{J}(\mathfrak{a}_{pq}^{\lambda/pq}).$$

By our choice of J, the first inclusion is an equality, hence the second inclusion gives $\mathcal{J}(\mathfrak{a}_q^{\lambda/q}) \subseteq J$. This completes the proof of the proposition. $\square$

EXAMPLE 2.134. If $\mathfrak{b}$ is a nonzero ideal on X and $\mathfrak{a}_\bullet$ is the graded sequence given by $\mathfrak{a}_m = \mathfrak{b}^m$ for all $m \in \mathbf{Z}_{\geq 0}$, then for every $\lambda \in \mathbf{R}_{\geq 0}$ we have $\mathcal{J}(\mathfrak{a}_m^{\lambda/m}) = \mathcal{J}(\mathfrak{b}^\lambda)$ for all positive integers m, hence $\mathcal{J}(\mathfrak{a}_\bullet^\lambda) = \mathcal{J}(\mathfrak{b}^\lambda)$.

All the properties that we discussed for multiplier ideals have analogues for asymptotic multiplier ideals of graded sequences. The proofs reduce immediately to the case of usual multiplier ideals. We refer to [**Laz04**, Chapter 11] for a detailed discussion of asymptotic multiplier ideals and their application to birational geometry; see also [**JoMu12**] for a discussion of invariants such as log canonical thresholds or orders of vanishing associated to graded sequences.

We only give here a sample application to the comparison between symbolic powers and regular powers, due to Ein, Lazarsfeld, and Smith [**ELS01**]. We begin with the following version of the Subadditivity Theorem for asymptotic multiplier ideals.

PROPOSITION 2.135. If $\mathfrak{a}_\bullet$ is a graded sequence of ideals on the smooth variety X, then for every $\lambda \in \mathbf{R}_{\geq 0}$ and every positive integer q, we have

$$\mathcal{J}(\mathfrak{a}_\bullet^{q\lambda}) \subseteq \mathcal{J}(\mathfrak{a}_\bullet^\lambda)^q.$$

PROOF. Let m be a positive integer such that

$$\mathcal{J}(\mathfrak{a}_\bullet^{q\lambda}) = \mathcal{J}(\mathfrak{a}_m^{q\lambda/m}).$$

Note now that

$$\mathcal{J}(\mathfrak{a}_m^{q\lambda/m}) \subseteq \mathcal{J}(\mathfrak{a}_m^{\lambda/m})^q \subseteq \mathcal{J}(\mathfrak{a}_\bullet^\lambda)^q,$$

where the first inclusion follows from Theorem 2.91 and the second one from the definition of $\mathcal{J}(\mathfrak{a}_\bullet^\lambda)$. We thus obtain the inclusion in the statement of the proposition. $\square$

We can now give the following application to symbolic powers.

THEOREM 2.136. *Let R be the coordinate ring of an n-dimensional smooth affine variety over an algebraically closed field of characteristic* 0. *If $\mathfrak{q}$ is a prime ideal in R of codimension r, then*

$$\mathfrak{q}^{(jr)} \subseteq \mathfrak{q}^j \quad \textit{for all} \quad j \geq 1.$$

PROOF. Let $\mathfrak{a}_\bullet$ be the graded sequence given by $\mathfrak{a}_p = \mathfrak{q}^{(p)}$ for all $p \geq 0$. Note that by the definition of asymptotic multiplier ideals, Exercise 2.58, and Proposition 2.135, for every q we have

$$\mathfrak{q}^{(jr)} \subseteq \mathcal{J}(\mathfrak{a}_{jr}) \subseteq \mathcal{J}(\mathfrak{a}_\bullet^{jr}) \subseteq \mathcal{J}(\mathfrak{a}_\bullet^r)^j.$$

To obtain the assertion in the theorem, it is thus enough to show that

$$\mathcal{J}(\mathfrak{a}_\bullet^r) \subseteq \mathfrak{q}. \tag{2.25}$$

Since $\mathfrak{q}$ is a prime ideal, in order to check (2.25) we may replace R by a localization R_f, with $f \not\in \mathfrak{q}$. We may thus assume that $\mathfrak{q}$ defines a smooth subvariety, in which case $\mathfrak{a}_m = \mathfrak{q}^m$ for all m, and thus using Examples 2.134 and 2.50 we get

$$\mathcal{J}(\mathfrak{a}_\bullet^r) = \mathcal{J}(\mathfrak{q}^r) = \mathfrak{q}.$$

This completes the proof of the theorem. $\square$

CHAPTER 3

Test ideals and F-pure thresholds

In this chapter we discuss analogues of multiplier ideals and log canonical thresholds in positive characteristic, whose definition makes use of the Frobenius morphism. The most interesting aspect of the story is the connection between the characteristic 0 and the positive characteristic invariants via reduction mod p.

Test ideals in rings of positive characteristic first arose in the work of Hochster and Huneke [**HH90**] on tight closure. A version for pairs was then introduced and studied by Hara, Takagi, and Yoshida in [**Tak04**], [**HY03**], and [**HT04**]. Like in the previous chapter, we here focus on pairs $(X, \mathfrak{a})$, in which X is a smooth and $\mathfrak{a}$ is an ideal on X, in which case we follow the simpler description of test ideals from [**BMS08**].

3.1. Frobenius powers in regular rings of positive characteristic

We begin by introducing some terminology. All rings are assumed to be commutative, with unit. If R is a ring of characteristic $p > 0$, we denote by $F = F_R \colon R \to R$ the *Frobenius homomorphism* given by $F(u) = u^p$. We write $F^e_* R$ for R, viewed as a left R-module via F^e_R.

Recall that a scheme X has characteristic p if $\mathcal{O}_X(U)$ has characteristic p for every open subset U of X. In this case we denote by $F = F_X \colon X \to X$ the (absolute) Frobenius morphism, given by the identity on the underlying topological space, and by $u \to u^p$ on sections of $\mathcal{O}_X(U)$. In this section all considerations will be of a local nature, so we will almost always consider affine schemes.

3.1.1. F-finite rings and schemes.

Definition 3.1. We say that a ring R of positive characteristic is F-finite if the Frobenius homomorphism $F \colon R \to R$ is finite (that is, if $F^e_* R$ is a finitely generated R-module, or equivalently, R is a finitely generated module over its subring R^p). More generally, a scheme X of characteristic $p > 0$ is *F-finite* if the Frobenius morphism $F \colon X \to X$ is a finite morphism.

Proposition 3.2. Let R be a ring of characteristic $p > 0$.

i) If R is F-finite, then every R-algebra of finite type is F-finite. In particular, an algebra of finite type over a perfect field is F-finite.
ii) If R is F-finite, then the formal power series $R[\![x_1, \ldots, x_n]\!]$ is F-finite.
iii) If R is F-finite, then every localization of R is F-finite.
iv) If k is a field and A is a k-algebra essentially of finite type which is F-finite, then k is F-finite.

Proof. Suppose that R is generated as a module over R^p by $b_1, \ldots, b_r$. If $A = R[a_1, \ldots, a_n]$, then A is generated over A^p by

$$\{b_k a_1^{j_1} \cdots a_n^{j_n} \mid 1 \le k \le r, 0 \le j_1, \ldots, j_n \le p-1\}.$$

This gives the assertion in i). Similarly, $S = R[\![x_1, \ldots, x_n]\!]$ is generated as a module over S^p by

$$\{b_k x_1^{j_1} \cdots x_n^{j_n} \mid 1 \leq k \leq r, 0 \leq j_1, \ldots, j_n \leq p-1\},$$

giving the assertion in ii). If S is a multiplicative system in R, then every element of $S^{-1}R$ can be written as $\frac{a}{s} = \frac{s^{p-1}a}{s^p}$, hence $S^{-1}R$ is generated as a module over $(S^{-1}R)^p$ by $\frac{b_1}{1}, \ldots, \frac{b_r}{1}$; hence we obtain iii).

If A is as in iv) and $K = A/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal in A, then $[K : k] < \infty$ by Nullstellensatz, hence $[K^p : k^p] < \infty$. Since A is F-finite, it follows from i) that $[K : K^p] < \infty$. Therefore $[K : k^p] < \infty$, and thus also $[k : k^p] < \infty$, so k is F-finite. $\square$

Example 3.3. It follows from Proposition 3.2 that if X is a scheme essentially of finite type over an F-finite field of positive characteristic (for example, over a perfect field), then X is F-finite. Moreover, all local rings and residue fields of an F-finite scheme are F-finite.

We will almost exclusively deal with the case when R is a regular ring. Globally, we will work with regular schemes (recall that a scheme is regular if it is Noetherian and all local rings are regular).

Lemma 3.4. *If X is a regular scheme of characteristic $p > 0$, then the Frobenius morphism $F \colon X \to X$ is flat.*

Proof. It is enough to show that if R is a local ring of X, then the Frobenius homomorphism $F_R \colon R \to R$ is flat. This follows directly from the fact that if $\varphi \colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local ring homomorphism, with R regular and S Cohen-Macaulay such that $\dim(S/\mathfrak{m}S) = \dim(S) - \dim(R)$, then φ is flat (see [**Mat89**, Theorem 23.1]). $\square$

Remark 3.5. It is result of Kunz [**Ku69**] that the converse holds: if R is a reduced local Noetherian ring of positive characteristic on which the Frobenius homomorphism is flat, then R is regular. However, we will not need this result.

Exercise 3.6. Show that if R is an F-finite regular local ring of positive characteristic, then its completion $\widehat{R}$ is F-finite.

Definition 3.7. If X is a scheme of characteristic $p > 0$ and $\mathfrak{a}$ is a coherent ideal in $\mathcal{O}_X$, for every $e \geq 1$, the e-th Frobenius power $\mathfrak{a}^{[p^e]}$ is the inverse image ideal by the e-th iterate F_X^e of the Frobenius morphism. Therefore, if $\mathfrak{a}$ is generated in some affine open subset U by $f_1, \ldots, f_r \in \mathcal{O}_X(U)$, then $\mathfrak{a}^{[p^e]}$ is generated in U by $f_1^{p^e}, \ldots, f_r^{p^e}$.

Remark 3.8. Note that if X is a regular scheme, since F_X is flat, we have a canonical isomorphism $(F_X^e)^*(\mathfrak{a}) \simeq \mathfrak{a}^{[p^e]}$.

For future reference, we record the following proposition. Recall that if $\mathfrak{a}$ is an ideal in R and $c \in R$, then

$$(\mathfrak{a} : c) = \{u \in R \mid uc \in \mathfrak{a}\}.$$

Proposition 3.9. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in the ring R of characteristic $p > 0$ and $q = p^e$, with $e \geq 1$, then the following hold:

i) $(\mathfrak{a} + \mathfrak{b})^{[q]} = \mathfrak{a}^{[q]} + \mathfrak{b}^{[q]}$.

ii) $(\mathfrak{a} \cdot \mathfrak{b})^{[q]} = \mathfrak{a}^{[q]} \cdot \mathfrak{b}^{[q]}$.

iii) If R is F-finite and regular and $(\mathfrak{a}_i)_{i \in I}$ is a family of ideals in R, then

$$\left(\bigcap_{i \in I} \mathfrak{a}_i \right)^{[q]} = \bigcap_{i \in I} \mathfrak{a}_i^{[q]}.$$

iv) If R is regular and $c \in R$, then

$$(\mathfrak{a} : c)^{[q]} = (\mathfrak{a}^{[q]} : c^q).$$

v) If R is regular, then $\mathfrak{b} \subseteq \mathfrak{a}$ if and only if $\mathfrak{b}^{[q]} \subseteq \mathfrak{a}^{[q]}$.

Proof. The assertions in i) and ii) are clear. For the assertion in iii), note that since R is F-finite and regular, it is a finitely generated projective module via F_R^e. The assertion thus follows by noticing that more generally, if M is a finitely generated projective module over a ring R and $(\mathfrak{a}_i)_{i \in I}$ is a family of ideals in R, then

$$\left(\bigcap_{i \in I} \mathfrak{a}_i \right) M = \bigcap_{i \in I} \mathfrak{a}_i M. \tag{3.1}$$

This is clear if M is a finitely generated free R-module. Moreover, if the assertion holds for $M_1 \oplus M_2$, then it holds for both M_1 and M_2. Since M is a direct summand of a finitely generated free R-module, we obtain the equality in (3.1).

We next prove iv). Since R is regular, the homomorphism F_R^e is flat, hence tensoring with R via F_R^e the exact sequence

$$0 \longrightarrow (\mathfrak{a} \colon c) \longrightarrow R \xrightarrow{\cdot c} R/\mathfrak{a}$$

gives the exact sequence

$$0 \longrightarrow (\mathfrak{a} \colon c)^{[q]} \longrightarrow R \xrightarrow{\cdot c^q} R/\mathfrak{a}^{[q]}.$$

This gives the assertion in iv).

In order to prove v), note first that if $\mathfrak{b} \subseteq \mathfrak{a}$, then we clearly have $\mathfrak{b}^{[q]} \subseteq \mathfrak{a}^{[q]}$, hence we only need to prove the reverse implication. Suppose that $\mathfrak{b}^{[q]} \subseteq \mathfrak{a}^{[q]}$ and let $u \in \mathfrak{b}$. The hypothesis gives $u^q \in \mathfrak{a}^{[q]}$, hence $1 \in (\mathfrak{a}^{[q]} : u^q) = (\mathfrak{a} : u)^{[q]}$, where the equality follows from iv). Since $(\mathfrak{a} : u)^{[q]} \subseteq (\mathfrak{a} : u)$, we conclude that $1 \in (\mathfrak{a} : u)$, hence $u \in \mathfrak{a}$. This proves that $\mathfrak{b} \subseteq \mathfrak{a}$. $\square$

We end this section with another useful property of the Frobenius homomorphism in regular rings.

Proposition 3.10. If R is a regular ring of characteristic $p > 0$, $\mathfrak{a}$ is an ideal in R, and $u \in R$ is such that there is a non-zero divisor c with $cu^{[p^e]} \in \mathfrak{a}^{[p^e]}$ for all $e \gg 0$, then $u \in \mathfrak{a}$.

In the terminology of [**HH90**], the hypothesis on u says that it lies in the *tight closure* of the ideal $\mathfrak{a}$. The assertion in the above proposition then says that every ideal in a regular F-finite ring is tightly closed.

Proof of Proposition 3.10. It is enough to show that $u \in \mathfrak{a}$ after localizing at every prime ideal. Note that c remains nonzero after localization, hence we may and will assume that R is local, with maximal ideal $\mathfrak{m}$. The hypothesis says that

$$c \in (\mathfrak{a}^{[p^e]} : u^{[p^e]}) = (\mathfrak{a} : u)^{[p^e]} \quad \text{for} \quad e \gg 0,$$

where the equality follows from Proposition 3.9iv). If $(\mathfrak{a} : u) \subseteq \mathfrak{m}$, then we conclude that $c \in \bigcap_{e>0} \mathfrak{m}^{[p^e]}$, hence $c = 0$ by Krull's Intersection Theorem, a contradiction. Therefore $(\mathfrak{a} : u) = R$, hence $u \in \mathfrak{a}$. □

3.1.2. Inverse Frobenius powers. In this section we fix an F-finite regular ring R of characteristic $p > 0$.

DEFINITION 3.11. If $\mathfrak{a}$ is an ideal in R and $q = p^e$, where e is a positive integer, then $\mathfrak{a}^{[1/q]}$ is the unique smallest ideal J with the property that $\mathfrak{a} \subseteq J^{[q]}$; in other words, for an ideal $\mathfrak{b}$ we have $\mathfrak{a} \subseteq \mathfrak{b}^{[q]}$ if and only if $\mathfrak{a}^{[1/q]} \subseteq \mathfrak{b}$.

REMARK 3.12. If $(\mathfrak{b}_i)_{i\in I}$ is a family of ideals in R such that $\mathfrak{a} \subseteq \mathfrak{b}_i^{[q]}$ for all $i \in I$ and $\mathfrak{b} = \cap_{i\in I}\mathfrak{b}_i$, then it follows from Proposition 3.9iii) that $\mathfrak{a} \subseteq \mathfrak{b}^{[q]}$. This implies that indeed, the set of ideals J with $\mathfrak{a} \subseteq J^{[q]}$ contains a unique smallest element.

In the next proposition we collect a few easy properties of the ideals $\mathfrak{a}^{[1/q]}$.

PROPOSITION 3.13. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in the F-finite regular ring R and let $q = p^e$, $q' = p^{e'}$, with $e, e' \geq 1$.

i) If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mathfrak{a}^{[1/q]} \subseteq \mathfrak{b}^{[1/q]}$.
ii) $(\mathfrak{a} \cap \mathfrak{b})^{[1/q]} \subseteq \mathfrak{a}^{[1/q]} \cap \mathfrak{b}^{1/q]}$
iii) $(\mathfrak{a} + \mathfrak{b})^{[1/q]} = \mathfrak{a}^{[1/q]} + \mathfrak{b}^{[1/q]}$.
iv) $(\mathfrak{a} \cdot \mathfrak{b})^{[1/q]} \subseteq \mathfrak{a}^{[1/q]} \cdot \mathfrak{b}^{[1/q]}$.
v) $(\mathfrak{a}^{[q']})^{[1/q]} = \mathfrak{a}^{[q'/q]} \subseteq \left(\mathfrak{a}^{[1/q]}\right)^{[q']}$.
vi) $\left(\mathfrak{a}^{[1/q]}\right)^{[1/q']} = \mathfrak{a}^{[1/qq']}$.

PROOF. If $\mathfrak{a} \subseteq \mathfrak{b}$, the inclusions

$$\mathfrak{a} \subseteq \mathfrak{b} \subseteq \left(\mathfrak{b}^{[1/q]}\right)^{[q]}$$

imply $\mathfrak{a}^{[1/q]} \subseteq \mathfrak{b}^{[1/q]}$. This proves i) and the assertion in ii) follows immediately from this one.

The inclusion "$\supseteq$" in iii) also follows from i). The reverse inclusion follows from the minimality in the definition of $(\mathfrak{a} + \mathfrak{b})^{[1/q]}$ and the inclusion

$$\mathfrak{a} + \mathfrak{b} \subseteq \left(\mathfrak{a}^{[1/q]}\right)^{[q]} + \left(\mathfrak{a}^{[1/q]}\right)^{[q]} = \left(\mathfrak{a}^{[1/q]} + \mathfrak{b}^{[1/q]}\right)^{[q]},$$

where the equality follows from Proposition 3.9i).

The inclusion in iv) follows from the minimality in the definition of $(\mathfrak{a} \cdot \mathfrak{b})^{[1/q]}$ and the inclusion

$$\mathfrak{a} \cdot \mathfrak{b} \subseteq \left(\mathfrak{a}^{[1/q]}\right)^{[q]} \cdot \left(\mathfrak{b}^{[1/q]}\right)^{[q]} = \left(\mathfrak{a}^{[1/q]} \cdot \mathfrak{b}^{[1/q]}\right)^{[q]},$$

where the equality follows from Proposition 3.9ii).

In order to prove v), we consider separately the cases when $q \geq q'$ and when $q' \geq q$. Suppose first that $q' \geq q$. Since $\mathfrak{a}^{[q']} = \left(\mathfrak{a}^{[q'/q]}\right)^{[q]}$, the minimality property in the definition of $\left(\mathfrak{a}^{[q']}\right)^{[1/q]}$ implies that

$$\left(\mathfrak{a}^{[q']}\right)^{[1/q]} \subseteq \mathfrak{a}^{[q'/q]}. \tag{3.2}$$

On the other hand, since

$$\left(\mathfrak{a}^{[q'/q]}\right)^{[q]} = \mathfrak{a}^{[q']} \subseteq \left(\left(\mathfrak{a}^{[q']}\right)^{[1/q]}\right)^{[q]},$$

it follows from Proposition 3.9iv) that

$$\left(\mathfrak{a}^{[q'/q]}\right) \subseteq \left(\mathfrak{a}^{[q']}\right)^{[1/q]}.$$

By combining this with (3.2), we obtain the equality in v). Note now that the inclusion

$$\mathfrak{a} \subseteq \left(\mathfrak{a}^{[1/q]}\right)^{[q]}$$

induces the inclusion

$$\mathfrak{a}^{[q'/q]} \subseteq \left(\left(\mathfrak{a}^{[1/q]}\right)^{[q]}\right)^{[q'/q]} = \left(\mathfrak{a}^{[1/q]}\right)^{[q']},$$

hence the inclusion in v).

Suppose now that $q \geq q'$. For any ideal J in R, it follows from Proposition 3.9iv) that $\mathfrak{a} \subseteq J^{[q/q']}$ if and only if $\mathfrak{a}^{[q']} \subseteq J^{[q]}$. This gives the equality in v). Since

$$\mathfrak{a} \subseteq \left(\mathfrak{a}^{[1/q]}\right)^{[q]} = \left(\left(\mathfrak{a}^{[1/q]}\right)^{[q']}\right)^{[q/q']},$$

the minimality property of $\mathfrak{a}^{[q'/q]}$ gives the inclusion in v).

Finally, we prove vi). Note first that we have the inclusions

$$\mathfrak{a} \subseteq \left(\mathfrak{a}^{[1/q]}\right)^{[q]} \subseteq \left(\left(\mathfrak{a}^{[1/q]}\right)^{[1/q']}\right)^{[qq']},$$

where the second inclusion follows from v). The minimality property of $\mathfrak{a}^{[1/qq']}$ thus gives

$$\mathfrak{a}^{[1/qq']} \subseteq \left(\mathfrak{a}^{[1/q]}\right)^{[1/q']}.$$

In order to prove the reverse inclusion, by the minimality property in the definition of $\left(\mathfrak{a}^{[1/q]}\right)^{[1/q']}$, it is enough to show that

$$\mathfrak{a}^{[1/q]} \subseteq \left(\mathfrak{a}^{[1/qq']}\right)^{[q']}.$$

This follows from v). This completes the proof of the proposition. $\square$

Proposition 3.14. Let $\varphi\colon R \to R'$ be a homomorphism of F-finite regular rings, $\mathfrak{a}$ an ideal in R, and $q = p^e$, with $e \geq 1$.

i) We have $(\mathfrak{a} \cdot R')^{[1/q]} \subseteq \mathfrak{a}^{[1/q]} \cdot R'$.
ii) If $R' = S^{-1}R$, where S is a multiplicative system in R, and φ is the canonical homomorphism, then we have equality in i).

Proof. The inclusion $\mathfrak{a} \subseteq \left(\mathfrak{a}^{[1/q]}\right)^{[q]}$ induces an inclusion

$$\mathfrak{a} \cdot R' \subseteq \left(\mathfrak{a}^{[1/q]}\right)^{[q]} \cdot R' = \left(\mathfrak{a}^{[1/q]} \cdot R'\right)^{[q]},$$

hence the inclusion in i) follows from the minimality in the definition of $(\mathfrak{a} \cdot R')^{[1/q]}$.

Suppose now that $R' = S^{-1}R$. If $\mathfrak{b}$ is an ideal in R such that $(S^{-1}\mathfrak{a})^{[1/q]} = S^{-1}\mathfrak{b}$, then the inclusion $S^{-1}\mathfrak{a} \subseteq (S^{-1}\mathfrak{b})^{[q]}$ implies that there is $s \in S$ such that $s \cdot \mathfrak{a} \subseteq \mathfrak{b}^{[q]}$. Therefore we have

$$\mathfrak{a} \subseteq (\mathfrak{b}^{[q]}\colon s^q) = (\mathfrak{b}\colon s)^{[q]},$$

where the equality follows from Proposition 3.9iv). This implies $\mathfrak{a}^{[1/q]} \subseteq (\mathfrak{b} : s)$ and thus

$$S^{-1}\mathfrak{a}^{[1/q]} \subseteq S^{-1}(\mathfrak{b} : s) = S^{-1}\mathfrak{b} = (S^{-1}\mathfrak{a})^{[1/q]}.$$

This completes the proof. $\square$

By the above proposition, the computation of $\mathfrak{a}^{[1/q]}$ can be done locally. Since the Frobenius homomorphism is finite and flat, we can thus reduce the computation to the case when R is free over R^q. In this case, the next proposition gives an explicit description of the ideal $\mathfrak{a}^{[1/q]}$.

PROPOSITION 3.15. Let R be a regular F-finite ring of characteristic $p > 0$ and let $q = p^e$, with $e \geq 1$. Suppose that R is free over R^q with a basis given by $e_1, \ldots, e_n$. If $\mathfrak{a}$ is an ideal of R generated by $f_1, \ldots, f_r$ and if we write

$$f_i = \sum_{j=1}^{n} h_{i,j}^q e_j \quad \text{for} \quad 1 \leq i \leq r, \tag{3.3}$$

then

$$\mathfrak{a}^{[1/q]} = (h_{i,j} \mid 1 \leq i \leq r, 1 \leq j \leq n). \tag{3.4}$$

PROOF. Let J denote the ideal on the right-hand side of (3.4). Since we clearly have $\mathfrak{a} \subseteq J^{[q]}$, we have $\mathfrak{a}^{[1/q]} \subseteq J$. In order to prove the reverse inclusion, let us choose generators $g_1, \ldots, g_m$ of $\mathfrak{a}^{[1/q]}$. Since $\mathfrak{a} \subseteq \left(\mathfrak{a}^{[1/q]}\right)^{[q]}$, we deduce that $f_i \in (g_1^q, \ldots, g_m^q)$ for all i, hence we can find $a_{i,k} \in R$ such that

$$f_i = \sum_{k=1}^{m} a_{i,k} g_k^q \quad \text{for} \quad 1 \leq i \leq r. \tag{3.5}$$

Suppose now that $e_1^*, \ldots, e_n^*$ is the dual basis of $e_1, \ldots, e_n$. Note first that it follows from (3.3) that $h_{i,j}^q = e_j^*(f_i)$ for all i and j. On the other hand, it follows from (3.5) that

$$e_j^*(f_i) = \sum_{k=1}^{m} g_k^q e_j^*(a_{i,k}).$$

We thus conclude that $h_{i,j}^q \in (g_1^q, \ldots, g_m^q) = \left(\mathfrak{a}^{[1/q]}\right)^{[q]}$. Using Lemma 3.9v), we conclude that $h_{i,j} \in \mathfrak{a}^{[1/q]}$ for all i and j, hence $J \subseteq \mathfrak{a}^{[1/q]}$. This completes the proof of the proposition. $\square$

3.2. Test ideals: definition and first properties

We begin by considering the affine case. Let R be an F-finite regular ring of characteristic $p > 0$. If $\mathfrak{a}$ is an ideal in R that is everywhere nonzero (that is, it is nonzero on each connected component of $\mathrm{Spec}(R)$), we make the convention that $\mathfrak{a}^0 = R$. If λ is a nonnegative real number, we will take the test ideal $\tau(\mathfrak{a}^\lambda)$ to be the largest ideal in the set $(\mathfrak{a}^r)^{[1/p^e]}$, where r and e are such that $\frac{r}{p^e} \geq \lambda$. In order to make this precise, we need the following

LEMMA 3.16. *With the above notation, if r, r', e, and e' are such that $\frac{r}{p^e} \leq \frac{r'}{p^{e'}}$ and $e \geq e'$, then*

$$(\mathfrak{a}^{r'})^{[1/p^{e'}]} \subseteq (\mathfrak{a}^r)^{[1/p^e]}.$$

PROOF. By hypothesis, we have $r \leq r' p^{e-e'}$, hence $\mathfrak{a}^r \supseteq \mathfrak{a}^{r' p^{e-e'}}$. We thus have the sequence of inclusions

$$(\mathfrak{a}^r)^{[1/p^e]} \supseteq \left((\mathfrak{a}^{r'})^{p^{e-e'}}\right)^{[1/p^e]} \supseteq \left(\mathfrak{a}^{r'})^{[p^{e-e'}]}\right)^{[1/p^e]} = (\mathfrak{a}^{r'})^{[1/p^{e'}]},$$

where the equality follows using Proposition 3.13v) and the second inclusion follows from the fact that for every ideal J, we have $J^{[p^e]} \subseteq J^{p^e}$. $\square$

Suppose now that $\mathfrak{a}$ is an ideal in R that is everywhere nonzero and $\lambda \in \mathbf{R}_{\geq 0}$. For every positive integer e, we consider the ideal $\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]}$. Since $\lambda p^{e+1} \leq \lceil \lambda p^e \rceil p$, it follows that $\lceil \lambda p^{e+1} \rceil \leq \lceil \lambda p^e \rceil p$ and thus $\frac{\lceil \lambda p^{e+1} \rceil}{p^{e+1}} \leq \frac{\lceil \lambda p^e \rceil}{p^e}$. Therefore Lemma 3.16 gives

$$\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} \subseteq \left(\mathfrak{a}^{\lceil \lambda p^{e+1} \rceil}\right)^{[1/p^{e+1}]}.$$

Since R is Noetherian, it follows that this sequence of ideals, when e goes to infinity, stabilizes.

Definition 3.17. The *test ideal* $\tau(\mathfrak{a}^\lambda)$ is the ideal of R with the property that

$$\tau(\mathfrak{a}^\lambda) = \left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} \quad \text{for} \quad e \gg 0.$$

Remark 3.18. Note that $\tau(\mathfrak{a}^\lambda)$ is the unique maximal element in the set of all ideals $(\mathfrak{a}^r)^{[1/p^e]}$, where r and e are such that $\frac{r}{p^e} \geq \lambda$. Indeed, it lies in this set, and for every such r and e, we have $r \geq \lceil \lambda p^e \rceil$, hence

$$(\mathfrak{a}^r)^{[1/p^e]} \subseteq \left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} \subseteq \tau(\mathfrak{a}^\lambda).$$

We next give some easy properties of test ideals.

Proposition 3.19. If S is a multiplicative system in R, $\mathfrak{a}$ is an ideal in R that is everywhere nonzero, and $\mathfrak{b} = S^{-1}\mathfrak{a} \subseteq S^{-1}R$, then for every $\lambda \in \mathbf{R}_{\geq 0}$, we have $\tau(\mathfrak{b}^\lambda) = S^{-1}\tau(\mathfrak{a}^\lambda)$.

Proof. For every $e \geq 1$, it follows from Proposition 3.14ii) that

$$\left(\mathfrak{b}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} = S^{-1}\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]}.$$

By taking $e \gg 0$, we get the assertion in the proposition. $\square$

Exercise 3.20. Show that if R is a regular F-finite local ring, $\mathfrak{a}$ is a nonzero ideal in R, and $\widehat{\mathfrak{a}} = \mathfrak{a} \cdot \widehat{R}$, then $\tau(\widehat{\mathfrak{a}}^\lambda) = \tau(\mathfrak{a}^\lambda) \cdot \widehat{R}$ for every $\lambda \in \mathbf{R}_{\geq 0}$.

Definition 3.21. If X is a regular F-finite scheme of characteristic $p > 0$ and $\mathfrak{a}$ is a (coherent) ideal in $\mathcal{O}_X$ that is everywhere nonzero, then we define the test ideal $\tau(\mathfrak{a}^\lambda)$ by defining it on affine open subsets of $U \subseteq X$:

$$\tau(\mathfrak{a}^\lambda)(U) := \tau\big(\mathfrak{a}(U)^\lambda\big).$$

It follows from Proposition 3.19 that we get in this way a coherent ideal of $\mathcal{O}_X$.

Proposition 3.22. Let X be a regular F-finite scheme of characteristic $p > 0$ and $\mathfrak{a}$, $\mathfrak{b}$ ideals in $\mathcal{O}_X$ that are everywhere nonzero.

i) If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\tau(\mathfrak{a}^\lambda) \subseteq \tau(\mathfrak{b}^\lambda)$ for every $\lambda \in \mathbf{R}_{\geq 0}$.

ii) If λ, $\mu \in \mathbf{R}_{\geq 0}$, with $\lambda \geq \mu$, then

$$\tau(\mathfrak{a}^\lambda) \subseteq \tau(\mathfrak{a}^\mu).$$

Proof. We may and will assume that $X = \operatorname{Spec}(R)$ is affine. If $\mathfrak{a} \subseteq \mathfrak{b}$, then for every $e \geq 1$, the inclusion $\mathfrak{a}^{\lceil \lambda p^e \rceil} \subseteq \mathfrak{b}^{\lceil \lambda p^e \rceil}$ induces the inclusion

$$\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} \subseteq \left(\mathfrak{b}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]}.$$

By taking $e \gg 0$, we obtain the assertion in i).

If $\lambda \geq \mu$, then for every $e \geq 1$, we have an inclusion $\mathfrak{a}^{\lceil \lambda p^e \rceil} \subseteq \mathfrak{a}^{\lceil \mu p^e \rceil}$, which induces the inclusion

$$\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} \subseteq \left(\mathfrak{a}^{\lceil \mu p^e \rceil}\right)^{[1/p^e]}.$$

By taking $e \gg 0$, we obtain the assertion in ii). $\square$

REMARK 3.23. For every $\lambda \in \mathbf{R}_{\geq 0}$, we have

$$\mathfrak{a} \subseteq \mathrm{rad}\big(\tau(\mathfrak{a}^\lambda)\big).$$

More precisely, if $m \in \mathbf{Z}$ with $m \geq \lambda$, then $\mathfrak{a}^m \subseteq \tau(\mathfrak{a}^\lambda)$. In order to check this, we may assume that $X = \mathrm{Spec}(R)$ is affine and use the fact that

$$\mathfrak{a}^m = \big((\mathfrak{a}^m)^{[p^e]}\big)^{[1/p^e]} \subseteq (\mathfrak{a}^{mp^e})^{[1/p^e]} \subseteq \big(\mathfrak{a}^{\lceil \lambda p^e \rceil}\big)^{[1/p^e]} \subseteq \tau(\mathfrak{a}^\lambda),$$

where the equality follows from Proposition 3.13v).

PROPOSITION 3.24. Let X be a regular F-finite scheme of characteristic $p > 0$. If $\mathfrak{a}$ is an ideal in $\mathcal{O}_X$ that is everywhere nonzero, then for every $\lambda \in \mathbf{R}_{\geq 0}$ there is $\epsilon > 0$ such that for all r, $e \geq 1$, with $\lambda < \frac{r}{p^e} < \lambda + \epsilon$, we have

$$(\mathfrak{a}^r)^{[1/p^e]} = \tau(\mathfrak{a}^\lambda).$$

In particular, we have $\tau(\mathfrak{a}^{\lambda'}) = \tau(\mathfrak{a}^\lambda)$ if $\lambda \leq \lambda' < \lambda + \epsilon$.

PROOF. If we cover X by affine open subsets U and we can find some ϵ_U on each of these subsets, the minimum of these ϵ_U works for X. We may thus assume that $X = \mathrm{Spec}(R)$ is affine and R is a domain.

We first consider all ideals of R of the form $(\mathfrak{a}^r)^{[1/p^e]}$, for r, $e \geq 1$, with $\lambda < \frac{r}{p^e}$. Using the fact that R is Noetherian, we pick one such ideal $J = (\mathfrak{a}^{r_0})^{[1/p^{e_0}]}$ which is not properly contained in any other such ideal. Let $\epsilon > 0$ be such that $\lambda + \epsilon < \frac{r_0}{p^{e_0}}$ and there is no $\frac{r}{p^e}$ in $(\lambda, \lambda + \epsilon)$ with $1 \leq e < e_0$. We first show that

$$(\mathfrak{a}^{r_1})^{[1/p^{e_1}]} = J \quad \text{for all} \quad r_1, e_1 \quad \text{with} \quad \tfrac{r_1}{p^{e_1}} \in (\lambda, \lambda + \epsilon). \tag{3.6}$$

Indeed, it follows from the choice of ϵ that $e_1 \geq e_0$ and $\frac{r_0}{p^{e_0}} > \frac{r_1}{p^{e_1}}$, hence Lemma 3.16 implies

$$J = (\mathfrak{a}^{r_0})^{[1/p^{e_0}]} \subseteq (\mathfrak{a}^{r_1})^{[1/p^{e_1}]}.$$

By the maximality in the choice of J, this inclusion must be an equality. We thus have (3.6).

In order to obtain the first assertion in the proposition, it is enough to show that $J = \tau(\mathfrak{a}^\lambda)$. Let $e \gg 0$, so that $\tau(\mathfrak{a}^\lambda) = \big(\mathfrak{a}^{\lceil \lambda p^e \rceil}\big)$ and $\frac{\lceil \lambda p^e \rceil + 1}{p^e} < \lambda + \epsilon$. If $\lambda p^e \not\in \mathbf{Z}$, then $\lambda < \frac{\lceil \lambda p^e \rceil}{p^e}$ and we are done by (3.6). Suppose now that $\lambda p^e \in \mathbf{Z}$. In this case (3.6) gives

$$J = (\mathfrak{a}^{\lambda p^e + 1})^{[1/p^e]} \subseteq (\mathfrak{a}^{\lambda p^e})^{[1/p^e]} = \tau(\mathfrak{a}^\lambda).$$

The reverse inclusion $(\mathfrak{a}^{\lambda p^e})^{[1/p^e]} \subseteq J$ is equivalent to $\mathfrak{a}^{\lambda p^e} \subseteq J^{[p^e]}$. Let $u \in \mathfrak{a}^{\lambda p^e}$. If $e' \geq e$, then $\frac{\lambda p^{e'} + 1}{p^{e'}} < \lambda + \epsilon$, hence (3.6) gives $J = \big(\mathfrak{a}^{\lambda p^{e'} + 1}\big)^{[1/p^{e'}]}$ and thus

$$u^{p^{e'-e}} \mathfrak{a} \subseteq \mathfrak{a}^{\lambda p^{e'} + 1} \subseteq J^{[p^{e'}]}.$$

If $c \in \mathfrak{a}$ is nonzero, we deduce that

$$c u^{p^{e'-e}} \in \big(J^{[p^e]}\big)^{[p^{e'-e}]} \quad \text{for all} \quad e' - e \geq 0.$$

We then deduce from Lemma 3.10 that $u \in J^{[p^e]}$. This completes the proof of the fact that $J = \tau(\mathfrak{a}^\lambda)$.

The last assertion in the proposition is clear: if $\lambda' \in (\lambda, \lambda+\epsilon)$, then for $e \gg 0$ we have $\lambda < \frac{\lceil \lambda' p^e \rceil}{p^e} < \lambda + \epsilon$, hence by what we have already proved

$$\tau(\mathfrak{a}^{\lambda'}) = \left(\mathfrak{a}^{\lceil \lambda' p^e \rceil}\right)^{[1/p^e]} = \tau(\mathfrak{a}^{\lambda}).$$

□

REMARK 3.25. Since it is clear that $\tau(\mathfrak{a}^0) = \mathcal{O}_X$, it follows from Proposition 3.24 that there is $\epsilon > 0$ such that $\tau(\mathfrak{a}^\lambda) = \mathcal{O}_X$ for all $\lambda \in [0, \epsilon)$.

PROPOSITION 3.26. Let X be a regular F-finite scheme of characteristic $p > 0$. If $\mathfrak{a}$ is an ideal in $\mathcal{O}_X$ that is everywhere nonzero, then for every positive integer m, we have

$$\tau\big((\mathfrak{a}^m)^\lambda\big) = \tau(\mathfrak{a}^{m\lambda}) \quad \text{for all} \quad \lambda \in \mathbf{R}_{\geq 0}.$$

PROOF. We may and will assume that X is affine. If $e \gg 0$, then

$$\tau\big((\mathfrak{a}^m)^\lambda\big) = \big((\mathfrak{a}^m)^{\lceil \lambda p^e \rceil}\big)^{[1/p^e]} = \big(\mathfrak{a}^{m\lceil \lambda p^e \rceil}\big)^{[1/p^e]}. \tag{3.7}$$

On the other hand, it follows from Proposition 3.24 that there is $\epsilon > 0$ such that

$$\tau(\mathfrak{a}^{m\lambda}) = (\mathfrak{a}^r)^{[1/p^{e'}]} \quad \text{if} \quad m\lambda < \tfrac{r}{p^e} < m\lambda + \epsilon.$$

For $e \gg 0$, we clearly have $m\lambda \leq \frac{m\lceil \lambda p^e \rceil}{p^e} < m\lambda + \epsilon$, hence

$$\tau(\mathfrak{a}^{m\lambda}) = \big(\mathfrak{a}^{m\lceil \lambda p^e \rceil}\big)^{[1/p^e]} \tag{3.8}$$

(if $\frac{m\lceil \lambda p^e \rceil}{p^e} = m\lambda$, then the equality in (3.8) follows from the definition of $\tau(\mathfrak{a}^{m\lambda})$, since $e \gg 0$). By combining (3.7) and (3.8), we obtain the equality in the proposition. □

REMARK 3.27. As we will see in the following sections, test ideals satisfy many of the general properties of multiplier ideals. An interesting aspect, however, is that while some of the more subtle properties of multiplier ideals have rather trivial proofs for test ideals, some of the straightforward properties of multiplier ideals that follow directly from the computation via a log resolution have more delicate proofs in the case of test ideals (as in the case of Propositions 3.24 and 3.26) or simply do not hold in this setting.

3.2.1. Mixed test ideals. As in the case of multiplier ideals, we can consider a mixed version of test ideals. Suppose that we have ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ that are everywhere nonzero and $\lambda_1, \ldots, \lambda_m \in \mathbf{R}_{\geq 0}$. Arguing as in the proof of Lemma 3.16, we see that if $X = \operatorname{Spec}(R)$, $e \geq e'$ are positive integers and $r_j, r'_j \in \mathbf{Z}_{\geq 0}$ for $1 \leq j \leq m$ are such that $\frac{r_j}{p^e} \leq \frac{r'_j}{p^{e'}}$ for all j, then

$$(\mathfrak{a}_1^{r'_1} \cdots \mathfrak{a}_m^{r'_m})^{[1/p^{e'}]} \subseteq (\mathfrak{a}_1^{r_1} \cdots \mathfrak{a}_m^{r_m})^{[1/p^e]}.$$

Using this, we see that for every $\lambda_1, \ldots, \lambda_m \in \mathbf{R}_{\geq 0}$, when $e \geq 1$, the ideals

$$\big(\mathfrak{a}_1^{\lceil \lambda_1 p^e \rceil} \cdots \mathfrak{a}_m^{\lceil \lambda_m p^e \rceil}\big)^{[1/p^e]}$$

form a non-decreasing sequence which stabilizes by the Noetherian property. The stable value is the *mixed test ideal* $\tau\big(\mathfrak{a}_1^{\lambda_1} \ldots \mathfrak{a}_m^{\lambda_m}\big)$. It is clear from the definition that if $\lambda_1 = \ldots = \lambda_m = \lambda$, then

$$\tau(\mathfrak{a}_1^{\lambda_1} \ldots \mathfrak{a}_m^{\lambda_m}) = \tau(\mathfrak{a}^\lambda),$$

where $\mathfrak{a} = \prod_i \mathfrak{a}_i$.

Arguing as in the proof of Proposition 3.19, we see that the definition of mixed test ideals commutes with localization and thus extends to regular F-finite schemes. Furthermore, the properties in Propositions 3.22, 3.24, and 3.26 extend to mixed test ideals, with similar proofs. For example, if $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ are everywhere nonzero ideals on a regular F-finite scheme X, then for every $\lambda_1, \ldots, \lambda_m \in \mathbf{R}_{\geq 0}$, there is $\epsilon > 0$ such that

$$\tau(\mathfrak{a}_1^{\lambda_1} \ldots \mathfrak{a}_m^{\lambda_m}) = \tau(\mathfrak{a}_1^{\lambda'_1} \ldots \mathfrak{a}_m^{\lambda'_m}) \quad \text{if} \quad \lambda'_i \in [\lambda_i, \lambda_i + \epsilon) \quad \text{for all} \quad i. \tag{3.9}$$

If $\ell_1, \ldots, \ell_m$ are positive integers, then

$$\tau(\mathfrak{a}_1^{\ell_1\lambda_1} \ldots \mathfrak{a}_r^{\ell_m\lambda_m}) = \tau\big((\mathfrak{a}_1^{\ell_1})^{\lambda_1} \ldots (\mathfrak{a}_m^{\ell_m})^{\lambda_m}\big).$$

We leave the proofs of these extensions to the mixed case as an exercise for the reader.

Note that using these properties, we may reduce the computation of mixed test ideals to the case of usual test ideals. More precisely, using (3.9), we reduce the computation of $\tau(\mathfrak{a}_1^{\lambda_1} \ldots \mathfrak{a}_m^{\lambda_m})$ to the case when $\lambda_1, \ldots, \lambda_m$ are rational numbers. In this case, if ℓ is a positive integer such that all $\ell\lambda_i$ are integers and $\mathfrak{a} = \prod_{i=1}^m \mathfrak{a}_i^{\ell\lambda_i}$, then

$$\tau(\mathfrak{a}_1^{\lambda_1} \ldots \mathfrak{a}_m^{\lambda_m}) = \tau(\mathfrak{a}^{1/\ell}).$$

Remark 3.28. Note that if $\mathfrak{a}$ is an everywhere nonzero ideal on X, then

$$\tau(\mathfrak{a}^{\lambda}\mathfrak{a}^{\mu}) = \tau(\mathfrak{a}^{\lambda+\mu}).$$

Indeed, we may assume that X is affine; if $e \gg 0$, then

$$\tau(\mathfrak{a}^{\lambda}\mathfrak{a}^{\mu}) = \big(\mathfrak{a}^{\lceil \lambda p^e \rceil + \lceil \mu p^e \rceil}\big)^{[1/p^e]} = \tau(\mathfrak{a}^{\lambda+\mu}).$$

The second equality is clear if $\lceil \lambda p^e \rceil + \lceil \mu p^e \rceil = \lceil (\lambda+\mu)p^e \rceil$ and if this is not the case it follows from Proposition 3.24, since we then have

$$\lambda + \mu < \frac{\lceil \lambda p^e \rceil + \lceil \mu p^e \rceil}{p^e} < \lambda + \mu + \frac{2}{p^e}.$$

Remark 3.29. If X is a regular F-finite scheme of characteristic $p > 0$ and $\mathfrak{a}$, $\mathfrak{b}$ are ideals on X that are everywhere nonzero, then for every $\lambda \in \mathbf{R}_{\geq 0}$, we have

$$\mathfrak{b} \cdot \tau(\mathfrak{a}^{\lambda}) \subseteq \tau(\mathfrak{b}\mathfrak{a}^{\lambda}). \tag{3.10}$$

In particular, we have

$$\mathfrak{a} \cdot \tau(\mathfrak{a}^{\lambda}) \subseteq \tau(\mathfrak{a}^{\lambda+1}). \tag{3.11}$$

Indeed, we may assume that $X = \operatorname{Spec}(R)$ is affine. Note first that it is enough to prove the assertion when $\mathfrak{b} = (h)$ is a principal ideal. Indeed, this would imply that for every $h \in \mathfrak{b}$, we have $h \cdot \tau(\mathfrak{a}^{\lambda}) \subseteq \tau(h\mathfrak{a}^{\lambda}) \subseteq \tau(\mathfrak{b}\mathfrak{a}^{\lambda})$, hence (3.10) holds.

The assertion we need follows by taking $e \gg 0$ if we show that

$$h \cdot \big(\mathfrak{a}^{\lceil \lambda p^e \rceil}\big)^{[1/p^e]} \subseteq \big(h^{p^e}\mathfrak{a}^{\lceil \lambda p^e \rceil}\big)^{[1/p^e]}$$

for all $e \geq 1$. If we put $m = \lceil \lambda p^e \rceil$, then the assertion we want to show is equivalent to

$$(\mathfrak{a}^m)^{[1/p^e]} \subseteq \Big((h^{p^e}\mathfrak{a}^m)^{[1/p^e]} \colon h\Big),$$

which in turn, by the definition of the left-hand side, is equivalent to

$$\mathfrak{a}^m \subseteq \big((h^{p^e}\mathfrak{a}^m)^{[1/p^e]} \colon h\big)^{[p^e]} = \Big(\big((h^{p^e}\mathfrak{a}^m)^{[1/p^e]}\big)^{[p^e]} \colon h^{p^e}\Big),$$

where the equality follows from Proposition 3.9iv). This is a consequence of the definition of $(h^{p^e}\mathfrak{a}^m)^{[1/p^e]}$. This completes the proof of (3.10) and we obtain the inclusion in (3.11) by taking $\mathfrak{a}=\mathfrak{b}$ in (3.10) and using Remark 3.28.

3.3. Properties of test ideals

In this section we begin the discussion of some basic properties of test ideals. Further properties as well as some pathological aspects (by comparison with multiplier ideals) are covered in Chapter 3.6 below.

We begin with the analogue of the Restriction Theorem for multiplier ideals (cf. Theorem 2.87).

Proposition 3.30. Let $f\colon Y\to X$ be a morphism of regular F-finite schemes of characteristic $p>0$. If $\mathfrak{a}$ is an ideal in $\mathcal{O}_X$ such that $\mathfrak{b}:=\mathfrak{a}\cdot\mathcal{O}_Y$ is everywhere nonzero, then

$$\tau(\mathfrak{b}^\lambda)\subseteq\tau(\mathfrak{a}^\lambda)\cdot\mathcal{O}_Y\quad\text{for all}\quad\lambda\in\mathbf{R}_{\geq 0}.$$

Proof. After covering X and Y by suitable affine open subsets, we may assume that $X=\operatorname{Spec}(R)$ and $Y=\operatorname{Spec}(S)$ are affine schemes and f corresponds to the homomorphism $\varphi\colon R\to S$. Since $\mathfrak{b}^{\lceil\lambda p^e\rceil}=\mathfrak{a}^{\lceil\lambda p^e\rceil}\cdot S$, it follows from Proposition 3.14 that for every $e\geq 1$, we have

$$\big(\mathfrak{b}^{\lceil\lambda p^e\rceil}\big)^{[1/p^e]}\subseteq\big(\mathfrak{a}^{\lceil\lambda p^e\rceil}\big)^{[1/p^e]}\cdot S.$$

For $e\gg 0$, this gives the inclusion in the proposition. □

We next show that for smooth morphisms, the inclusion in Proposition 3.30 is an equality. We will not use this result in an essential way, so the reader not familiar with formally smooth morphisms could safely skip it.

Proposition 3.31. If $f\colon Y\to X$ is a smooth morphism between regular F-finite schemes of characteristic $p>0$, $\mathfrak{a}\subseteq\mathcal{O}_X$ is an ideal that is everywhere nonzero, and $\mathfrak{b}=\mathfrak{a}\cdot\mathcal{O}_Y$, then

$$\tau(\mathfrak{b}^\lambda)=\tau(\mathfrak{a}^\lambda)\cdot\mathcal{O}_Y\quad\text{for all}\quad\lambda\geq 0.\tag{3.12}$$

More generally, if $\varphi\colon(A,\mathfrak{m},k)\to(B,\mathfrak{n},K)$ is a local homomorphism of regular F-finite local rings of characteristic $p>0$ and B is $\mathfrak{n}$-formally smooth over A, then for every nonzero ideal $\mathfrak{a}$ in A, if $\mathfrak{b}=\mathfrak{a}\cdot B$, then

$$\tau(\mathfrak{b}^\lambda)=\tau(\mathfrak{a}^\lambda)\cdot B\quad\text{for all}\quad\lambda\geq 0.\tag{3.13}$$

Proof. The first assertion follows from the second one: indeed, in order to prove the equality in (3.12), it is enough to prove the corresponding equality in $\mathcal{O}_{Y,y}$ for every $y\in Y$. Since test ideals commute with localization by Proposition 3.19, the equality in (3.12) follows from the one in (3.13), applied for the homomorphism $\mathcal{O}_{X,x}\to\mathcal{O}_{Y,y}$.

In order to prove the second assertion, after replacing φ by $\widehat{A}\to\widehat{B}$ (which satisfies the same hypotheses), and using the fact that test ideals commute with completion (see Exercise 3.20), we may assume that A and B are complete. Since B is $\mathfrak{n}$-smooth over A, it follows that the field extension K/k is separable and $B/\mathfrak{m}B$ is geometrically regular (in particular, it is a regular ring). If we choose a regular system of parameters $x_1,\ldots,x_n$ of A and $y_1,\ldots,y_m\in\mathfrak{n}$ that induce a regular system of parameters of $B/\mathfrak{m}B$, we see that $x_1, \ldots, x_n, y_1, \ldots, y_m$ is a regular system of parameters of B and we have

$$A \simeq k[\![x_1, \ldots, x_n]\!] \quad \text{and} \quad B \simeq K[\![x_1, \ldots, x_n, y_1, \ldots, y_m]\!].$$

If $(a_i)_{i \in I}$ is a basis of k over k^p, since K/k is separable, it follows that we can complete this to a basis $(a_i)_{i \in J}$ of K over K^p (see [**Mat89**, Theorem 26.4]). Note that for every $e \geq 1$, A is free over A^{p^e}, with a basis given by $a_i x_1^{u_1} \cdots x_n^{u_n}$ for $i \in I$ and $0 \leq u_j \leq p^e - 1$ for all j. We have a similar description for a basis of B over B^{p^e}; in particular, we see that we can complete a basis of A over A^{p^e} to a basis of B over B^{p^e}. In this case, the description of inverse Frobenius powers in Proposition 3.15 implies that for every $m \geq 0$ and $e \geq 1$ we have

$$(\mathfrak{b}^m)^{[1/p^e]} = (\mathfrak{a}^m)^{[1/p^e]} B.$$

The equality in (3.13) is then an immediate consequence of the definition of test ideals. $\square$

As in characteristic 0, Proposition 3.30 (in fact, an obvious extension to the mixed case) implies a subadditivity statement. For simplicity, we only give the following version:

PROPOSITION 3.32. Let X be a regular scheme of finite type over a perfect field k of characteristic $p > 0$. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals that are everywhere nonzero, then

$$\tau(\mathfrak{a}^\lambda \mathfrak{b}^\mu) \subseteq \tau(\mathfrak{a}^\lambda) \cdot \tau(\mathfrak{b}^\mu) \quad \text{for all} \quad \lambda, \mu \in \mathbf{R}_{\geq 0}.$$

Again, the key case to understand is that in the following lemma:

LEMMA 3.33. *Let X and Y be regular schemes of finite type over a perfect field k of characteristic $p > 0$. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals on X and Y, respectively, that are everywhere nonzero, and if $\widetilde{\mathfrak{a}} = \mathfrak{a} \cdot \mathcal{O}_{X \times Y}$ and $\widetilde{\mathfrak{b}} = \mathfrak{b} \cdot \mathcal{O}_{X \times Y}$, then*

$$\tau(\widetilde{\mathfrak{a}}^\lambda \widetilde{\mathfrak{b}}^\mu) = \big(\tau(\mathfrak{a}^\lambda) \cdot \mathcal{O}_{X \times Y}\big) \cdot \big(\tau(\mathfrak{b}^\mu) \cdot \mathcal{O}_{X \times Y}\big) \quad \textit{for all} \quad \lambda, \mu \in \mathbf{R}_{\geq 0}.$$

PROOF. Note that since k is perfect, a scheme of finite type over k is regular if and only if it is smooth over k. This implies in particular that the product $X \times Y$ (where the product is over $\mathrm{Spec}(k)$) is again regular.

The assertion in the lemma follows directly from the definition of test ideals if we show that for every $r, s \in \mathbf{Z}_{\geq 0}$ and every positive integer e, we have

$$(\widetilde{\mathfrak{a}}^r \widetilde{\mathfrak{b}}^s)^{[1/p^e]} = \big((\mathfrak{a}^r)^{[1/p^e]} \cdot \mathcal{O}_{X \times Y}\big) \cdot \big((\mathfrak{b}^s)^{[1/p^e]} \cdot \mathcal{O}_{X \times Y}\big). \tag{3.14}$$

In order to prove this, we may assume that $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ are affine and that R and S are free over R^{p^e} and S^{p^e}, respectively. In this case $X \times Y = \mathrm{Spec}(T)$, where $T = R \otimes_k S$. If $(a_i)_{1 \leq i \leq m}$ and $(b_j)_{1 \leq j \leq n}$ are bases of R and S over R^{p^e} and S^{p^e}, respectively, then $(a_i \otimes b_j)_{i,j}$ is a basis of T over T^{p^e}. Let $f_1, \ldots, f_d$ be generators of $\mathfrak{a}^r$ and $g_1, \ldots, g_\ell$ be generators of $\mathfrak{b}^s$. If we write

$$f_u = \sum_{i=1}^m P_{u,i}^{p^e} a_i \quad \text{and} \quad g_v = \sum_{j=1}^n Q_{v,j}^{p^e} b_j,$$

with $P_{u,i} \in R$ and $Q_{v,j} \in S$, then it follows from Proposition 3.15 that

$$(\mathfrak{a}^r)^{[1/p^e]} = (P_{u,i} \mid 1 \leq u \leq d, 1 \leq i \leq m) \quad \text{and} \quad (\mathfrak{b}^s)^{[1/p^e]} = (Q_{v,j} \mid 1 \leq v \leq \ell, 1 \leq j \leq n).$$

On the other hand, $\widetilde{\mathfrak{a}}^r\widetilde{\mathfrak{b}}^s$ is generated by $(f_u\otimes g_v)_{u,v}$ and we have

$$f_u\otimes g_v=\sum_{i=1}^m\sum_{j=1}^n(P_{u,i}\otimes Q_{v,j})^{p^e}(a_i\otimes b_j).$$

Therefore Proposition 3.15 gives

$$(\widetilde{\mathfrak{a}}^r\widetilde{\mathfrak{b}}^s)^{[1/p^e]}=(P_{u,i}\otimes Q_{v,j}\mid 1\leq u\leq d, 1\leq v\leq \ell, 1\leq i\leq m, 1\leq j\leq n).$$

The equality in (3.14) is now clear. □

PROOF OF PROPOSITION 3.32. The argument is the same as in characteristic 0 (see the proof of Theorem 2.91), by considering the diagonal embedding $X\hookrightarrow X\times X$ and combining Proposition 3.30 and Lemma 3.33. □

We next give the analogue of Skoda's theorem for multiplier ideals (cf. 2.110). This is due to Hara and Takagi [**HT04**]

THEOREM 3.34. *If X is a regular F-finite scheme of characteristic $p>0$ and $\mathfrak{a}$ is an ideal on X that is everywhere nonzero and it is locally generated by r elements, then*

$$\tau(\mathfrak{a}^\lambda)=\mathfrak{a}\cdot\tau(\mathfrak{a}^{\lambda-1})\quad \textit{for all}\quad \lambda\geq r.$$

PROOF. We may assume that $X=\mathrm{Spec}(R)$ is affine and $\mathfrak{a}=(f_1,\ldots,f_r)$. The key observation is that for every $e\geq 1$, if $m\geq r(p^e-1)+1$, then

$$\mathfrak{a}^m=\mathfrak{a}^{[p^e]}\cdot\mathfrak{a}^{m-p^e}.$$

Suppose now that $\lambda\geq r$ and $e\gg 0$, so that

$$\tau(\mathfrak{a}^\lambda)=\left(\mathfrak{a}^{\lceil\lambda p^e\rceil}\right)^{[1/p^e]}.$$

The above observation gives

$$\mathfrak{a}^{\lceil\lambda p^e\rceil}=\mathfrak{a}^{[p^e]}\cdot\mathfrak{a}^{\lceil(\lambda-1)p^e\rceil},$$

and using assertions iv) and v) in Proposition 3.13, we obtain

$$\tau(\mathfrak{a}^\lambda)=\left(\mathfrak{a}^{\lceil\lambda p^e\rceil}\right)^{[1/p^e]}=\left(\mathfrak{a}^{[p^e]}\cdot\mathfrak{a}^{\lceil(\lambda-1)p^e\rceil}\right)^{[1/p^e]}$$
$$=\left(\mathfrak{a}^{[p^e]}\right)^{[1/p^e]}\cdot\left(\mathfrak{a}^{\lceil(\lambda-1)p^e\rceil}\right)^{[1/p^e]}=\mathfrak{a}\cdot\left(\mathfrak{a}^{\lceil(\lambda-1)p^e\rceil}\right)^{[1/p^e]}.$$

Since $e\gg 0$, the right-most term is equal to $\mathfrak{a}\cdot\tau(\mathfrak{a}^{\lambda-1})$. This completes the proof of the theorem. □

As in characteristic 0, we can use Theorem 3.34 to relate the integral closure of suitable powers of an ideal to the original ideal via the following result.

PROPOSITION 3.35. If X is an F-finite regular scheme of characteristic $p>0$ and $\mathfrak{a}$ is an ideal on X that is everywhere nonzero, then

$$\tau(\mathfrak{a}^\lambda)=\tau(\overline{\mathfrak{a}}^\lambda)\quad\text{for all}\quad\lambda\in\mathbf{R}_{\geq 0}.$$

PROOF. We note that the integral closure $\overline{\mathfrak{a}}$ is defined separately on each connected component of X (since each such component is a normal variety, the definition and basic properties discussed in Chapter 2.6.1 apply). Since $\mathfrak{a}\subseteq\overline{\mathfrak{a}}$, the inclusion $\tau(\mathfrak{a}^\lambda)\subseteq\tau(\overline{\mathfrak{a}}^\lambda)$ follows from Proposition 3.22i).

In order to prove the reverse inclusion, note first that by Proposition 3.24 there is $\epsilon>0$ such the following two conditions hold:

a) $\tau(\overline{\mathfrak{a}}^{\lambda})=\tau(\overline{\mathfrak{a}}^{\lambda+\epsilon})$.
b) $\tau(\mathfrak{a}^{\lambda})=(\mathfrak{a}^{r})^{[1/p^e]}$ if $\lambda<\frac{r}{p^e}\leq\lambda+\epsilon$.

We now use the fact that by Proposition 2.106, there is a positive integer m such that for every $i\geq m$, we have $\overline{\mathfrak{a}^i}\subseteq\mathfrak{a}^{i-m}$ and thus $\overline{\mathfrak{a}}^i\subseteq\mathfrak{a}^{i-m}$. If $e\gg 0$, we thus conclude that

$$\tau(\overline{\mathfrak{a}}^{\lambda})=\tau(\overline{\mathfrak{a}}^{\lambda+\epsilon})=\left(\overline{\mathfrak{a}}^{\lceil(\lambda+\epsilon)p^e\rceil}\right)^{[1/p^e]}\subseteq\left(\mathfrak{a}^{\lceil(\lambda+\epsilon)p^e\rceil-m}\right)^{[1/p^e]}. \tag{3.15}$$

Note next that

$$\frac{\lceil(\lambda+\epsilon)p^e\rceil-m}{p^e}\leq\lambda+\epsilon$$

and since $e\gg 0$, we also have

$$\frac{\lceil(\lambda+\epsilon)p^e\rceil-m}{p^e}\geq\lambda+\epsilon-\frac{m}{p^e}>\lambda.$$

Condition b) above thus implies

$$\left(\mathfrak{a}^{\lceil(\lambda+\epsilon)p^e\rceil-m}\right)^{[1/p^e]}=\tau(\mathfrak{a}^{\lambda}),$$

which together with (3.15) completes the proof. $\square$

We obtain the following corollary (which, for simplicity, we don't state in its most general form).

COROLLARY 3.36. If X is a regular scheme of finite type over an infinite F-finite field, with $\dim(X)=n$, then for every ideal $\mathfrak{a}$ on X that is everywhere nonzero, we have $\overline{\mathfrak{a}^n}\subseteq\mathfrak{a}$.

PROOF. Since the statement is local, it follows from Proposition 2.109 that we may assume that we have an ideal $\mathfrak{b}\subseteq\mathfrak{a}$ that is generated by n elements such that $\overline{\mathfrak{b}}=\overline{\mathfrak{a}}$ (here is where we use the fact that the ground field is infinite). This implies $\overline{\mathfrak{b}^n}=\overline{\mathfrak{a}^n}$: this follows, for example, from the fact that $\overline{\mathfrak{a}^n}\subseteq\overline{\overline{\mathfrak{a}}^n}\subseteq\overline{\overline{\mathfrak{a}^n}}=\overline{\mathfrak{a}^n}$. Note now that we have

$$\overline{\mathfrak{a}^n}\subseteq\tau(\overline{\mathfrak{a}^n})=\tau(\overline{\mathfrak{b}^n})$$

by Remark 3.29,

$$\tau(\overline{\mathfrak{b}^n})=\tau(\mathfrak{b}^n)$$

by Proposition 3.35, and

$$\tau(\mathfrak{b}^n)\subseteq\mathfrak{b}$$

by Theorem 3.34. By combining these, we obtain the inclusion in the corollary. $\square$

We next turn to the following version of the Summation Theorem for multiplier ideals, due to Takagi [**Tak06**].

THEOREM 3.37. *If X is a regular F-finite scheme of characteristic $p>0$ and $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ ideals on X that are everywhere nonzero, then for every λ, $\mu\in\mathbf{R}_{\geq 0}$, we have*

$$\tau\big((\mathfrak{a}+\mathfrak{b})^{\lambda}\mathfrak{c}^{\mu}\big)=\sum_{\alpha+\beta=\lambda}\tau(\mathfrak{a}^{\alpha}\mathfrak{b}^{\beta}\mathfrak{c}^{\mu}),$$

where the sum on the right-hand side is over all $\alpha,\beta\in\mathbf{R}_{\geq 0}$ with $\alpha+\beta=\gamma$.

REMARK 3.38. Unlike in the case of multiplier ideals, it is not clear (and probably not true in general) that the sum on the right-hand side involves only finitely many distinct ideals.

PROOF OF THEOREM 3.37. If α, $\beta \in \mathbf{R}_{\geq 0}$ are such that $\alpha + \beta = \lambda$, since we have $\mathfrak{a} \subseteq \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{b} \subseteq \mathfrak{a} + \mathfrak{b}$, we have

$$\tau(\mathfrak{a}^\alpha \mathfrak{b}^\beta \mathfrak{c}^\mu) \subseteq \tau\big((\mathfrak{a}+\mathfrak{b})^\alpha (\mathfrak{a}+\mathfrak{b})^\beta \mathfrak{c}^\mu\big) = \tau\big((\mathfrak{a}+\mathfrak{b})^\lambda \mathfrak{c}^\mu\big),$$

where the equality follows from (a variant of) Remark 3.28. We thus get

$$\sum_{\alpha+\beta=\lambda} \tau(\mathfrak{a}^\alpha \mathfrak{b}^\beta \mathfrak{c}^\mu) \subseteq \tau\big((\mathfrak{a}+\mathfrak{b})^\lambda \mathfrak{c}^\mu\big).$$

In order to prove the reverse inclusion, let $e \gg 0$, so that

$$\tau\big((\mathfrak{a}+\mathfrak{b})^\lambda \mathfrak{c}^\mu\big) = \big((\mathfrak{a}+\mathfrak{b})^{\lceil \lambda p^e \rceil} \mathfrak{c}^{\lceil \mu p^e \rceil}\big)^{[1/p^e]}.$$

Since we clearly have

$$(\mathfrak{a}+\mathfrak{b})^{\lceil \lambda p^e \rceil} = \sum_{i+j=\lceil \lambda p^e \rceil} \mathfrak{a}^i \mathfrak{b}^j,$$

we deduce using Proposition 3.13iii) that

$$\tau\big((\mathfrak{a}+\mathfrak{b})^\lambda \mathfrak{c}^\mu\big) = \sum_{i+j=\lceil \lambda p^e \rceil} \big(\mathfrak{a}^i \mathfrak{b}^j \mathfrak{c}^{\lceil \mu p^e \rceil}\big)^{[1/p^e]}.$$

In order to complete the proof of the theorem, it is thus enough to show that for every nonnegative integers i, j, with $i + j = \lceil \lambda p^e \rceil$, we have

$$\big(\mathfrak{a}^i \mathfrak{b}^j \mathfrak{c}^{\lceil \mu p^e \rceil}\big)^{[1/p^e]} \subseteq \sum_{\alpha+\beta=\lambda} \tau(\mathfrak{a}^\alpha \mathfrak{b}^\beta \mathfrak{c}^\mu). \tag{3.16}$$

This is clear if $j = 0$, hence from now on we assume $j \geq 1$. It follows that if we put $\alpha = \frac{i}{p^e}$, then $\alpha \leq \frac{\lceil \lambda p^e \rceil - 1}{p^e} \leq \lambda$, hence if we put $\beta = \lambda - \alpha$, we have $\beta \geq 0$. Note that

$$\beta p^e = \lambda p^e - i \leq \lceil \lambda p^e \rceil - i = j,$$

hence $j \geq \lceil \beta p^e \rceil$ and thus

$$\big(\mathfrak{a}^i \mathfrak{b}^j \mathfrak{c}^{\lceil \mu p^e \rceil}\big)^{[1/p^e]} \subseteq \big(\mathfrak{a}^{\lceil \alpha p^e \rceil} \mathfrak{b}^{\lceil \beta p^e \rceil} \mathfrak{c}^{\lceil \mu p^e \rceil}\big)^{[1/p^e]} \subseteq \tau(\mathfrak{a}^\alpha \mathfrak{b}^\beta \mathfrak{c}^\mu).$$

We thus have (3.16), which completes the proof of the theorem. □

REMARK 3.39. Note that while the results in Proposition 3.30 and Theorems 3.34 and 3.37 simply follow from the definition of test ideals, the corresponding results for multiplier ideals all rely on Relative Vanishing.

3.4. F-jumping numbers and F-thresholds

Our goal in this section is to study those $\lambda \in \mathbf{R}_{\geq 0}$ where the test ideals jump. We fix a regular F-finite scheme X of characteristic $p > 0$ and let $\mathfrak{a}$ be an everywhere nonzero ideal in $\mathcal{O}_X$.

DEFINITION 3.40. An *F-jumping number* of $\mathfrak{a}$ is a positive real number λ such that for every $\mu < \lambda$, we have $\tau(\mathfrak{a}^\lambda) \subsetneq \tau(\mathfrak{a}^\mu)$. If $\mathfrak{a}$ is generated by $f \in \mathcal{O}_X(X)$, then we simply say *jumping number of f*.

REMARK 3.41. It follows from Proposition 3.24 that if λ is not an F-jumping number of $\mathfrak{a}$, then the test ideal $\tau(\mathfrak{a}^\mu)$ is constant for μ in a neighborhood of λ.

REMARK 3.42. It follows from the definition that if $X = \bigcup_{i\in I} U_i$ is an open cover, then the set of F-jumping numbers of $\mathfrak{a}$ is the union of the sets of F-jumping numbers of $\mathfrak{a}|_{U_i}$. This allows us to reduce the study of F-jumping numbers to the affine case.

DEFINITION 3.43. The *F-pure threshold* of $\mathfrak{a}$ is

$$\mathrm{fpt}(\mathfrak{a}) := \sup\{\lambda \in \mathbf{R}_{\geq 0} \mid \tau(\mathfrak{a}^{\lambda}) = \mathcal{O}_X\}.$$

For a point $x \in X$, the *F-pure threshold* of $\mathfrak{a}$ at x is

$$\mathrm{fpt}_x(\mathfrak{a}) := \sup\{\lambda \in \mathbf{R}_{\geq 0} \mid \tau(\mathfrak{a}^{\lambda})_x = \mathcal{O}_{X,x}\}.$$

If $\mathfrak{a}$ is generated by $f \in \mathcal{O}_X(X)$, then we simply write $\mathrm{fpt}(f)$ and $\mathrm{fpt}_x(f)$.

REMARK 3.44. It is clear that if $\mathrm{fpt}(\mathfrak{a}) < \infty$, then this is the smallest F-jumping number of $\mathfrak{a}$. Note that if $\mathfrak{a} = \mathcal{O}_X$, then $\mathrm{fpt}(\mathfrak{a}) = \infty$ (and the set of F-jumping numbers is empty); we will see shortly that the converse holds too.

REMARK 3.45. It follows directly from the definition that for every $x \in X$, we have

$$\mathrm{fpt}_x(\mathfrak{a}) = \max_{U \ni x}\{\mathrm{fpt}(\mathfrak{a}|_U)\},$$

where the maximum is over the open neighborhoods U of X. Similarly, we have

$$\mathrm{fpt}(\mathfrak{a}) = \min_{x\in X}\{\mathrm{fpt}_x(\mathfrak{a})\},$$

where the minimum is over all $x \in X$ (it is enough to only consider the closed points if X is affine or it is of finite type over a field).

LEMMA 3.46. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals on X that are everywhere nonzero.*

i) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mathrm{fpt}(\mathfrak{a}) \leq \mathrm{fpt}(\mathfrak{b})$ and $\mathrm{fpt}_x(\mathfrak{a}) \leq \mathrm{fpt}_x(\mathfrak{b})$ for all $x \in X$.*
ii) *For every positive integer m, we have $\mathrm{fpt}(\mathfrak{a}^m) = \frac{\mathrm{fpt}(\mathfrak{a})}{m}$ and $\mathrm{fpt}_x(\mathfrak{a}^m) = \frac{\mathrm{fpt}_x(\mathfrak{a})}{m}$ for all $x \in X$.*

PROOF. The assertion in i) follows from the fact that $\tau(\mathfrak{a}^{\lambda}) \subseteq \tau(\mathfrak{b}^{\lambda})$ for all $\lambda \in \mathbf{R}_{\geq 0}$, see Proposition 3.22i). The one in ii) follows from the fact that $\tau\big((\mathfrak{a}^m)^{\lambda}\big) = \tau(\mathfrak{a}^{m\lambda})$ for all $\lambda \in \mathbf{R}_{\geq 0}$, see Proposition 3.26. $\square$

3.4.1. F-jumping numbers as F-thresholds. In this section we give a different description of F-jumping numbers following [**MTW05**]. Let R be an F-finite regular ring of characteristic $p > 0$ and $\mathfrak{a} \subseteq R$ an ideal that is everywhere nonzero. If $J \subsetneq R$ is an ideal such that $\mathfrak{a} \subseteq \mathrm{rad}(J)$, then for every positive integer e, we denote by $\nu_{\mathfrak{a}}^J(p^e)$ the largest $r \in \mathbf{Z}_{\geq 0}$ such that $\mathfrak{a}^r \not\subseteq J^{[p^e]}$ (note that this is well-defined by our assumptions on J). If $\mathfrak{a} = (f)$, then we simply write $\nu_f^J(p^e)$. Note that for every e, we have

$$\frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} \leq \frac{\nu_{\mathfrak{a}}^J(p^{e+1})}{p^{e+1}}. \tag{3.17}$$

Indeed, if $\mathfrak{a}^r \not\subseteq J^{[p^e]}$, then $\mathfrak{a}^{pr} \not\subseteq J^{[p^{e+1}]}$: otherwise we have

$$(\mathfrak{a}^r)^{[p]} \subseteq \mathfrak{a}^{pr} \subseteq J^{[p^{e+1}]}$$

and we conclude that $\mathfrak{a}^r \subseteq J^{[p^e]}$ by Proposition 3.9v), a contradiction. It follows from (3.17) that we have

$$c^J(\mathfrak{a}) := \sup_{e\geq 1} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} = \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}.$$

This is the F-*threshold* of $\mathfrak{a}$ with respect to J.

REMARK 3.47. In fact, $c^J(\mathfrak{a}) < \infty$. Indeed, suppose that $\mathfrak{a}$ is generated by r elements, so that $\mathfrak{a}^{r(p^e-1)+1} \subseteq \mathfrak{a}^{[p^e]}$ for every $e > 0$. If $m \geq 1$ is such that $\mathfrak{a}^m \subseteq J$, then

$$\mathfrak{a}^{m(r(p^e-1)+1)} \subseteq \left(\mathfrak{a}^{[p^e]}\right)^m = (\mathfrak{a}^m)^{[p^e]} \subseteq J^{[p^e]},$$

hence $\nu_{\mathfrak{a}}^J(p^e) \leq m(r(p^e-1)+1)$ for all $e \geq 1$ and thus $c^J(\mathfrak{a}) \leq mr$.

REMARK 3.48. It follows from the definition that we have

$$\frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} \leq c^J(\mathfrak{a}) \quad \text{for all} \quad e \geq 1. \tag{3.18}$$

In fact, this is a strict inequality. Indeed, if this is not the case, then there is $e \geq 1$ such that $\nu_{\mathfrak{a}}^J(p^{e'}) = cp^{e'}$ for all $e' \geq e$, where $c = c^J(\mathfrak{a})$. If $f \in \mathfrak{a}$ is not a zero-divisor, then for every $e' \geq e$, we have

$$f \cdot \left(\mathfrak{a}^{cp^e}\right)^{[p^{e'-e}]} \subseteq \mathfrak{a}^{cp^{e'}+1} \subseteq J^{[p^{e'}]} = \left(J^{[p^e]}\right)^{[p^{e'-e}]}.$$

We deduce from Proposition 3.10 that $\mathfrak{a}^{cp^e} \subseteq J^{[p^e]}$, a contradiction.

REMARK 3.49. If $J_1 \subseteq J_2$ are ideals in R, with $J_2 \neq R$ and $\mathfrak{a} \subseteq \mathrm{rad}(J_1)$, then it is clear that for every $e \geq 1$, we have

$$\nu_{\mathfrak{a}}^{J_2}(p^e) \leq \nu_{\mathfrak{a}}^{J_1}(p^e).$$

Dividing by p^e and letting e go to infinity, we get $c^{J_2}(\mathfrak{a}) \leq c^{J_1}(\mathfrak{a})$.

The next lemma will allow us to relate the F-jumping numbers of $\mathfrak{a}$ with the F-thresholds.

PROPOSITION 3.50. Let $\mathfrak{a}$ be an ideal of R that is everywhere nonzero.

i) If J is a proper ideal of R such that $\mathfrak{a} \subseteq \mathrm{rad}(J)$, then

$$\tau\left(\mathfrak{a}^{c^J(\mathfrak{a})}\right) \subseteq J.$$

ii) If $\lambda \in \mathbf{R}_{\geq 0}$ is such that $\tau(\mathfrak{a}^\lambda) \neq \mathcal{O}_X$, then

$$c^{\tau(\mathfrak{a}^\lambda)}(\mathfrak{a}) \leq \lambda.$$

PROOF. In order to show i), let us put $c = c^J(\mathfrak{a})$. For every $e \geq 1$, it follows from Remark 3.48 that $\nu_{\mathfrak{a}}^J(p^e) < cp^e \leq \lceil cp^e \rceil$, hence

$$\mathfrak{a}^{\lceil cp^e \rceil} \subseteq J^{[p^e]}.$$

We thus have

$$\left(\mathfrak{a}^{\lceil cp^e \rceil}\right)^{[1/p^e]} \subseteq J.$$

By taking $e \gg 0$, this gives the inclusion in i).

For ii), note first that $\mathfrak{a} \subseteq \mathrm{rad}\big(\tau(\mathfrak{a}^\lambda)\big)$ by Remark 3.23, hence $c^{\tau(\mathfrak{a}^\lambda)}(\mathfrak{a})$ is defined. Let $\mathfrak{b} = \tau(\mathfrak{a}^\lambda)$. By definition of the test ideal, for every $e \geq 1$, we have $\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} \subseteq \mathfrak{b}$, hence $\mathfrak{a}^{\lceil \lambda p^e \rceil} \subseteq \mathfrak{b}^{[p^e]}$. We thus have $\nu_{\mathfrak{a}}^{\mathfrak{b}}(p^e) \leq \lceil \lambda p^e \rceil - 1$. Dividing by p^e and taking the limit when e goes to infinity, gives $c^{\mathfrak{b}}(\mathfrak{a}) \leq \lambda$. $\square$

COROLLARY 3.51. Let $\mathfrak{a}$ be a proper ideal in R that is everywhere nonzero.

i) If λ is an F-jumping number of $\mathfrak{a}$ and $J = \tau(\mathfrak{a}^\lambda)$, then $\lambda = c^J(\mathfrak{a})$.
ii) If J is a proper ideal in R such that $\mathfrak{a} \subseteq \mathrm{rad}(J)$, then $c^J(\mathfrak{a})$ is an F-jumping number of $\mathfrak{a}$.

In particular, the set of F-jumping numbers of $\mathfrak{a}$ is equal to the set of F-thresholds $c^J(\mathfrak{a})$, where J runs over the proper ideals of R with $\mathfrak{a} \subseteq \mathrm{rad}(J)$.

PROOF. We first prove i). The inequality $c := c^J(\mathfrak{a}) \leq \lambda$ follows from Proposition 3.50ii). If this inequality is strict, since λ is an F-jumping number of $\mathfrak{a}$, it follows that

$$J = \tau(\mathfrak{a}^\lambda) \subsetneq \tau(\mathfrak{a}^c) \subseteq J,$$

a contradiction, where the second inclusion follows from Proposition 3.50i). We thus have $c = \lambda$.

We next prove ii). Note first that $c^J(\mathfrak{a}) > 0$: this follows, for example, from Remark 3.48. If $c^J(\mathfrak{a})$ is not an F-jumping number, then there is $\mu < c^J(\mathfrak{a})$ such that

$$\tau\big(\mathfrak{a}^{c^J(\mathfrak{a})}\big) = \tau(\mathfrak{a}^\mu).$$

Using Proposition 3.50i), we deduce

$$\tau(\mathfrak{a}^\mu) \subseteq J$$

and thus, using Remark 3.49, we get

$$c^J(\mathfrak{a}) \leq c^{\tau(\mathfrak{a}^\mu)}(\mathfrak{a}) \leq \mu < c^J(\mathfrak{a}),$$

a contradiction, where the second inequality follows from Proposition 3.50ii). This gives the assertion in ii). Finally, the last assertion in the corollary follows from i) and ii). $\square$

REMARK 3.52. Using similar arguments to those in the proof of Corollary 3.51, we see that if $X = \mathrm{Spec}(R)$ is a regular F-finite affine scheme of characteristic $p > 0$ and $\mathfrak{a}$ is an ideal in R that is everywhere nonzero, then for every $x \in V(\mathfrak{a})$ corresponding to $\mathfrak{p} \supseteq \mathfrak{a}$,

$$\mathrm{fpt}_x(\mathfrak{a}) = c^{\mathfrak{p}}(\mathfrak{a}).$$

Indeed, note first that it follows from Proposition 3.50i) that $\tau\big(\mathfrak{a}^{c^{\mathfrak{p}}(\mathfrak{a})}\big) \subseteq \mathfrak{p}$, hence $\mathrm{fpt}_x(\mathfrak{a}) \leq c^{\mathfrak{p}}(\mathfrak{a})$. On the other hand, if $\tau(\mathfrak{a}^\lambda) \subseteq \mathfrak{p}$, then it follows from Remark 3.49 and Proposition 3.50ii) that

$$c^{\mathfrak{p}}(\mathfrak{a}) \leq c^{\tau(\mathfrak{a}^\lambda)}(\mathfrak{a}) \leq \lambda,$$

hence $\mathrm{fpt}_x(\mathfrak{a}) \geq c^{\mathfrak{p}}(\mathfrak{a})$.

In particular, this shows that if $x \in V(\mathfrak{a})$, then $\mathrm{fpt}_x(\mathfrak{a}) < \infty$. We also note that since test ideals commute with localization (see Proposition 3.19), we have $\mathrm{fpt}_x(\mathfrak{a}) = \mathrm{fpt}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathfrak{a}R_{\mathfrak{p}})$, and thus

$$\mathrm{fpt}_x(\mathfrak{a}) = c^{\mathfrak{p}R_{\mathfrak{p}}}(\mathfrak{a}R_{\mathfrak{p}}).$$

EXAMPLE 3.53. If R is a domain and $\mathfrak{a} \subseteq R$ defines a regular subscheme of pure codimension r, then $\mathrm{fpt}(\mathfrak{a}) = r$ and $\tau(\mathfrak{a}^\lambda) = \mathfrak{a}^{\lfloor \lambda - r + 1 \rfloor}$ for all $\lambda \geq r$. Indeed, it is enough to prove these assertions after localizing at each point in $V(\mathfrak{a})$, hence we may assume that $(R, \mathfrak{m}, k)$ is local and $\mathfrak{a} \subseteq \mathfrak{m}$. Since $\mathfrak{a}$ defines a regular subscheme of codimension r, there is a regular system of parameters $x_1, \ldots, x_n$ of R such that $\mathfrak{a} = (x_1, \ldots, x_r)$. Using the fact that $\widehat{R} = k[\![x_1, \ldots, x_n]\!]$, it is easy to see that a monomial $x_1^{a_1}\cdots x_n^{a_n}$ lies in $\mathfrak{m}^{[p^e]}=(x_1^{p^e},\ldots,x_n^{p^e})$ if and only if $i_\alpha\geq p^e$ for some α, with $1\leq\alpha\leq n$. This implies that $\mathfrak{a}^s\subseteq\mathfrak{m}^{[p^e]}$ if and only if $s\geq r(p^e-1)+1$. Therefore $\nu_{\mathfrak{a}}^{\mathfrak{m}}(p^e)=r(p^e-1)$ and thus

$$\mathrm{fpt}(\mathfrak{a})=c^{\mathfrak{m}}(\mathfrak{a})=\lim_{e\to\infty}\frac{r(p^e-1)}{p^e}=r.$$

The formula for $\tau(\mathfrak{a}^\lambda)$ now follows from Theorem 3.34.

Remark 3.54. If X is a regular F-finite scheme and $\mathfrak{a}$ is a proper, everywhere nonzero ideal that is locally generated by r elements, then $\mathrm{fpt}(\mathfrak{a})\leq r$. Indeed, it is enough to prove that $\mathrm{fpt}_x(\mathfrak{a})\leq r$ for every $x\in V(\mathfrak{a})$. After replacing X by $\mathrm{Spec}(\mathcal{O}_{X,x})$, we may assume that $X=\mathrm{Spec}(R)$, where $(R,\mathfrak{m})$ and $\mathfrak{a}\subseteq\mathfrak{m}$. Arguing as in Remark 3.47, we see that $\mu_{\mathfrak{a}}^{\mathfrak{m}}(p^e)\leq r(p^e-1)$ for every $e\geq 1$. Dividing by p^e and taking the limit, we conclude that $\mathrm{fpt}(\mathfrak{a})=c^{\mathfrak{m}}(\mathfrak{a})\leq r$.

Lemma 3.55. *Let $\mathfrak{a}$ be an ideal in R that is everywhere nonzero. If J is a proper ideal of R such that $\mathfrak{a}\subseteq\mathrm{rad}(J)$, then*

$$c^{J^{[p]}}(\mathfrak{a})=p\cdot c^J(\mathfrak{a}).$$

Proof. Since $\mathfrak{a}\subseteq\mathrm{rad}(J^{[p]})=\mathrm{rad}(J)$, it follows that $c^{J^{[p]}}(\mathfrak{a})$ is well-defined. It follows from definition that for every $e\geq 1$, we have

$$\nu_{\mathfrak{a}}^{J^{[p]}}(p^e)=\nu_{\mathfrak{a}}^J(p^{e+1}).$$

Dividing by p^e and taking the limit when e goes to infinity, gives the equality in the lemma. $\square$

The following consequence is one of the special features of F-jumping numbers by comparison with the jumping numbers for test ideals.

Corollary 3.56. If X is a regular F-finite scheme of characteristic $p>0$ and $\mathfrak{a}$ is an ideal in $\mathcal{O}_X$ that is everywhere nonzero, then the set of F-jumping numbers of $\mathfrak{a}$ is closed under multiplication by p.

Proof. After taking a suitable affine open cover of X, we may assume that X is affine. In this case the assertion follows by combining Corollary 3.51 and Lemma 3.55. $\square$

Remark 3.57. For locally principal ideals, the F-threshold $c^J(\mathfrak{a})$ determines all numbers $\nu_{\mathfrak{a}}^J(p^e)$. Indeed, suppose that $\mathfrak{a}$ is a locally principal ideal in R that is everywhere nonzero and J is a proper ideal such that $\mathfrak{a}\subseteq\mathrm{rad}(J)$. In this case, for every $e\geq 1$, we have

$$\frac{\nu_{\mathfrak{a}}^J(p^{e+1})+1}{p^{e+1}}\leq\frac{\nu_{\mathfrak{a}}^J(p^e)+1}{p^e}. \tag{3.19}$$

Indeed, note that $\nu_{\mathfrak{a}}^J(p^e)+1$ is the smallest r such that $\mathfrak{a}^r\subseteq J^{[p^e]}$ and since $\mathfrak{a}$ is locally principal, if $\mathfrak{a}^r\subseteq J^{[p^e]}$, then $\mathfrak{a}^{pr}\subseteq J^{[p^{e+1}]}$. It thus follows from (3.19) that

$$\inf_{e\geq 1}\frac{\nu_{\mathfrak{a}}^J(p^e)+1}{p^e}=\lim_{e\to\infty}\frac{\nu_{\mathfrak{a}}^J(p^e)+1}{p^e}=c^J(\mathfrak{a}).$$

Using also Remark 3.48, we thus see that for every $e\geq 1$, we have

$$\frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}<c^J(\mathfrak{a})\leq\frac{\nu_{\mathfrak{a}}^J(p^e)+1}{p^e}$$

and thus

$$\nu_{\mathfrak{a}}^J(p^e) = \lceil c^J(\mathfrak{a})p^e \rceil - 1.$$

REMARK 3.58. Recall that in the context of multiplier ideals there isn't a big difference between invariants of principal ideals and invariants of arbitrary ideals (see Proposition 2.59). For example, if X is a smooth affine variety in characteristic 0, any $c \in (0,1)$ is the log canonical threshold of an ideal on X if and only if it is the F-pure threshold of a principal ideal on X. On the other hand, in positive characteristic there is a significant difference between invariants of arbitrary ideals and invariants of principal ideals. One such instance is given in Remark 3.57. Another one is given by the following restriction on F-pure thresholds of principal ideals: if R is an F-finite regular ring of characteristic $p > 0$ and $f \in R$ is not a zero-divisor, then for every positive integers a and e, with $a \leq p^e - 1$ we have

$$\mathrm{fpt}(f) \notin \left(\frac{a}{p^e}, \frac{a}{p^e - 1} \right).$$

Indeed, if $c = \mathrm{fpt}(f) \in \left(\frac{a}{p^e}, \frac{a}{p^e-1} \right)$, then

$$a < p^e c < a + c.$$

By combining Corollary 3.56 and Theorem 3.34, we see that since c is an F-jumping number of f, we conclude that also $cp^e - a$ is a jumping number of f. However, it lies in $(0, c)$, contradicting the fact that $\mathrm{fpt}(f)$ is the smallest F-jumping number of f.

3.4.2. Discreteness and rationality of F-jumping numbers.

Our goal in this section is to prove the following result from [**BMS08**] concerning F-jumping numbers for schemes essentially of finite type over a field.

THEOREM 3.59. *Let k be an F-finite field of characteristic $p > 0$ and X a regular scheme essentially of finite type over k. If $\mathfrak{a}$ is an ideal in $\mathcal{O}_X$ that is everywhere nonzero, then the set of F-jumping numbers of $\mathfrak{a}$ is a discrete set*[1] *of rational numbers.*

For principal ideals, discreteness and rationality of F-jumping numbers for *any* F-finite regular scheme was proved in [**BMS09**]. These results were extended (replacing regular schemes by normal, $\mathbf{Q}$-Gorenstein schemes, with index non-divisible by p) in [**BSTZ10**].

In the case of affine space, we will deduce discreteness from the following more precise result:

PROPOSITION 3.60. Let k be an F-finite field of characteristic $p > 0$. If $\mathfrak{a}$ is a nonzero ideal in $R = k[x_1, \ldots, x_n]$ that is generated by polynomials of degree $\leq d$, then for every $\lambda \in \mathbf{R}_{\geq 0}$, the test ideal $\tau(\mathfrak{a}^\lambda)$ is generated by polynomials of degree $\leq \lfloor \lambda d \rfloor$.

PROOF. Let us fix a positive integer e and let us estimate the degrees of the generators of $(\mathfrak{a}^r)^{[1/p^e]}$.

Let $a_1, \ldots, a_m$ be a basis of k over k^{p^e}, so a basis of R over R^{p^e} is given by $Q_{u,i} = a_i x^u$, where $1 \leq i \leq m$ and $u = (u_1, \ldots, u_n) \in \mathbf{Z}^n_{\geq 0}$ with $0 \leq u_j \leq p^e - 1$ for all j (where $x^u = x_1^{u_1} \cdots x_n^{u_n}$). Since $\mathfrak{a}$ is generated by polynomials of degree $\leq d$,

[1] This means that in every bounded interval there are only finitely many elements of this set.

it follows that $\mathfrak{a}^r$ is generated by polynomials of degree $\leq dr$. Let us choose such generators $f_1, \ldots, f_s$ for $\mathfrak{a}^{\lceil \lambda p^e \rceil}$. If we write

$$f_j = \sum_{i,u} P_{j,u,i}^{p^e} Q_{u,i} \quad \text{for} \quad 1 \leq j \leq s, \tag{3.20}$$

with $P_{j,u,i} \in R$, then it follows from Proposition 3.15 that $(\mathfrak{a}^r)^{[1/p^e]}$ is generated by the $P_{j,u,i}$. Note also that equation (3.20) implies that for every i, j, and u, we have $\deg(P_{j,u,i}) \leq dr/p^e$.

By definition of test ideals, if $e \gg 0$, then $\tau(\mathfrak{a}^\lambda) = \left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]}$, hence by the previous discussion, $\tau(\mathfrak{a}^\lambda)$ is generated by polynomials of degree $\leq \lfloor dr/p^e \rfloor$, where $r = \lceil \lambda p^e \rceil$. For $e \gg 0$, we have

$$\frac{dr}{p^e} = \frac{d\lceil \lambda p^e \rceil}{p^e} < \frac{d(\lambda p^e + 1)}{p^e} = d\lambda + \frac{d}{p^e} < \lfloor d\lambda \rfloor + 1,$$

and thus $\tau(\mathfrak{a}^\lambda)$ is generated by polynomials of degree $\leq \lfloor \lambda d \rfloor$. $\square$

The next proposition will allow us to reduce the proof of Theorem 3.59 to the case of the affine space.

Proposition 3.61. Let Y be a regular, connected F-finite scheme of characteristic $p > 0$ and X a regular closed subscheme of pure codimension r. If $\mathfrak{a}$ is a nonzero ideal in $\mathcal{O}_X$ and $\mathfrak{b}$ is its inverse image in $\mathcal{O}_Y$, then $\tau(\mathfrak{b}^\lambda) = \mathcal{O}_Y$ for $\lambda < r$ and

$$\tau(\mathfrak{b}^\lambda) \cdot \mathcal{O}_X = \tau(\mathfrak{a}^{\lambda - r}) \quad \text{for all} \quad \lambda \geq r.$$

Proof. It is enough to show that the assertions hold after localizing and completing at each $x \in X$. Since taking test ideals commutes with localization and completion (see Proposition 3.19 and Exercise 3.20), it follows that we may assume that $Y = \mathrm{Spec}(R)$, where R is a complete local ring. Since Y is a regular scheme and X is a regular closed subscheme of Y, it follows that we may assume that $R = k[\![x_1, \ldots, x_n]\!]$ and X is defined by $(x_1 \ldots, x_r)$. Note that since k is a quotient of R, it is F-finite.

Arguing by induction on r, we see that it is enough to treat the case when $r = 1$. If $\mathfrak{a} \subseteq S = R/(x_1) = k[\![x_2, \ldots, x_n]\!]$, we can write $\mathfrak{b} = \widetilde{\mathfrak{a}} + (x_1)$, where $\widetilde{\mathfrak{a}} = \mathfrak{a} \cdot R$ (via the obvious injective homomorphism $S \to R$). It follows from Theorem 3.37 that for every $\lambda \in \mathbf{R}_{\geq 0}$, we have

$$\tau(\mathfrak{b}^\lambda) = \sum_{\alpha + \beta = \lambda} \tau\left(\widetilde{\mathfrak{a}}^\alpha x_1^\beta\right).$$

Moreover, a similar argument to that used in the proof of Lemma 3.33, together with Example 3.53, give

$$\tau(\widetilde{\mathfrak{a}}^\alpha x_1^\beta) = \tau(\mathfrak{a}^\alpha) \cdot (x_1^{\lfloor \beta \rfloor}).$$

It is then clear that $\tau(\mathfrak{b}^\lambda) = R$ if $\lambda < 1$ and for $\lambda \geq 1$ we have

$$\tau(\mathfrak{b}^\lambda) \cdot R/(x_1) = \sum_{\alpha + \beta = \lambda, \beta < 1} \tau(\mathfrak{a}^\alpha) \cdot R/(x_1) = \tau(\mathfrak{a}^{\lambda - 1}),$$

where for the last equality we use the fact that by Proposition 3.24, the largest element among the ideals $\tau(\mathfrak{a}^\alpha)$ with $\alpha > \lambda - 1$ is $\tau(\mathfrak{a}^{\lambda - 1})$. $\square$

We can now prove the main result of this section.

Proof of Theorem 3.59. If $X = U_1 \cup \ldots \cup U_r$ is an affine open cover, then the set of F-jumping numbers of $\mathfrak{a}$ is the union of the sets of jumping numbers of $\mathfrak{a}|_{U_i}$. Therefore it is enough to prove the theorem when $X = \mathrm{Spec}(R)$ is affine and connected (hence irreducible). By assumption, we can write $R = S^{-1}T$, where T is a k-algebra of finite type and S is a multiplicative system in T. If we write $\mathfrak{a} = S^{-1}\mathfrak{b}$ for an ideal $\mathfrak{b}$ in T, then it follows from Proposition 3.19 that the F-jumping numbers of $\mathfrak{a}$ are among the F-jumping numbers of $\mathfrak{b}$. Therefore we may and will assume that R is a k-algebra of finite type.

Let us consider a closed immersion $X \hookrightarrow Y = \mathbf{A}_k^n$, for some n, and let $c = \mathrm{codim}_Y(X)$. If $\mathfrak{b}$ is the inverse image of $\mathfrak{a}$ in $\mathcal{O}_Y$, then it follows from Proposition 3.61 that if λ is an F-jumping number of $\mathfrak{a}$, then $\lambda + c$ is an F-jumping number of $\mathfrak{b}$. Therefore it is enough to know the assertion of the theorem for $\mathfrak{b}$, hence we may and will assume that $X = \mathbf{A}_k^n$.

Let r and d be such that $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ can be generated by r polynomials of degrees $\leq d$. We first show that for every M, the set of F-jumping numbers of $\mathfrak{a}$ in $[0, M]$ is finite. For every $\lambda \in [0, M]$, let V_λ be the intersection of $\tau(\mathfrak{a}^\lambda)$ with the vector space W of polynomials in $k[x_1, \ldots, x_n]$ of degree $\leq \lfloor dM \rfloor$.

If $0 \leq \lambda \leq \mu \leq M$, we have $\tau(\mathfrak{a}^\mu) \subseteq \tau(\mathfrak{a}^\lambda)$ and thus $V_\mu \subseteq V_\lambda$. Moreover, Proposition 3.61 implies that for every $\lambda \leq M$, the ideal $\tau(\mathfrak{a}^\lambda)$ is generated by V_λ. Therefore λ is a jumping number if and only if $V_\lambda \subsetneq V_{\lambda'}$ for every $\lambda' < \lambda$. Since $\dim_k(W) < \infty$, it follows that the number of F-jumping numbers of $\mathfrak{a}$ in $[0, M]$ is $\leq \dim_k(W) = \binom{\lfloor dM \rfloor + n}{n}$.

We next prove that every F-jumping number of $\mathfrak{a}$ is rational. This follows from the discreteness of this set, together with the following two properties:

i) If λ is an F-jumping number of $\mathfrak{a}$, then $p\lambda$ too is an F-jumping number of $\mathfrak{a}$ (see Corollary 3.56).
ii) If $\lambda > r$ is an F-jumping number of $\mathfrak{a}$, then $\lambda - 1$ too is an F-jumping number of $\mathfrak{a}$ (this is a consequence of Theorem 3.34).

For any real number β we define $\{\beta\}$ as follows: if $\beta > r$, then $\{\beta\}$ is the unique real number in $(r-1, r]$ such that $\beta - \{\beta\} \in \mathbf{Z}$; on the other hand, if $\beta \leq r$, we put $\{\beta\} = \beta$. Then it follows from properties i) and ii) above that for every $e \geq 1$, we know that $\{p^e\lambda\}$ is an F-jumping number of $\mathfrak{a}$ in the interval $(0, r]$. Since we already know that this set of F-jumping numbers has $\leq \dim_k(W)$ elements, it follows that there are $1 \leq e_1 < e_2 \leq \dim_k(W) + 1$ such that $p^{e_2}\lambda - p^{e_1}\lambda \in \mathbf{Z}$. We deduce that $\lambda \in \mathbf{Q}$, completing the proof of the theorem. □

Remark 3.62. The argument in the proof of Theorem 3.59 shows that for every prime p and every positive integers r, d, and n, there is a positive integer $N = N(p, r, d, n)$ such that for every F-finite field k of characteristic $p > 0$ and every nonzero ideal $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ generated by $\leq r$ polynomials of degree $\leq d$, all F-jumping numbers of $\mathfrak{a}$ lie in $\frac{1}{N}\mathbf{Z}_{\geq 0}$. Indeed, note first that it follows from Theorem 3.34 that if $\lambda > r$ is an F-jumping number of $\mathfrak{a}$, then $\lambda - 1$ is an F-jumping number of $\mathfrak{a}$ as well. Therefore it is enough to only consider the F-jumping numbers $\leq r$. If we run the argument in the proof of Theorem 3.59 with $M = r$, we see that for every F-jumping number λ of $\mathfrak{a}$ in $(0, r]$, there are a and b with $a + b \leq \dim_k(W)$ such that $p^a(p^b - 1)\lambda \in \mathbf{Z}$. Since $\dim_k(W) = \binom{\lfloor dr \rfloor + n}{n}$, we see that the $p^a(p^b - 1)$ that appear can take only finitely many values. If we take N to be the least common multiple of these values, this satisfies our condition.

EXERCISE 3.63. Let X be a regular F-finite scheme of positive characteristic. Show that if $\mathfrak{a}$ is an everywhere nonzero ideal on X and $x \in X$ is a point with $\mathrm{ord}_x(\mathfrak{a}) = d$ and $\dim(\mathcal{O}_{X,x}) = n$, then

$$\frac{1}{d} \leq \mathrm{fpt}_x(\mathfrak{a}) \leq \frac{n}{d}.$$

EXERCISE 3.64. Let X be a regular F-finite scheme of positive characteristic. Show that $\mathfrak{a}$ and $\mathfrak{b}$ are everywhere nonzero ideals on X, then for every $x \in X$, we have

$$\mathrm{fpt}_x(\mathfrak{a}+\mathfrak{b}) \leq \mathrm{fpt}_x(\mathfrak{a}) + \mathrm{fpt}_x(\mathfrak{b}).$$

3.5. Test ideals and F-pure thresholds: examples

In this chapter we discuss some computations of test ideals and F-pure thresholds. We will see that while there are many parallels with the characteristic 0 invariants, there are some new arithmetic phenomena that come in the picture.

REMARK 3.65. If X is a regular F-finite scheme and $\mathfrak{a}$ is a proper ideal of $\mathcal{O}_X$ that is everywhere nonzero and locally generated by r-elements and whose zero-locus is nonempty, then it follows from Theorem 3.34 that $\mathrm{fpt}(\mathfrak{a}) \leq r$. In particular, if $\mathfrak{a}$ is locally principal, we see that $\mathrm{fpt}(\mathfrak{a}) \leq 1$.

REMARK 3.66. Let k be an F-finite field of characteristic 0 and let us consider on $R = k[x_1, \ldots, x_n]$ the grading given by $\deg(x_i) = a_i \in \mathbf{Z}_{>0}$ for all i. If $0 \neq f \in R$ is homogeneous with respect to this grading (that is, all monomials in f have the same degree), then every ideal $(f^r)^{[1/p^e]}$ is homogeneous: by Proposition 3.15, this is generated by polynomials of the form $\sum_{i\in I} c_i x^{(u_i - v)/p^e}$, where the x^{u_i} are monomials that appear in f^r and v is a fixed monomial (of degree $\leq p^e - 1$ in each variable); since the monomials x^{u_i} have the same degree, it follows that the monomials $x^{(u_i-v)/p^e}$ have the same degree. We thus deduce from the definition of test ideals that all $\tau(\mathfrak{a}^\lambda)$ are homogeneous. In particular, we have $\tau(f^\lambda) = R$ if and only if this equality holds in a neighborhood of 0. Therefore we have $\mathrm{fpt}(f) = \mathrm{fpt}_0(f)$.

EXAMPLE 3.67. Let k be an F-finite field of characteristic $p > 0$ and $f = x_1^{a_1} \cdots x_r^{a_r} \in k[x_1, \ldots, x_n]$ for some $r \leq n$, where $a_1, \ldots, a_r$ are positive integers. We claim that

$$\mathrm{fpt}(f) = \mathrm{fpt}_0(f) = \min_{1\leq i\leq r} \frac{1}{a_i}.$$

Indeed, let $\mathfrak{m} = (x_1, \ldots, x_n)$. For every $e \geq 1$, we have $f^r \in \mathfrak{m}^{[p^e]} = (x_1^{p^e}, \ldots, x_n^{p^e})$ if and only if $ra_i \geq p^e$ for some i, with $1 \leq i \leq r$. We thus conclude that

$$\nu_f^{\mathfrak{m}}(p^e) = \min_{1\leq i\leq r} \lceil p^e/a_i \rceil - 1.$$

We thus conclude that

$$\mathrm{fpt}_0(f) = c^{\mathfrak{m}}(f) = \lim_{e\to\infty} \frac{\min_{1\leq i\leq r}\lceil p^e/a_i\rceil - 1}{p^e} = \min_{1\leq i\leq r} \frac{1}{a_i}.$$

The fact that $\mathrm{fpt}(f) = \mathrm{fpt}_0(f)$ follows Remark 3.66, since f is a homogeneous polynomial. We will generalize this example to a computation of test ideals of arbitrary monomial ideals in Proposition 3.79 below.

EXAMPLE 3.68. Let k be an algebraically closed field of characteristic $p > 0$, X a smooth surface over k, and $\mathfrak{a} \subseteq \mathcal{O}_X$ the ideal defining a curve in X, having at most nodes as singularities. In this case $\mathrm{fpt}(\mathfrak{a}) = 1$. Indeed, since test ideals commute with localization and completion (see Proposition 3.19 and Exercise 3.20), this follows from the fact that the F-pure threshold of an ideal defining a smooth hypersurface is 1 (see Example 3.53) and $\mathrm{fpt}(f) = 1$ if $f = xy \in k[x, y]$ (see Example 3.67).

A useful result for certain computations of F-pure thresholds is the following theorem of Lucas (see [**Luc78**] and also [**Gra97**]). If p is a positive prime integer and if we write two positive integers a and b is base-p expansion $a = \sum_{i=0}^r a_i p^i$ and $b = \sum_{i=0}^r b_i p^i$ (that is, we have $a_i, b_i \in \{0, 1, \ldots, p-1\}$ for all i), then

$$\binom{m}{n} \equiv \prod_{i=0}^{r} \binom{a_i}{b_i} \pmod p.$$

As usual, we use the convention that $\binom{m}{n} = 0$ if $m < n$. The interesting consequence for us is that $\binom{m}{n} \not\equiv 0 \pmod p$ if and only if $b_i \leq a_i$ for all i.

EXAMPLE 3.69. Let k be an F-finite field of characteristic $p > 0$ and let $f = x^2 + y^3 \in k[x, y]$. Our goal is to compute $\mathrm{fpt}_0(f)$. For simplicity, we write $\nu(p^e)$ for $\nu_f^{\mathfrak{m}}(p^e)$, where $\mathfrak{m} = (x, y)$.

Let's start by computing $\nu(p)$, which is the largest r such that $f^r \notin (x^p, y^p)$. Of course, we need to have $r \leq p - 1$, hence in the binomial expansion

$$f^r = \sum_{i+j=r}^{r} \binom{r}{i} x^{2i} y^{3j},$$

all binomial coefficients are nonzero in k. Therefore $f^r \notin (x^p, y^p)$ if and only if there are i and j with $i + j = r$ such that $2i \leq p - 1$ and $3j \leq p - 1$. We thus conclude that

$$\nu(p) = \lfloor (p-1)/2 \rfloor + \lfloor (p-1)/3 \rfloor. \tag{3.21}$$

For simplicity, let's assume from now on that $p > 3$. As suggested by the formula for $\nu(p)$, we need to distinguish two cases depending on the congruence class of p mod 3.

We first show that

$$\nu(p^e) = \frac{5}{6}(p^e - 1) \quad \text{for} \quad e \geq 1,\ p \equiv 1 \pmod 3. \tag{3.22}$$

The case $e = 1$ is covered by (3.21), but we now give an argument that is valid for all e. Since $\frac{5}{6}(p^e - 1) = \frac{p^e-1}{2} + \frac{p^e-1}{3}$, it follows that in the binomial expansion of $f^{5(p^e-1)/6}$ we have the term

$$\binom{5(p^e-1)/6}{(p^e-1)/2} x^{p^e-1} y^{p^e-1}. \tag{3.23}$$

Since we have the decomposition

$$\frac{5(p^e-1)}{6} = \sum_{i=0}^{e-1} \frac{5(p-1)}{6} p^e \quad \text{and} \quad \frac{p^e-1}{2} = \sum_{i=0}^{e-1} \frac{p-1}{2} p^i,$$

it follows from Lucas' Theorem that the coefficient in (3.23) is nonzero. Therefore $f^{5(p^e-1)/6} \notin (x^p, y^p)$ and thus $\nu(p^e) \geq \frac{5}{6}(p^e - 1)$. The fact that this is an equality follows from the fact that if $r > \frac{5}{6}(p^e-1)$ and $i+j=r$, then either $2i \geq p^e$ or $3j \geq p^e$. We thus have (3.22).

We next show that

$$\nu(p^e) = \frac{5p^e - p^{e-1}}{6} - 1 \quad \text{for} \quad e \geq 2,\ p \equiv 2\,(\mathrm{mod}\,3). \tag{3.24}$$

Note that it follows from Remark 3.57 that for every $e \geq 1$, we have

$$\nu(p^{e+1}) \leq p \cdot \nu(p^e) + p - 1.$$

Using the formula for $\nu(p)$ when $p \equiv 2$ (mod 3), this implies that for every $e \geq 2$, we have

$$\nu(p^e) \leq \nu(p) \cdot p^{e-1} + \sum_{i=0}^{e-2}(p-1)p^i = \frac{5p-7}{6}p^{e-1} + p^{e-1} - 1 = \frac{5p^e - p^{e-1}}{6} - 1.$$

In order to show that this is an equality, it is enough to show that if we put $a = \frac{5p^e - p^{e-1}}{6} - 1$ and $b = \frac{p^e-1}{2}$ (so that $a - b = \frac{2p^e - p^{e-1} - 3}{6}$), then the coefficient of the term

$$\binom{a}{b} x^{2b} y^{3(a-b)} = \binom{a}{b} x^{p^e-1} y^{p^e - \frac{p^{e-1}+3}{2}}$$

does not vanish. This follows from Lucas' Theorem, using the base-p expansions:

$$a = \frac{5p-7}{6}p^{e-1} + \sum_{i=0}^{e-2}(p-1)p^i \quad \text{and} \quad b = \sum_{i=0}^{e-1}\frac{p-1}{2}p^i.$$

We thus conclude that

$$\mathrm{fpt}_0(x^2+y^3) = \begin{cases} \frac{5}{6}, & \text{if } p \equiv 1\,(\mathrm{mod}\,3); \\ \frac{5}{6} - \frac{1}{6p}, & \text{if } p \equiv 2\,(\mathrm{mod}\,3),\ p > 2. \end{cases}$$

One should compare this with the formula for the log canonical threshold of the cusp: over a field of characteristic 0 (in fact, over *any* field), we have $\mathrm{lct}_0(x^2+y^3) = \frac{5}{6}$ (see Example 2.53).

The case of F-pure thresholds of diagonal hypersurfaces has been systematically studied in [**Her15**].

Exercise 3.70. With the notation in Example 3.69, show that if $p=2$, then $\mathrm{fpt}_0(f) = \frac{1}{2}$ and if $p=3$, then $\mathrm{fpt}_0(f) = \frac{2}{3}$. Show that in general, if $c = \mathrm{fpt}_0(f)$, then $\tau(f^\lambda) = (x,y)$ for $\lambda \in [c,1)$.

Before we discuss the next example, we give a result characterizing principal ideals with F-pure threshold 1.

Proposition 3.71. Let $(R,\mathfrak{m})$ be a local regular F-finite ring of characteristic 0. If $f \in \mathfrak{m}$ is nonzero, then the following are equivalent:

i) $\mathrm{fpt}(f) = 1$.
ii) For every $e \geq 1$, we have $f^{p^e-1} \not\in \mathfrak{m}^{[p^e]}$ (equivalently, $\nu_f^{\mathfrak{m}}(p^e) = p^e - 1$).
iii) We have $f^{p-1} \not\in \mathfrak{m}^{[p]}$ (equivalently, $\nu_f^{\mathfrak{m}}(p) = p-1$).

Proof. The implication i)⇒ii) follows from the equality $\mathrm{fpt}(f) = c^{\mathfrak{m}}(f)$ (see Remark 3.52) and Remark 3.57: indeed, if $c^{\mathfrak{m}}(f) = 1$, then $\nu_f^{\mathfrak{m}}(e) = p^e - 1$ for all $e \geq 1$, hence $f^{p^e-1} \not\in \mathfrak{m}^{[p^e]}$. The implication ii)⇒iii) is trivial, hence let's prove iii)$\Rightarrow$i). If $\mathrm{fpt}(f) < 1$, then it follows from Remark 3.58 that $\mathrm{fpt}(f) \leq 1 - \frac{1}{p}$. Using again Remark 3.57, we conclude that

$$\nu_f^{\mathfrak{m}}(p) = \lceil p \cdot \mathrm{fpt}(f) \rceil - 1 \leq p - 2,$$

contradicting iii). This completes the proof. □

EXAMPLE 3.72. Let k be an F-finite field of characteristic $p > 0$, $n \geq 3$, and $f \in R = k[x_1, \ldots, x_n]$ homogeneous of degree $d \leq n$ that defines a smooth hypersurface in $\mathbf{P}^{n-1}$. Recall that if we are over a field of characteristic 0 (in fact, this remains true over *any* field), the log canonical threshold of such f is 1 (see Example 2.54). The case when $d < n$ is easier: we will see in Remark 3.78 below that $\mathrm{fpt}(f) = 1$ if p is large enough (depending on n). From now on we assume $d = n$, which is the interesting case.

It follows from Proposition 3.71 that $\mathrm{fpt}(f) = 1$ if and only if $f^{p-1} \not\in (x_1^p, \ldots, x_n^p)$ (note that since f is homogeneous, we have $\mathrm{fpt}(f) = \mathrm{fpt}_0(f)$ by Remark 3.66). Note that all monomials in f^{p-1} have degree $n(p-1)$ and there is only one such monomial which is not in $(x_1^p, \ldots, x_n^p)$, namely $(x_1 \cdots x_n)^{p-1}$. We thus see that $\mathrm{fpt}(f) = 1$ if and only if $(x_1 \cdots x_n)^{p-1}$ appears with nonzero coefficient in f^{p-1}.

We next give a cohomological description of this condition. Note that if X is any scheme of characteristic p, the Frobenius morphism $F_X \colon X \to X$ induces a map, the *Frobenius map on cohomology*

$$F \colon H^i(X, \mathcal{O}_X) \to H^i\big(X, (F_X)_*\mathcal{O}_X\big) = H^i(X, \mathcal{O}_X),$$

where the equality follows from the fact that F_X is an affine morphism. If X is a scheme over k, then $H^i(X, \mathcal{O}_X)$ is a k-vector space, but the map F is not linear, but p-*linear*: it satisfies $F(au) = a^p F(u)$ for every $a \in k$.

Returning to our set-up, consider the hypersurface X in $\mathbf{P}_k^{n-1}$ defined by f. Recall that $H^i(X, \mathcal{O}_X) = 0$ for $1 \leq i < \dim(X) = n-2$ and $H^{n-2}(X, \mathcal{O}_X) \simeq k$. Let us describe the action of F on $H^{n-2}(X, \mathcal{O}_X)$. Recall that we have an isomorphism

$$H^{n-2}(X, \mathcal{O}_X) \simeq H_{\mathfrak{m}}^{n-1}\big(R/(f)\big)_0,$$

where on the right-hand side we have the $(n-1)$ local cohomology group of $R/(f)$ with respect to the ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. Moreover, this is compatible with the Frobenius action, where on the right-hand side the action is induced by $F_{R/(f)}$.

Note that we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & R(-n) \arrow[r, "\cdot f"] \arrow[d, "F'"'] & R \arrow[r] \arrow[d, "F_R"'] & R/(f) \arrow[r] \arrow[d, "F_{R/(f)}"'] & 0 \\ 0 \arrow[r] & R(-n) \arrow[r, "\cdot f"] & R \arrow[r] & R/(f) \arrow[r] & 0, \end{tikzcd}$$

where $F'(u) = f^{p-1}F_R(u) = f^{p-1}u^p$. The long exact sequence for local cohomology gives an isomorphism

$$H_{\mathfrak{m}}^{n-1}\big(R/(f)\big) \simeq \{u \in H_{\mathfrak{m}}^n(R)(-n) \mid fu = 0\},$$

such that the action of Frobenius on the left corresponds to the action induced by F' on the right. We thus get an isomorphism

$$H^{n-2}(X, \mathcal{O}_X) \simeq H_{\mathfrak{m}}^{n-1}\big(R/(f)\big)_0 \simeq \{u \in H_{\mathfrak{m}}^n(R)_{-n} \mid fu = 0\} = k \cdot \left[\tfrac{1}{x_1 \cdots x_n}\right],$$

where the last equality follows from the fact that

$$H^n_{\mathfrak{m}}(R) = R_{x_1\cdots x_n}/\sum_{i=1}^n R_{x_1\cdots\widehat{x_i}\cdots x_n} = \bigoplus_{a_1,\ldots,a_n\geq 1} k\left[\tfrac{1}{x_1^{a_1}\cdots x_n^{a_n}}\right].$$

Moreover, the Frobenius action maps the generator $\left[\frac{1}{x_1\cdots x_n}\right]$ to $\left[\frac{f^{p-1}}{(x_1\cdots x_n)^{p^e}}\right]$. We see that this is 0 precisely when $f^{p-1}\in(x_1^p,\ldots,x_n^p)$, that is, when $\mathrm{fpt}(f)<1$. If this is not the case, then after base-change to a perfect field, the Frobenius action on $H^n(X,\mathcal{O}_X)$ is bijective. In the former case, we call X *supersingular*, while in the latter case we call X *ordinary*, extending classical terminology from the case $n=3$, when X is an elliptic curve.

We next describe the possible values of the F-pure threshold in the case of affine cones over smooth Calabi-Yau hypersurfaces in projective space. Let $R=k[x_1,\ldots,x_n]$, where k is an F-finite field of characteristic $p>0$ and $\mathfrak{m}=(x_1,\ldots,x_n)$. For a homogeneous polynomial $f\in R$ of degree $d\geq 2$, we consider the Jacobian ideal $J_f=\left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right)$ of f. Note that if k is perfect, that f defines a smooth hypersurface in $\mathbf{P}^{n-1}$ if and only if $V(f,J_f)=\{0\}$; if p does not divide d, then Euler's formula implies $f\in J_f$, hence this condition is equivalent with J_f being $\mathfrak{m}$-primary. The following result is due to Bhatt and Singh [**BS15**].

THEOREM 3.73. *Let k be an F-finite field of characteristic $p>0$. If $0\neq f\in R=k[x_1,\ldots,x_n]$, with $p\geq n-2\geq 4$, is a homogeneous polynomial of degree n such that J_f is $\mathfrak{m}$-primary, then* $\mathrm{fpt}(f)=1-\frac{h}{p}$ *for some integer h, with $0\leq h\leq\min\{n-2,p-1\}$.*

In fact, it is shown in [**BS15**] that if $p\geq n^2-n-1$, then the integer h in the theorem admits a geometric interpretation, being the order of vanishing of the so-called Hasse invariant on the versal deformation space of $V(f)\subseteq\mathbf{P}^{n-1}$. We also note that a generalization of Theorem 3.73 to weighted homogeneous Calabi-Yau hypersurfaces was obtained in [**Mül18**].

EXAMPLE 3.74. It is shown in [**Her15**, Corollary 3.5] that if $f=x_1^n+\ldots+x_n^n$ and we have $p>n$ and $p\equiv h+1\ (\mathrm{mod}\, n)$, where $0\leq h\leq n-2$, then $\mathrm{fpt}(f)=1-\frac{h}{p}$. In particular, for every $n\geq 3$ and every h with $0\leq h\leq n-2$ such that $h+1$ is relatively prime to n, there are infinitely many primes p such that $\mathrm{fpt}(f)=1-\frac{h}{p}$ (this follows from Dirichlet's theorem on primes in arithmetic progressions).

We begin with some easy lemmas.

LEMMA 3.75. *If $f\in R$ is a homogeneous polynomial of degree $d\geq 2$ such that J_f is an $\mathfrak{m}$-primary ideal, then $\mathfrak{m}^{(d-2)n+1}\subseteq J_f$.*

PROOF. Since J_f is an $\mathfrak{m}$-primary ideal generated by n elements, it follows that these elements form a regular sequence. Since $\deg(\partial f/\partial x_i)=d-1$ for all i, it follows that the Hilbert series of R/J_f is

$$H_{R/J_f}(t)=\frac{(1-t^{d-1})^n}{(1-t)^n}=(1+t+\ldots+t^{d-2})^n.$$

In particular, we see that $(R/J_f)_i=0$ for $i\geq(d-2)n+1$, which gives the assertion in the lemma. $\square$

LEMMA 3.76. *For every $k \in \mathbf{Z}_{\geq 0}$, we have*

$$\mathfrak{m}^{[q]} : \mathfrak{m}^k = \mathfrak{m}^{[q]} + \mathfrak{m}^{nq-n-k+1},$$

with the convention that $\mathfrak{m}^j = R$ if $j < 0$.

PROOF. The inclusion "$\supseteq$" follows from the fact that $\mathfrak{m}^{n(q-1)+1} \subseteq \mathfrak{m}^{[q]}$. In order to prove the prove the reverse inclusion, we may and will assume that we have $n(q-1) - k \geq 0$. Note first that the ideal $\mathfrak{m}^{[q]} : \mathfrak{m}^k$ is a monomial ideal, being an intersection of monomial ideals. Suppose that $x^u \not\in \mathfrak{m}^{[q]}$ is such that $x^u \cdot \mathfrak{m}^k \in \mathfrak{m}^{[q]}$. Consider the element

$$w = \left[\frac{x^u}{x_1^q \cdots x_n^q}\right] \in H^n_{\mathfrak{m}}(R).$$

It is straightforward to see that for every nonzero monomial $w \in H^n_{\mathfrak{m}}(R)_m$, where $m < -n$, there is i such that $x_i w$ is nonzero. The hypothesis on x^u implies that $w \neq 0$, but $\mathfrak{m}^k \cdot w = 0$; therefore we have $w \in H^n_{\mathfrak{m}}(R)_{\geq 1-n-k}$. Therefore $\deg(x^u) \geq nq - n - k + 1$. This completes the proof. ☐

LEMMA 3.77. *Suppose that $f \in R$ is a homogeneous polynomial of degree $d \geq 2$ such that J_f is $\mathfrak{m}$-primary. Let $q = p^e$, for some $e \geq 1$, and let r be the largest integer such that $f^r \not\in \mathfrak{m}^{[q]}$. If p does not divide $r+1$, then*

$$r \geq \frac{n(q+1)}{d} - n.$$

PROOF. By hypothesis, we have $f^{r+1} \in \mathfrak{m}^{[q]}$. Note that the ideal $\mathfrak{m}^{[q]}$ is closed under the action of the partial derivatives ∂_{x_i}, hence

$$(r+1) f^r \frac{\partial f}{\partial x_i} \in \mathfrak{m}^{[q]} \quad \text{for} \quad 1 \leq i \leq n.$$

By assumption, we have $r + 1 \neq 0$ in k, hence we deduce $f^r \cdot J_f \subseteq \mathfrak{m}^{[q]}$. Since $\mathfrak{m}^{(d-2)n+1} \subseteq J_f$ by Lemma 3.75, we obtain

$$f^r \cdot \mathfrak{m}^{(d-2)n+1} \subseteq \mathfrak{m}^{[q]},$$

hence $f^r \in \mathfrak{m}^{[q]} + \mathfrak{m}^N$ by Lemma 3.76, where

$$N = nq - n - (d-2)n = n(q-d+1).$$

This implies that $rd \geq N$: indeed, if we write $f^r = g + h$, with $g \in \mathfrak{m}^{[q]}$ and $h \in \mathfrak{m}^N$, since f^r is homogeneous and $\mathfrak{m}^{[q]}$ and $\mathfrak{m}^N$ are homogeneous ideals, we may assume that g and h are homogeneous of degree $\deg(f^r) = rd$. Since $f^r \not\in \mathfrak{m}^{[q]}$, we have $h \neq 0$ and thus $h \in \mathfrak{m}^N$ implies $rd \geq N$. An easy computation then gives the inequality in the statement. ☐

PROOF OF THEOREM 3.73. As we have already discussed, since $p > n$ and f defines a smooth hypersurface in $\mathbf{P}^{n-1}$, the ideal J_f is $\mathfrak{m}$-primary. We also note that since f is homogeneous, it follows from Remarks 3.66 and 3.52 that $\mathrm{fpt}(f) = \mathrm{fpt}_0(f) = c^{\mathfrak{m}}(f)$. We put $\nu = \nu_f^{\mathfrak{m}}$ and begin by considering $\nu(p) \in \{0, \ldots, p-1\}$. Since $d = n$, it follows from Lemma 3.77 that if $\nu(p) \neq p-1$, then $\nu(p) \geq p - n + 1$. We thus see that in any case, we have $\nu(p) = p - 1 - h$, with $0 \leq h \leq \min\{n-2, p-1\}$.

If $h = 0$, then it follows from Proposition 3.71 that $\mathrm{fpt}(f) = 1$, hence we are done. We next assume that $h \geq 1$ and prove by induction that

$$\nu(p^e) = p^{e-1}(p-h) - 1 \quad \text{for all} \quad e \geq 1. \tag{3.25}$$

For $e = 1$ this is clear. Suppose now that we know (3.25) for e and let's deduce it for $e + 1$. We have seen in Remark 3.57 that

$$\nu(p^{e+1}) = p \cdot \nu(p^e) + j \quad \text{for some} \quad j,\ 0 \le j \le p - 1. \tag{3.26}$$

Ij $j \neq p - 1$, then it follows from Lemma 3.77 that

$$\nu(p^{e+1}) \geq p^{e+1} - n + 1.$$

On the other hand, it follows from (3.25) and (3.26) that

$$\nu(p^{e+1}) \leq p \cdot \nu(p^e) + p - 2 = p^e(p - h) - p + p - 2 = p^e(p - h) - 2.$$

By combining these two inequalities, we get

$$p^{e+1} - n + 1 \leq p^e(p - h) - 2,$$

hence $n - 3 \geq p^e h \geq p$ (recall that we assume $h \geq 1$), a contradiction with our hypothesis.

We thus conclude that $j = p - 1$, hence (3.25) gives

$$\nu(p^{e+1}) = p^e(p - h) - p + p - 1 = p^e(p - h) - 1,$$

which completes the proof of the induction step. We conclude using (3.25) that

$$c^{\mathfrak{m}}(f) = \lim_{e \to \infty} \frac{p^{e-1}(p - h) - 1}{p^e} = 1 - \frac{h}{p}.$$

This completes the proof of the theorem. $\square$

Remark 3.78. Similarly, it follows from Lemma 3.77 that if $f \in k[x_1, \ldots, x_n]$ is homogeneous with $\deg(f) = d < n$ such that J_f is $\mathfrak{m}$-primary and we have $p > n^2 - 4n + 2$, then $\mathrm{fpt}(f) = 1$. Indeed, if $\mathrm{fpt}(f) = \mathrm{fpt}_0(f) < 1$, then it follows from Proposition 3.71 that $\nu_f^{\mathfrak{m}}(p) \leq p - 2$. On the other hand, we may apply Lemma 3.77 to get

$$\nu_f^{\mathfrak{m}}(p) \geq \frac{n(p+1)}{d} - n \geq \frac{n(p+1)}{n-1} - n > p - 2,$$

where the last inequality follows easily from our lower bound on p. This contradiction implies that $\mathrm{fpt}(f) = 1$.

We end this section with a result due to Hara and Yoshida [**HY03**] which gives the analogue of Howald's Theorem 2.60 in the setting of test ideals. We use the notation related to monomial ideals that was introduced before Theorem 2.60.

Proposition 3.79. Let k be an F-finite field of characteristic $p > 0$. If $\mathfrak{a} \subseteq R = k[x_1, \ldots, x_n]$ is a monomial ideal, then for every $\lambda \in \mathbf{R}_{\geq 0}$, we have

$$\tau(\mathfrak{a}^\lambda) = \big(x^u \mid u + \mathbf{1} \in \mathrm{Int}\big(\lambda \cdot P(\mathfrak{a})\big)\big).$$

Proof. Note that if $\mathfrak{b} \subseteq R$ is any monomial ideal, then it follows from Proposition 3.15 that for every $e \geq 1$, the ideal $\mathfrak{b}^{[1/p^e]}$ is monomial as well; in fact, we have

$$\mathfrak{b}^{[1/p^e]} = (x^{\lfloor (1/p^e)u \rfloor} \mid x^u \in \mathfrak{b}), \tag{3.27}$$

where for $v = (v_1, \ldots, v_n) \in \mathbf{R}^n$, we put $\lfloor u \rfloor = \big(\lfloor u_1 \rfloor, \ldots, \lfloor u_n \rfloor\big)$. On the other hand, it is clear that since $\mathfrak{a}$ is a monomial ideal, then every $\mathfrak{a}^{\lceil \lambda p^e \rceil}$ is a monomial ideal. Therefore it follows from the definition of $\tau(\mathfrak{a}^\lambda)$ that this is a monomial ideal. In order to prove the proposition, we just need to show that

$$x^u \in \tau(\mathfrak{a}^\lambda) \quad \text{if and only if} \quad u + \mathbf{1} \in \mathrm{Int}\big(\lambda \cdot P(\mathfrak{a})\big). \tag{3.28}$$

Suppose first that $x^u \in \tau(\mathfrak{a}^\lambda)$, so that for $e \gg 0$, we have $x^u \in \big(\mathfrak{a}^{\lceil \lambda p^e \rceil}\big)^{[1/p^e]}$. It follows from (3.27) that there are $v_1, \ldots, v_N$ with $x^{v_i} \in \mathfrak{a}$ and $N = \lceil \lambda p^e \rceil$ such that if $v = \sum_i v_i$, we have $u - \lfloor (1/p^e)v \rfloor \in \mathbf{Z}^n_{\geq 0}$. Since $v_i \in P(\mathfrak{a})$ for all i, we have $v \in \lceil \lambda p^e \rceil \cdot P(\mathfrak{a})$ and thus $(1/p^e)v \in \lambda \cdot P(\mathfrak{a})$. Since $\big(\lfloor (1/p^e)v \rfloor + \mathbf{1}\big) - (1/p^e)v \in \mathbf{R}^n_{>0}$, it follows that

$$u+\mathbf{1} = \big(u - \lfloor (1/p^e)v \rfloor\big) + \big(\lfloor (1/p^e)v \rfloor + \mathbf{1} - (1/p^e)v\big) + (1/p^e)v \in \lambda \cdot P(\mathfrak{a}) + \mathbf{R}^n_{>0} \subseteq \mathrm{Int}\big(\lambda \cdot P(\mathfrak{a})\big).$$

Conversely, suppose that $u \in \mathbf{Z}^n_{\geq 0}$ is such that $u + \mathbf{1} \in \mathrm{Int}\big(\lambda \cdot P(\mathfrak{a})\big)$ and we want to show that $x^u \in \tau(\mathfrak{a}^\lambda)$. By Proposition 2.106, there is a positive integer m such that $\overline{\mathfrak{a}^i} \subseteq \mathfrak{a}^{i-m}$ for all $i \geq m$. The hypothesis on u implies that there are $\epsilon_1, \epsilon_2 > 0$ such that $u + \mathbf{1} \in \epsilon_1 \cdot \mathbf{1} + (\lambda + \epsilon_2) \cdot P(\mathfrak{a})$. Let $e \gg 0$ be such that $p^e \epsilon_1 \geq 1$ and $p^e(\lambda + \epsilon_2) \geq \lceil \lambda p^e \rceil + m$. Therefore we can write

$$p^e(u + \mathbf{1}) - \mathbf{1} = w, \quad \text{for some} \quad w \in \mathbf{Z}^n_{\geq 0} \cap \big(\lceil \lambda p^e \rceil + m\big) \cdot P(\mathfrak{a}). \tag{3.29}$$

It follows from Example 2.103 that

$$w \in \overline{\mathfrak{a}^{\lceil \lambda p^e \rceil + m}} \subseteq \mathfrak{a}^{\lceil \lambda p^e \rceil},$$

hence $x^w \in \mathfrak{a}^{\lceil \lambda p^e \rceil}$. Since it follows from (3.29) that $\lfloor (1/p^e)w \rfloor = u$, we conclude that

$$x^u \in \big(\mathfrak{a}^{\lceil \lambda p^e \rceil}\big)^{[1/p^e]} \subseteq \tau(\mathfrak{a}^\lambda).$$

This completes the proof. $\square$

3.6. Further properties and examples

In this section we discuss further properties of test ideals, focussing on those that are somewhat peculiar by comparison with what happens for multiplier ideals in characteristic 0. We begin with the following easy proposition which shows that for principal ideals, the test ideals with exponents having the denominator a power of p are easy to describe.

Proposition 3.80. Let R be a regular F-finite ring of characteristic $p > 0$. If $\mathfrak{a}$ is a locally principal ideal in R that is everywhere nonzero, then for every $m \in \mathbf{Z}_{\geq 0}$, we have

$$\tau(\mathfrak{a}^{m/p^e}) = (\mathfrak{a}^m)^{[1/p^e]}.$$

Proof. Given $e' \geq e$, since $\mathfrak{a}$ is locally principal, the ideal $\mathfrak{a}^m$ is locally principal as well, and thus

$$\mathfrak{a}^{mp^{e'-e}} = (\mathfrak{a}^m)^{[p^{e'-e}]}.$$

It follows that if $\lambda = m/p^e$, then

$$\big(\mathfrak{a}^{\lceil \lambda p^{e'} \rceil}\big)^{[1/p^{e'}]} = \big(\mathfrak{a}^{mp^{e'-e}}\big)^{[1/p^{e'}]} = \big((\mathfrak{a}^m)^{[p^{e'-e}]}\big)^{[1/p^{e'}]} = (\mathfrak{a}^m)^{[1/p^e]},$$

where the last equality follows from Proposition 3.13v). Since this holds for all $e' \geq e$, we obtain the formula in the proposition using the definition of $\tau(\mathfrak{a}^\lambda)$. $\square$

Recall that in characteristic 0, multiplier ideals are integrally closed (see Remark 2.100). On the other hand, the next proposition from [**MY09**] shows that in positive characteristic, under rather mild conditions, every ideal is a test ideal.

Proposition 3.81. If R is a regular domain of characteristic $p>0$ such that R is a finitely generated free module over R^p, then for every nonzero ideal $\mathfrak{a}$ in R, there is a nonzero $f\in R$ and $\lambda\in \mathbf{Q}_{\geq 0}$ such that $\mathfrak{a}=\tau(f^{\lambda})$.

Proof. Let $N=\mathrm{rank}_{R^p}(R)$. If $N=1$, then R is a field, in which case the assertion in the proposition is trivial. Hence from now on we assume that $N>1$. Note that for every $e\geq 1$, R is free over R^{p^e} of rank N^e.

Let $f_1,\ldots,f_m$ be generators of $\mathfrak{a}$ and let $e\geq 1$ be such that $N^e\geq m$. If $g_1,\ldots,g_{N^e}$ is a basis of R over R^{p^e}, let

$$f=\sum_{i=1}^m f_i^{p^e}g_i \quad\text{and}\quad \lambda=\frac{1}{p^e}.$$

Since $\mathfrak{a}\neq 0$, it follows that some f_i is nonzero, and thus $f\neq 0$. We deduce using Propositions 3.80 and 3.15

$$\tau(f^{\lambda})=(f)^{[1/p^e]}=\mathfrak{a}.$$

$\square$

Unlike in the case of multiplier ideals, computing test ideals in positive characteristic (say, of principal ideals in a polynomial ring) is conceptually rather simple. We now explain how this can be done based on our results so far, though this is not very efficient in practice.

Remark 3.82. Suppose that $f\in R=k[x_1,\ldots,x_n]$ is nonzero, where k is an F-finite field of characteristic $p>0$. Note that by Theorem 3.34, we only need to understand the test ideals $\tau(f^{\lambda})$ with $\lambda\leq 1$ and only the F-jumping numbers in the interval $(0,1]$.

Let $d=\deg(f)$. It follows from Remark 3.62 that if we take $N=N(p,1,d,n)$, then every F-jumping number of f is of the form $\frac{i}{N}$ for some $i\in\mathbf{Z}_{>0}$. If e is such that $p^e>N$, then it follows that for every i, with $1\leq i\leq N$, if we take $a_i=\lfloor p^e i/N\rfloor$, then

$$\frac{i-1}{N}<\frac{a_i}{p^e}\leq\frac{i}{N}.$$

By Proposition 3.80, for every i, we have $\tau(f^{a_i/p^e})=(f^{a_i})^{[1/p^e]}$ and this can be easily computed via Proposition 3.15. We then conclude that $\tau(f^{a_i/p^e})=\tau(f^{\lambda})$ for all $\lambda\in\left[\frac{i-1}{N},\frac{i}{N}\right)$. Moreover, $\frac{i}{N}$ is an F-jumping number if and only if $\tau(f^{a_i/p^e})\neq\tau(f^{a_{i+1}/p^e})$.

We next turn to two other instances of peculiar behavior of test ideals in positive characteristic. While we have seen in Theorem 3.59 that the F-jumping numbers of an ideal form a discrete set of rational numbers, the following example from [**Per13**] shows that the picture is considerably more complicated when considering the constancy regions of mixed test ideals.

Example 3.83. Let k be a field of characteristic $p>2$ and let $f_1,\ f_2\in R=k[x,y]$, with $f_1=x+y$ and $f_2=xy$. We consider some mixed test ideals $\tau(f_1^{\lambda_1}f_2^{\lambda_2})$

with $\lambda_1+\lambda_2=1$. We first note that the argument in the proof of Proposition 3.80 extends to give that for every $a,b\in\mathbf{Z}_{\geq 0}$ and every $e\geq 1$, we have

$$\tau(f_1^{a/p^e}f_2^{b/p^e})=(f_1^af_2^b)^{[1/p^e]}.$$

Using Proposition 3.15, we get

$$\tau\big(f_1^{\frac{1}{p^e}}f_2^{1-\frac{1}{p^e}}\big)=\big((x+y)(xy)^{p^e-1}\big)^{[1/p^e]}=(x^{p^e}y^{p^e-1}+x^{p^e-1}y^{p^e})^{[1/p^e]}=(x,y)$$

while

$$\tau\big(f_1^{\frac{2}{p^e}}f_2^{1-\frac{2}{p^e}}\big)=\big((x+y)^2(xy)^{p^e-2}\big)^{[1/p^e]}=(x^{p^e}y^{p^e-2}+2(xy)^{p^e-1}+x^{p^e-2}y^{p^e})^{[1/p^e]}=R.$$

Since for every $\frac{1}{p^e}$ there is some $\frac{2}{p^{e'}}<\frac{1}{p^e}$, it follows that we can't write the segment

$$\{(\lambda_1,\lambda_2)\mid\lambda_1,\lambda_2\geq 0,\lambda_1+\lambda_2=1\}$$

as a finite union of segments such that on the interior of each of these the test ideal is constant. For a detailed analysis of the constancy regions of these test ideals in the case when $k=\mathbf{F}_2$, we refer to [**Per13**, Example 5.3]. What can be said in general for mixed test ideals is that there is a *p-fractal map* such that the constancy regions are the fibers of this map, see [**Per13**, Theorem 4.6].

The next example from [**MY09**] shows that for test ideals we don't have an analogue of Proposition 2.93; that is, test ideals do not commute with restriction to general fibers in a family. However, it is now understood that in order to study test ideals in families one can define a relative version of test ideals on the total space of the family, see [**PSZ18**], though we do not discuss this.

EXAMPLE 3.84. Let k be an algebraically closed field of characteristic $p>0$ and $f\colon X=\mathbf{A}_k^3\to T=\mathbf{A}_k^1$ corresponding to the homomorphism $k[y]\hookrightarrow k[x_1,x_2,y]$. Consider on X the ideal $\mathfrak{a}=(f)$, where $f=x_1^p+x_2^py$. In this case it follows from Propositions 3.80 and 3.15 that

$$\tau(\mathfrak{a}^{1/p})=\mathfrak{a}^{[1/p]}=(x_1,x_2).$$

On the other hand, for every closed point $t\in T$, if we consider the fiber $X_t=f^{-1}(t)$ and the restriction $\mathfrak{a}_t=\mathfrak{a}\cdot\mathcal{O}_{X_t}$, we see that if $s\in k$ is such that $s^p=t$, then $\mathfrak{a}_t$ is generated by $(x_1+sx_2)^p$, hence

$$\tau(\mathfrak{a}_t^{1/p})=\tau(x_1+sx_2)=(x_1+sx_2).$$

In particular, we see that $\tau(\mathfrak{a}_t^{1/p})\neq\tau(\mathfrak{a}^{1/p})\cdot\mathcal{O}_{X_t}$ for all closed points $t\in T$.

We end this section with a result from [**MY09**] showing that test ideals still behave well in families in the sense that the analogue of Corollary 2.95 holds, that is, the F-pure threshold is lower semicontinuous.

THEOREM 3.85. *Let k be an F-finite field of characteristic $p>0$. Suppose that $f\colon X\to T$ is a morphism of regular schemes of finite type over k such that all fibers of f are regular, of pure dimension d, and $\mathfrak{a}$ is an ideal on X such that the restriction $\mathfrak{a}_t$ of $\mathfrak{a}$ to $X_t=f^{-1}(t)$ is everywhere nonzero for every $t\in T$. If $s\colon T\to X$ is such that $\pi\circ s=\mathrm{id}_T$, then for every $\lambda\in\mathbf{R}_{\geq 0}$, the set*

$$W_\lambda:=\{t\in T\mid \mathrm{fpt}_{s(t)}(\mathfrak{a}_t)\geq\lambda\}$$

is open in T.

PROOF. Note first that we may and will assume that X and T are affine. Indeed, for every $t \in T$, we can choose an affine open neighborhood V of t and an affine open neighborhood $U \subseteq f^{-1}(V)$ of $s(t)$. If $V_h \subseteq s^{-1}(U)$ is a principal affine open subset of V containing t, then $U \cap f^{-1}(V_h)$ is an affine open neighborhood of $s(t)$ and f and s induce maps $U \cap f^{-1}(V_h) \to V_h$ and $V_h \to U \cap f^{-1}(V_h)$. Since it is enough to show that for every such choices $W_\lambda \cap V_h$ is open, we may and will assume that $T = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(S)$, for finite type k-algebras R and S, that are integral domains.

If we consider a surjective R-algebra homomorphism $R[x_1, \ldots, x_n] \to S$ and if $\mathfrak{b} \subseteq R[x_1, \ldots, x_n]$ is the inverse image of $\mathfrak{a}$, then it follows from Proposition 3.61 that for every $t \in T$ and $x \in X_t$, we have $\mathrm{fpt}_x(\mathfrak{a}_t) = \mathrm{fpt}_x(\mathfrak{b}_t) - c$, where $c = \mathrm{codim}_{\mathbf{A}^n_R}(X)$. Therefore we may and will assume that $S = R[x_1, \ldots, x_n]$. Finally, if $\varphi\colon R[x_1, \ldots, x_n] \to R$ is the homomorphism corresponding to s and $\varphi(x_i) = b_i$, we have an isomorphism of R-algebras $\rho\colon S \to S$, with $\rho(x_i) = x_i + b_i$. After replacing $\mathfrak{a}$ by $\rho(\mathfrak{a})$, we may thus assume that $b_i = 0$ for all i, that is, s maps any $t \in T$ to $0 \in \mathbf{A}^n_{k(t)}$.

Note now that the set

$$\Lambda := \{\mathrm{fpt}_0(\mathfrak{a}_t) \mid t \in T\}$$

is a finite set. Indeed, if $\mathfrak{a} \subseteq S$ is generated by r polynomials of degree $\leq d$, then for every $t \in T$ corresponding to $\mathfrak{p} \subseteq R$, the ideal $\mathfrak{a}_t \subseteq k(\mathfrak{p})[x_1, \ldots, x_n]$ is generated by r polynomials of degree $\leq d$ and $\mathrm{fpt}_0(\mathfrak{a})$ is an F-jumping number of $\mathfrak{a}_t$ (though possibly not the F-pure threshold), which is $\leq r$ (see Remark 3.65). Therefore the finiteness statement follows from Remark 3.62.

We may thus choose $\lambda' \in \Lambda$ that is largest with the property that $\lambda' < \lambda$, so that

$$W_\lambda = \{t \in T \mid \mathrm{fpt}_0(\mathfrak{a}_t) > \lambda'\}.$$

On the other hand, it follows from Remark 3.52 that if $t \in T$ corresponds to the prime ideal $\mathfrak{q} \subseteq R$, then $\mathrm{fpt}_0(\mathfrak{a}_t) \leq \lambda'$ if and only if

$$\mathfrak{a}^{\lfloor \lambda' p^e \rfloor + 1} \cdot k(\mathfrak{q})[x_1, \ldots, x_n] \subseteq (x_1^{p^e}, \ldots, x_n^{p^e}) k(\mathfrak{q})[x_1, \ldots, x_n] \quad \text{for all} \quad e \geq 1.$$

This condition says that for every $e \geq 1$ and every $g \in \mathfrak{a}^{\lfloor \lambda' p^e \rfloor + 1}$, the prime ideal $\mathfrak{q}$ contains all coefficients in g for the monomials $x_1^{a_1} \cdots x_n^{a_n}$ with $a_i \leq p^e - 1$ for all i. This clearly defines a closed subset of T and thus W_λ is an open set. This completes the proof of the theorem. $\square$

CHAPTER 4

Comparison between multiplier and test ideals

Suppose that $\mathfrak{a}$ is an ideal on a smooth variety in characteristic 0. Our goal is to relate via reduction to positive characteristic the multiplier ideals of $\mathfrak{a}$ and the test ideals of the reduction of $\mathfrak{a}$. We begin by reviewing the general framework for reduction mod p.

4.1. Reduction to positive characteristic

Let k be a fixed field of characteristic 0. Consider the set $\mathrm{FG}_{\mathbf{Z}}(k)$ of finite type $\mathbf{Z}$-subalgebras of k, which is a filtering set with respect to the order given by inclusion. Note that by definition, every element of $\mathrm{FG}_{\mathbf{Z}}(k)$ is a domain and that if $A \in \mathrm{FG}_{\mathbf{Z}}(k)$, then $A_a \in \mathrm{FG}_{\mathbf{Z}}(k)$ for every nonzero $a \in A$.

Given an algebra R of finite type over k, we can find $A \in \mathrm{FG}_{\mathbf{Z}}(k)$ and a flat A-algebra of finite type R_A, together with an isomorphism

$$\varphi_A^R \colon R_A \otimes_A k \xrightarrow{\sim} R.$$

Indeed, if we choose an isomorphism $R \simeq k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, we may take any $A \in \mathrm{FG}_{\mathbf{Z}}(k)$ that contains all coefficients of $f_1, \ldots, f_m$ and define $R_A = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, with φ_A^R being the induced isomorphism. It is a consequence of Generic Flatness (see [**Eis95**, Theorem 14.4]) that after possibly replacing A by a localization A_a, with $a \in A \smallsetminus \{0\}$, we may assume that R_A is flat over A. We call such R_A a *model* of R over A.

Given such A, R_A, and φ_A^R, for every $B \in \mathrm{FG}_{\mathbf{Z}}(k)$ with $A \subseteq B$, we get a model of R over B by taking $R_B = R_A \otimes_A B$ and φ_B^R the induced isomorphism. In particular, given finitely many k-algebras $R_1, \ldots, R_d$, we can choose one A such that we have a model $(R_i)_A$ over A for each R_i.

Suppose that R_A is a model of R over A. Note that φ_A^R induces a ring homomorphism $R_A \to R$; for every $b \in R$, we can find $B \in \mathrm{FG}_{\mathbf{Z}}(k)$ containing A, such that after replacing A by B and R_A by $R_B = R_A \otimes_A B$, u lies in the image of R_B. Indeed, if $u = \sum_{i=1}^r \varphi_R^A(a_i \otimes b_i)$, it is enough to choose B such that $A[b_1, \ldots, b_r] \subseteq B$.

Suppose now that $f \colon R \to S$ is a morphism of k-algebras of finite type and that we have $A \in \mathrm{FG}_{\mathbf{Z}}(k)$ and models R_A and S_A for R and S, respectively, over A. After possibly replacing A by some $B \in \mathrm{FG}_{\mathbf{Z}}(k)$ containing A and R_A and S_A by $R_B = R_A \otimes_A B$ and $S_B = S_A \otimes_A B$, respectively, we may assume that there is an A-algebra homomorphism $f_A \colon R_A \to S_A$ such that $f \otimes_A k$ is equal to f via the identifications given by φ_A^R and φ_A^S (such u_A is a *model* of u over A). Indeed, if we choose a surjective A-algebra homomorphism $A[x_1, \ldots, x_n]$, with kernel $(g_1, \ldots, g_m)$, we may first enlarge A so that for every i with $1 \leq i \leq n$, the image of x_i in S is equal to the image of some $v_i \in S_A$. Since for every j with $1 \leq j \leq m$ the image of $g_j(v_1, \ldots, v_n)$ in S is 0 and the map $S_A \otimes \mathrm{Frac}(A) \to S_A \otimes_A k$ is injective, it follows that after possibly replacing A with some A_a, for some nonzero $a \in A$, we may assume that $g_j(v_1, \ldots, v_n) = 0$ for all j. Therefore the homomorphism $A[x_1, \ldots, x_n] \to S_A$ that maps each x_i to v_i induces the homomorphism $u_A \colon R_A \to S_A$ with the required property. We similarly see that if $f_A, f'_A \colon R_A \to S_A$ both induce f, then they are equal as morphisms $R_A \otimes_A A_a \to S_A \otimes_A A_a$ for some nonzero $a \in A$. This implies that given finitely many commutative diagrams of k-algebra morphisms, after the localization of A at a suitable element, we may assume that we have a similar commutative diagram of A-algebra homomorphisms between the corresponding models over A. In particular, this shows that an isomorphism between finitely generated k-algebras can be lifted to an isomorphism between the corresponding models.

A similar construction works for modules, though we now skip some of the details. Suppose that R is a finitely generated k-algebra and M is a finitely generated R-module. A *model* of M over A is given by a model R_A of R over A and by a finitely generated R_A-module M_A, flat over A, together with an isomorphism

$$\varphi_A^M \colon M_A \otimes_{R_A} R = M_A \otimes_A k \xrightarrow{\sim} M.$$

In order to find such R_A and M_A we choose an isomorphism

$$k[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \simeq R$$

as before, as well as a presentation $M \simeq \mathrm{Coker}(T \colon R^{\oplus r} \to R^{\oplus s})$. We also choose polynomials $g_{i,j} \in k[x_1, \ldots, x_n]$ that map to the entries of the matrix defining T. If $A \in \mathrm{FG}_{\mathbf{Z}}(k)$ contains all coefficients of the f_i and of the $g_{i,j}$, then we can define R_A as before and take M_A to be the cokernel of the map defined by the matrix with entries the images of the $g_{i,j}$ in R_A. It is a consequence of Generic Flatness that after replacing A by some A_a, with $a \in A$ nonzero, we may assume that both R_A and M_A are flat over A. As before, we see that given finitely many such pairs (R_i, M_i), we can find one A such that each (R_i, M_i) has a model over A.

Arguing as before, we see that given a morphism of finitely generated R-modules $u \colon M \to N$ and models R_A, M_A, and N_A for R, M, and N, respectively, over A, after possibly replacing A by a larger element of $\mathrm{FG}_{\mathbf{Z}}(k)$, we may assume that there is an R_A-linear map $u_A \colon M_A \to N_A$ such that $u_A \otimes_A k = u$ via the identifications given by φ_A^M and φ_A^N (such a map u_A is a *model* of u over A). If two morphisms $u_A, u'_A \colon M_A \to N_A$ are models of u, then after replacing A by a suitable A_a, we get $u_A = u'_A$. We deduce that given finitely many commutative diagrams of R-modules, after the localization of A at a suitable nonzero element, we may assume that we have corresponding commutative diagrams of R_A-linear maps between the corresponding A-models. In particular, this shows that an isomorphisms of R-modules can be lifted to isomorphisms between the corresponding models.

We can now globalize this construction. Given a scheme[1] X of finite type over k, we can find $A \in \mathrm{FG}_{\mathbf{Z}}(k)$ and a separated scheme X_A flat and of finite type over A, together with an isomorphism $\varphi_A^X \colon X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \to X$. Indeed, we can choose an affine open cover $X = U_1 \cup \ldots \cup U_n$ and find $A \in \mathrm{FG}_{\mathbf{Z}}(k)$ such that we have over A models for the k-algebras $\mathcal{O}_X(U_i)$ and $\mathcal{O}_X(U_i \cap U_j)$, as well as for the glueing morphisms (note that each $U_i \cap U_j$ is affine since we assumed that X is separated). By gluing these in the obvious way we get X_A (note that X_A is separated by construction). We call X_A a *model* of X over A. If $B \in \mathrm{FG}_{\mathbf{Z}}(k)$ is such

[1] In this section we assume that all schemes are separated.

that $A \subseteq B$, then we get a model of X over B by taking $X_B = X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ and φ_B^X to be the induced isomorphism. This implies that given finitely many such schemes $X_1, \ldots, X_n$, we can find one A and models $(X_i)_A$ of X_i over A for each i. If $f\colon X \to Y$ is a morphism of schemes of finite type over k, then we can find $A \in \mathrm{FG}_{\mathbf{Z}}(k)$ together with models X_A and Y_A for X and Y, respectively, such that we have a morphism $f_A\colon X_A \to Y_A$ of schemes over A which induces f after base-change to k via the identifications given by φ_A^X and φ_A^Y. Furthermore, if f'_A is a morphism with the same properties, then f_A and f'_A become equal after base change to $\mathrm{Spec}(A_a)$ for a suitable nonzero $a \in A$. The previous considerations regarding commutative diagrams extend to this global case.

Similarly, if $\mathcal{F}$ is a coherent sheaf on X, then we can find $A \in \mathrm{FG}_{\mathbf{Z}}(k)$ such that we have a model X_A of X over A and a coherent sheaf $\mathcal{F}_A$ on X_A, flat over A, together with an isomorphism $\varphi_A^{\mathcal{F}}\colon \nu^*(\mathcal{F}_A) \to \mathcal{F}$, where $\nu\colon X \to X_A$ is the canonical morphism induced by φ_A^X (in this case $\mathcal{F}_A$ is a *model* of $\mathcal{F}$). Moreover, given given models $\mathcal{F}_A$ and $\mathcal{G}_A$ of coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, respectively, over A, and a morphism of coherent sheaves $u\colon \mathcal{F} \to \mathcal{G}$ on X, after possibly replacing A by a suitable localization A_a, there is a morphism of coherent sheaves $u_A\colon \mathcal{F}_A \to \mathcal{G}_A$ on X_A that pulls-back to u via base-change to $\mathrm{Spec}(k)$ (this is a *model* of u over A). Again, if u'_A is another such model of u over A, then $u_A = u'_A$ after base-change to $\mathrm{Spec}(A_a)$ for a suitable nonzero $a \in A$. Therefore finitely many commutative diagrams of morphisms of coherent sheaves give commutative diagrams between the corresponding models over a suitable element of $\mathrm{FG}_{\mathbf{Z}}(k)$.

If X_A is a model of X over A and $\mathcal{F}_A$ is a model of $\mathcal{F}$, then for a point $s \in \mathrm{Spec}(A)$ we denote by X_s the fiber of X_A over s and by $\mathcal{F}_s$ the restriction of $\mathcal{F}$ to X_s. Note that X_s is a scheme of finite type over the residue field $k(s)$ of s. If s is the generic point of $\mathrm{Spec}(A)$, then $k(s)$ is a subfield of k. We will be interested in the case when s is a closed point of $\mathrm{Spec}(A)$; then $k(s)$ is a finite field by the following well-known lemma.

LEMMA 4.1. *If $\mathfrak{m}$ is a maximal ideal in a finitely generated $\mathbf{Z}$-algebra A, then $A/\mathfrak{m}$ is a finite field.*

PROOF. The intersection $\mathfrak{m} \cap \mathbf{Z}$ is a prime ideal in $\mathbf{Z}$. If this is equal to $p\mathbf{Z}$, for some prime p, then $A/\mathfrak{m}$ is an algebra of finite type over the finite field $\mathbf{F}_p$. By Nullstellensatz, we have $[A/\mathfrak{m} : \mathbf{F}_p] < \infty$, hence $A/\mathfrak{m}$ is a finite field.

Let's suppose now that $\mathfrak{m} \cap \mathbf{Z} = 0$ and show that we get a contradiction. Let us write $K = A/\mathfrak{m}$. By assumption, we have $\mathbf{Q} \subseteq K$ and K is a finitely generated $\mathbf{Z}$-algebra, hence also a finitely generated $\mathbf{Q}$-algebra. By Nullstellensatz, we have $[K : \mathbf{Q}] < \infty$, hence if $a_1, \ldots, a_n$ generate K as a $\mathbf{Z}$-algebra, then each a_i is algebraic over $\mathbf{Q}$. If we consider the minimal polynomials of $a_1, \ldots, a_n$ over $\mathbf{Q}$ and d is a positive integer such that all coefficients of these minimal polynomials lie in $\mathbf{Z}[1/d]$, then it follows that the ring extension $\mathbf{Z}[1/d] \hookrightarrow K$ is integral. Since K is a field, we deduce that $\mathbf{Z}[1/d]$ is a field, which is a contradiction (a prime in $\mathbf{Z}$ that does not divide d has no inverse in $\mathbf{Z}[1/d]$). This completes the proof. $\square$

REMARK 4.2. Suppose now that $u\colon \mathcal{F} \to \mathcal{G}$ is a morphism of coherent sheaves on X and let us choose models X_A, $\mathcal{F}_A$, $\mathcal{G}_A$, and u_A for X, $\mathcal{F}$, $\mathcal{G}$, and u, respectively. If $s \in \mathrm{Spec}(A)$, then we get a morphism $u_s\colon \mathcal{F}_s \to \mathcal{G}_s$. Since we may assume that

$\mathrm{Coker}(u_A)$ and $\mathrm{Im}(u_A)$ are flat over A, it follows that we have

$$\mathrm{Coker}(u_s) = \mathrm{Coker}(u)_s, \quad \mathrm{Im}(u_s) = \mathrm{Im}(u)_s, \quad \text{and} \quad \mathrm{Ker}(u_s) = \mathrm{Ker}(u)_s.$$

In particular, if u is injective or surjective, then so are all u_s. We deduce that if $\mathcal{F}$ is an ideal or it is locally free, then the same holds for all $\mathcal{F}_s$.

REMARK 4.3. Similarly, if $f\colon X \to Y$ is a morphism of schemes of finite type over k and we consider models X_A, Y_A, and f_A over A, then for every $s \in \mathrm{Spec}(A)$, we get a morphism $f_s\colon X_s \to Y_s$. If f has one of the following properties: projective, finite, or it is an open or closed immersion, then we may assume that the same property holds for f_A, and thus for all f_s.

REMARK 4.4. Suppose now that f is projective, $\mathcal{F}$ is a coherent sheaf on X, and we also have a model $\mathcal{F}_A$ of $\mathcal{F}$. Arguing as in [**Har77**, Section III. 12], we see that $\mathcal{F}_A$ satisfies generic base-change: after possibly replacing A by A_a for some nonzero $a \in A$, we may assume that for all $s \in \mathrm{Spec}(A)$, the canonical morphism

$$\big(R^i(f_A)_*(\mathcal{F}_A)\big)_s \to R^i(f_s)_*(\mathcal{F}_s) \tag{4.1}$$

is an isomorphism for all $i \geq 0$.

REMARK 4.5. Given a model X_A of X over A, it is easy to deduce from Noether's Normalization Theorem that we may assume that $\dim(X_s) \leq \dim(X)$ for all $s \in \mathrm{Spec}(A)$. If X is smooth and irreducible over k, then the Jacobian criterion for smoothness implies that we may assume that X_A is smooth over $\mathrm{Spec}(A)$, of relative dimension equal to $\dim(X)$. In particular, each X_s is smooth over $\mathrm{Spec}\big(k(s)\big)$, of pure dimension equal to $\dim(X)$. If k is algebraically closed, since the geometric generic fiber of X_A over $\mathrm{Spec}(A)$ is connected, it follows that all X_s are connected as well.

We will consider properties $\mathcal{P}$ of schemes of finite type over finite fields such that given a scheme W over k, and a finite field extension k' of k, $\mathcal{P}(W)$ holds if and only if $P\big(W \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')\big)$ holds. If X_A is a model of X over A, we say that $\mathcal{P}(X_s)$ *holds for general closed points* in $\mathrm{Spec}(A)$ if there is an open subset U of $\mathrm{Spec}(A)$ such that $\mathcal{P}(X_s)$ holds for all closed points $s \in U$. In this case, after replacing A by a suitable localization A_a, we may assume that $\mathcal{P}(X_s)$ holds for all closed points s. We note that we will also be interested in properties that are expected to only hold for a dense set of closed points $s \in \mathrm{Spec}(A)$.

REMARK 4.6. With $\mathcal{P}$ as above, note that both conditions

i) $\mathcal{P}(X_s)$ holds for general closed points $s \in \mathrm{Spec}(A)$
ii) $\mathcal{P}(X_s)$ holds for a dense set of closed points $s \in \mathrm{Spec}(A)$

are independent of the choice of a model. Indeed, if $\alpha\colon \mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is induced by the inclusion $A \hookrightarrow C$ of finitely generated $\mathbf{Z}$-algebras, then the following hold:

a) By Lemma 4.1, α maps closed points to closed points (and the corresponding morphism between the residue fields is an extension of finite fields).
b) The image of α contains a (dense) open subset.
c) The image or inverse image of a dense subset has the same property.

By combining these facts, we get the independence of model.

REMARK 4.7. Suppose that X_A and $\mathcal{F}_A$ are models over A for the scheme X of finite type over k and for the coherent sheaf $\mathcal{F}$ on X. Note that if $\mathcal{F}_s = 0$ for general closed points $s \in \operatorname{Spec}(A)$, then it follows from Nakayama's Lemma that after possibly doing base-change to some localization A_a, we have $\mathcal{F}_A = 0$. Therefore we conclude that $\mathcal{F} = 0$. In particular, if $f\colon X \to Y$ is a projective morphism and we have a model $f_A\colon X_A \to Y_A$ over A such that for some $i \geq 0$ and general closed points $s \in \operatorname{Spec}(A)$ we have $R^i(f_s)_*(\mathcal{F}_s)$, then $R^i f_*(\mathcal{F}) = 0$ (in other words, we can prove vanishing of higher direct images by reduction to prime characteristic).

We will also need the following uniform vanishing statement in a setting where we deal with infinitely many sheaves at the same time.

LEMMA 4.8. *Let $f\colon Y \to X$ be a projective morphism of schemes of finite type over k, $\mathcal{F}$ a coherent sheaf on Y, and $\mathcal{L} \in \operatorname{Pic}(Y)$ that is f-ample. If $f_A\colon Y_A \to X_A$, $\mathcal{F}_A$, and $\mathcal{L}_A$ are models over $A \in \mathrm{FG}_{\mathbf{Z}}(k)$ of f, $\mathcal{F}$, and $\mathcal{L}$, respectively, then there is m_0 such that after possibly replacing A by the localization at some nonzero element, we have*

$$R^i f_*\big(\mathcal{F}_s \otimes \mathcal{L}_s^m\big) = 0 \quad \textit{for all} \quad s \in \operatorname{Spec}(A),\ i \geq 1,\ m \geq m_0.$$

PROOF. We may and will assume that X and X_A are affine. Furthermore, we may assume that $\mathcal{L}$ is very ample over X: indeed, if m is a positive integer such that $\mathcal{L}^m$ is very ample, it is enough to prove the assertion in the lemma for the line bundle $\mathcal{L}^m$ and each of the sheaves $\mathcal{F} \otimes \mathcal{L}^j$, for $0 \leq j \leq m-1$.

Suppose now that $\mathcal{L}$ and $\mathcal{L}_A$ are very ample. We use asymptotic Serre vanishing to choose m_0 such that $H^i\big(Y_A, \mathcal{F}_A \otimes \mathcal{L}_A^m\big) = 0$ for all $i \geq 1$ and $m \geq m_0$. By Remark 4.4, after possibly inverting a nonzero element in A, we may assume that for every $s \in \operatorname{Spec}(A)$, we have

$$H^i\big(Y_s, \mathcal{F}_s \otimes \mathcal{L}_s^m\big) = 0 \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad m_0 \leq m \leq m_0 + n, \tag{4.2}$$

where $n = \dim(Y)$. This implies that for every such s, the sheaf $\mathcal{F}_s$ is (m_0+n)-regular with respect to $\mathcal{L}_s$ in the sense of Castelnuovo-Mumford regularity (we refer to [**Laz04**, Chapter 1.8] for the basic facts on Castelnuovo-Mumford regularity[2]). This implies that $\mathcal{F}_s$ is m-regular for every $m \geq m_0 + n$, and we conclude that

$$H^i\big(Y_s, \mathcal{F}_s \otimes \mathcal{L}_s^m\big) = 0 \quad \text{for} \quad i \geq 1 \quad \text{and} \quad m \geq m_0.$$

This completes the proof of the lemma. ☐

4.2. The Cartier isomorphism

We now fix a perfect field k of characteristic $p > 0$. All schemes are assumed to be of finite type over k.

[2] The results on Castelnuovo-Mumford regularity are usually stated for very ample line bundles on projective varieties. We here need the relative version of those results, which is proved in the same way.

LEMMA 4.9. *If $f\colon X \to Y$ is an étale morphism, then for every $e \geq 1$, the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "F_X^e"'] & Y \arrow[d, "F_Y^e"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

is Cartesian.

PROOF. Consider the Cartesian diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "q"'] & Y \arrow[d, "F_Y"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

and the induced morphism $h\colon X \to W$, so that $f = g \circ h$ and $F_X = q \circ h$. Since f is étale, it follows that g is étale, hence the factorization $f = g \circ h$ implies that h is étale. On the other hand, the factorization $F_X = q \circ h$ implies that h is injective and for every $x \in X$, the field extension $k\big(h(x)\big) \hookrightarrow k(x)$ is purely inseparable. Since h is étale, this field extension is also separable, hence it is an isomorphism. Furthermore, since F_Y is finite, q is finite; since F_X is finite, too, we conclude that h is finite.

Therefore h is a finite étale morphism, which is injective and induces isomorphisms between the corresponding residue fields. It is easy to see that in this case h is an isomorphism onto a subset U that is both open and closed in W. In fact, we need to have $U = W$: if $y \in W \smallsetminus h(X)$, then we get a contradiction with the fact that q induces an isomorphism between $g^{-1}\big(g(y)\big)$ and $f^{-1}\big(g(y)\big)$. Therefore h is an isomorphism. □

From now on we fix a smooth, irreducible scheme X over k, of dimension n. Note that for every closed point $x \in X$, the residue field $k(x)$ of x is finite over k, hence it is perfect (this follows since we clearly have $\big[k(x) : k\big] = \big[k(x)^p : k^p\big]$).

COROLLARY 4.10. If U is an affine open subset of X and $x_1, \ldots, x_n \in \mathcal{O}_X(U)$ are algebraic coordinates, then for every $q = p^e$, with $e \geq 1$, $\mathcal{O}_X(U)$ is free over $\mathcal{O}_X(U)^q$, with basis

$$\{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i \leq q-1 \text{ for all } i\}.$$

PROOF. We have an étale morphism $\varphi = (x_1, \ldots, x_n)\colon U \to \mathbf{A}_k^n$. The assertion then follows using Lemma 4.9 from the fact that it holds when $x_1, \ldots, x_n$ are the standard coordinates on $\mathbf{A}_k^n$. □

REMARK 4.11. The following fact is often useful: if X is a scheme of characteristic p and $F\colon X \to X$ is the Frobenius morphism, then for every line bundle $\mathcal{L}$ on X, we have a canonical isomorphism $F^*(\mathcal{L}) \simeq \mathcal{L}^{\otimes p}$. Indeed, it is clear that if $\mathcal{L}$ is described by the Čech cocycle $(\varphi_{i,j})$, then $F^*(\mathcal{L})$ is described by the Čech cocycle $(\varphi_{i,j}^p)$, which also describes $\mathcal{L}^{\otimes p}$.

We next turn to the de Rham complex of X. Since X is smooth over k, the sheaves $\Omega_X^i = \Omega_{X/k}^i$ are locally free $\mathcal{O}_X$-modules. We will consider the de Rham complex $\Omega_X^\bullet$:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \ldots \xrightarrow{d} \Omega_X^n \longrightarrow 0,$$

placed in cohomological degrees $0, \ldots, n$. We note that $\oplus_{i\geq 0}\Omega_X^i$ has the structure of a *differential graded algebra*: exterior multiplication gives a (graded-commutative) $\mathcal{O}_X$-algebra structure such that

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge d\beta.$$

We note that while the maps are not $\mathcal{O}_X$-linear, they still induce corresponding maps between the corresponding sheaves of rational differential forms

$$0 \longrightarrow k(X) \xrightarrow{d} \Omega_X^1 \otimes_{\mathcal{O}_X} k(X) \xrightarrow{d} \ldots \xrightarrow{d} \Omega_X^n \otimes_{\mathcal{O}_X} k(X) \longrightarrow 0. \tag{4.3}$$

Indeed, this follows from the "quotient rule": we can extend the exterior derivative to rational differential forms such that if η is a differential form and h is a regular function, then $d\big(\frac{1}{h}\eta\big) = \frac{1}{h}d\eta - \frac{1}{h^2}dh \wedge \eta$.

If $U \subseteq X$ is open and $0 \neq f \in \mathcal{O}_X(U)$, we denote by $\mathrm{dlog}(f)$ the rational 1-form $\frac{1}{f}df$. Note that if $0 \neq g \in \mathcal{O}_X(U)$, then

$$\mathrm{dlog}(fg) = \mathrm{dlog}(f) + \mathrm{dlog}(g). \tag{4.4}$$

Suppose now that E is a reduced SNC divisor on X. We define the subsheaf $\Omega_X^1(\log E) \subseteq \Omega_X^1 \otimes_{\mathcal{O}_X} k(X)$ as follows. Suppose that U is an open subset of X and $x_1, \ldots, x_n$ are algebraic coordinates on U such that $E = \mathrm{div}_X(x_1 \cdots x_r)$. In this case $\Omega_X^1(\log E)|_U$ is generated by $\mathrm{dlog}(x_1), \ldots, \mathrm{dlog}(x_r), dx_{r+1}, \ldots, dx_n$. Note that the definition is independent of the choice of coordinates: if $y_1, \ldots, y_n$ is another such system of coordinates, after relabeling we may assume that x_i and y_i define the same divisor E_i, hence we can write $y_i = h_i x_i$ for $1 \leq i \leq r$, for some $h_i \in \mathcal{O}_X(U_i)^*$. In this case it follows from (4.4) that we have

$$\mathrm{dlog}(y_i) = \mathrm{dlog}(x_i) + \mathrm{dlog}(h_i) \subseteq \mathcal{O}_U \cdot \mathrm{dlog}(x_i) + \Omega_U^1.$$

We can thus glue the local definitions to get the subshsheaf $\Omega_X^1(\log E)$ of the sheaf $\Omega_X^1 \otimes_{\mathcal{O}_X} k(X)$. It follows from the definition that this is a locally free $\mathcal{O}_X$-module, of rank n. Note that if f is a local section of $\mathcal{O}_X$ such that the closed set $V(f)$ is contained in $\mathrm{Supp}(E)$, then $\mathrm{dlog}(f)$ is a local section of $\Omega_X^1(\log E)$. For every i, with $1 \leq i \leq n$, we put $\Omega_X^i(\log E) := \wedge^i\Omega_X^1(\log E)$. In particular, we have $\Omega_X^0(\log E) = \mathcal{O}_X$ and it is easy to see, using the definition, that $\Omega_X^n(\log E) = \omega_X(E)$.

Note that the differential in the complex (4.3) maps $\Omega_X^i(\log E)$ to $\Omega_X^{i+1}(\log E)$: this follows easily from the definition, using the fact that $d\big(\mathrm{dlog}(f)\big) = 0$ for every f. We call this the *log de Rham complex* of the pair (X, E). Note that $\oplus_{i=0}^n\Omega_X^i(\log E)$ is again a differential graded algebra.

While the differentials in the log de Rham complex $\Omega_X^\bullet(\log E)$ are not $\mathcal{O}_X$-linear, in characteristic p they are linear over $\mathcal{O}_X^p$: this follows from the fact that if g is a regular function and η is a differential form, both of them defined on some $U \subseteq X$, then $d(g^p\eta) = g^p d(\eta)$, since $d(g^p) = 0$. In other words, $F_*\Omega_X^\bullet(\log E)$ is a complex of $\mathcal{O}_X$-modules (here, for simplicity, we write F for F_X). Therefore $\oplus_{i=0}^n\mathcal{H}^i\big(F_*\Omega_X^\bullet(\log E)\big)$ is a graded $\mathcal{O}_X$-algebra (the multiplicative structure being induced from the one on the de Rham complex). We have the following important result:

THEOREM 4.12 (Cartier). *If X is a smooth, irreducible n-dimensional scheme over the perfect field k, and if E is a reduced SNC divisor on X, then there is a unique morphism of graded $\mathcal{O}_X$-algebras*

$$C^{-1} = C^{-1}_{X,E} \colon \bigoplus_{i=0}^{n} \Omega^i_X(\log E) \to \bigoplus_{i=0}^{n} \mathcal{H}^i\big(F_*\Omega^\bullet_X(\log E)\big) \tag{4.5}$$

such that the following conditions hold:

$$C^{-1}(df) = [f^{p-1}df] \quad \text{for every local section} \quad f \in \mathcal{O}_X \quad \text{and} \tag{4.6}$$

$$C^{-1}\big(\mathrm{dlog}(g)\big) = \big[\mathrm{dlog}(g)\big] \quad \text{for every local section } g \in \mathcal{O}_X \text{ with } V(g) \subseteq \mathrm{Supp}(E). \tag{4.7}$$

Moreover, C^{-1} is an isomorphism.

The inverse C of C^{-1} is known as the *Cartier isomorphism.*

REMARK 4.13. Once we know that the morphism C^{-1} is an isomorphism, this implies that the right-hand side of (4.5) is a locally free $\mathcal{O}_X$-module. In particular, it has no torsion and thus condition (4.7) in the theorem follows from condition (4.6).

LEMMA 4.14. *If R is a k-algebra, then for every $m < p$ and for every $f, g \in R$, we have*

$$(f+g)^m d(f+g) - (f^m df + g^m dg) = 0 \quad \text{in} \quad \Omega^1_{R/k}/d(R).$$

PROOF. For every $a, b \geq 0$, we have $d(f^a g^b) = 0$ in $\Omega^1_{R/k}/d(R)$, hence

$$af^{a-1}g^b df + bf^a g^{b-1} dg = 0 \quad \text{in} \quad \Omega^1_{R/k}/d(R).$$

It follows that in this quotient we have

$$(f+g)^m d(f+g) = \sum_{i=0}^{m} \binom{m}{i} f^i g^{m-i} df + \sum_{i=0}^{m} \binom{m}{i} f^i g^{m-i} dg$$

$$= f^m df + \sum_{i=0}^{m-1} \binom{m}{i} f^i g^{m-i} df - \sum_{i=1}^{m} i(m+1-i)^{-1} \binom{m}{i} f^{i-1} g^{m-i+1} df + g^m dg.$$

Since $i(m+1-i)^{-1}\binom{m}{i} = \binom{m}{i-1}$, we see that in the above expression the two sums cancel out and we obtain the equality in the lemma. $\square$

PROOF OF THEOREM 4.12. Note first that if f is a local section of $\mathcal{O}_X$, then $d(f^{p-1}df) = 0$, hence we have indeed a corresponding local section $[f^{p-1}df]$ of $\mathcal{H}^1\big(F_*\Omega^\bullet_X(\log E)\big)$. Similarly, if g is a local section of $\mathcal{O}_X$ such that $V(g) \subseteq \mathrm{Supp}(E)$, then $d\big(\mathrm{dlog}(g)\big) = 0$, hence we have a local section $[\mathrm{dlog}(g)]$ of $\mathcal{H}^1\big(F_*\Omega^\bullet_X(\log E)\big)$. Since $\bigoplus_{i=0}^n \Omega^i_X(\log E)$ is generated as an $\mathcal{O}_X$-algebra by $\Omega^1_X(\log E)$, which in turn is generated as an $\mathcal{O}_X$-module by df and $\mathrm{dlog}(g)$ as above, the uniqueness of such C^{-1} is clear.

Because of uniqueness, it is enough to construct C locally. Moreover, proving that C^{-1} is an isomorphism can also be done locally, hence we may and will assume that X is affine and we have $x_1, \ldots, x_n \in \mathcal{O}_X(X)$ giving an algebraic system of coordinates on X such that $E = \mathrm{div}_X(x_1 \cdots x_r)$. Suppose that we have a morphism C^{-1} of $\mathcal{O}_X$-algebras as in (4.5). By Corollary 4.10, the monomials x^u, with $0 \leq u_i \leq p-1$ for all i give a basis of $\mathcal{O}_X(U)$ over $\mathcal{O}_X(U)^p$.

Note first that condition (4.6) in the theorem holds for all f if it holds for these monomials. Indeed, this follows from the following two observations:

a) If condition (4.6) holds for f_1 and f_2, then it also holds for $f_1 + f_2$. This follows from Lemma 4.14.
b) If condition (4.6) holds for f, then it also holds for $h^p f$: indeed, then

$$C^{-1}(h^p df) = [h^{p^2} f^{p-1} df] = \big[(h^p f)^{p-1} d(h^p f)\big].$$

Similarly, if we know (4.6), then in order to check condition (4.7) for all g with $V(g) \subseteq E$, it is enough to check it when $g = x_i$, with $1 \leq i \leq r$. Indeed, in general we have a subset $J \subseteq \{1, \ldots, r\}$ such that $g = h \cdot \prod_{i\in J} x_i^{a_i}$, for an invertible function h, hence

$$\mathrm{dlog}(g) = \mathrm{dlog}(h) + \sum_{i\in J} a_i \cdot \mathrm{dlog}(x_i).$$

Since $C^{-1}(dh) = [h^{p-1}dh]$ and h is invertible, we deduce that $C^{-1}\big(\mathrm{dlog}(h)\big) = \big[\mathrm{dlog}(h)\big]$. It is then clear that if condition (4.7) holds for $x_1, \ldots, x_r$, then the condition also holds for g.

Since $x_1, \ldots, x_n$ form an algebraic system of coordinates, the morphism $\varphi = (x_1, \ldots, x_n)\colon X \to \mathbf{A}_k^n$ is étale. If we also denote by $x_1, \ldots, x_n$ the standard coordinates on $\mathbf{A}_k^n$ and D is the divisor defined by $x_1 \cdots x_r$ on $\mathbf{A}_k^n$, then it is clear that we have a canonical isomorphism

$$\Omega_X^\bullet(\log E) \simeq \varphi^* \Omega_{\mathbf{A}_k^n}^\bullet(\log D).$$

Furthermore, it follows from Lemma 4.9, flat base-change, and the fact that φ is flat that we have canonical isomorphisms

$$\begin{aligned}\bigoplus_{i=0}^n \mathcal{H}^i\big(F_*\Omega_X^\bullet(\log E)\big) &\simeq \bigoplus_{i=0}^n \mathcal{H}^i\big(F_*\varphi^*\Omega_{\mathbf{A}_k^n}^\bullet(\log D)\big)\\ \simeq \bigoplus_{i=0}^n \mathcal{H}^i\big(\varphi^* F_*\Omega_{\mathbf{A}_k^n}^\bullet(\log D)\big) &= \bigoplus_{i=0}^n \varphi^*\mathcal{H}^i\big(F_*\Omega_{\mathbf{A}_k^n}^\bullet(\log D)\big).\end{aligned}$$

It follows that if we prove the existence of $C_{\mathbf{A}_k^n,D}^{-1}$ for $(\mathbf{A}_k^n, D)$, and the fact that it is an isomorphism, then $C_{X,E}^{-1} := \varphi^*(C_{\mathbf{A}_k^n,D}^{-1})$ satisfies the same properties with respect to (X, E). Indeed, the fact that it is an isomorphism of graded $\mathcal{O}_X$-algebras is clear. Moreover, the fact that condition (4.6) holds when f is a monomial in $x_1, \ldots, x_n$ is clear and the fact that condition (4.7) holds when $g = x_1, \ldots, x_r$ is clear as well. By the above discussion, this shows that $C_{X,E}^{-1}$ has the desired properties.

From now one we assume that $X = \mathbf{A}_k^n$ and $E = D$. Note that we have an isomorphism of differential graded $\mathcal{O}_X$-algebras

$$\Omega_{\mathbf{A}_k^n}^\bullet(\log D) \simeq \otimes_{i=1}^r \mathrm{pr}_i^* \Omega_{\mathbf{A}_k^1}^\bullet(\log\{0\}) \otimes \otimes_{i=r+1}^n \mathrm{pr}_i^* \Omega_{\mathbf{A}_k^1}^\bullet,$$

where $\mathrm{pr}_i\colon \mathbf{A}_k^n \to \mathbf{A}_k^1$ is the projection onto the i-th component. We thus get a corresponding tensor product decomposition for $\bigoplus_{i=0}^n \mathcal{H}^i\big(\Omega_{\mathbf{A}_k^n}^\bullet(\log D)\big)$ via the Künneth theorem. It is thus straightforward to check that it is enough to prove the theorem when $X = \mathbf{A}_k^1$ and when $E = \{0\}$ or $E = 0$.

In the former case, the complex $\Omega_{\mathbf{A}_k^1}^\bullet(\log\{0\})$ consists of

$$k[t] \xrightarrow{d} k[t]\frac{dt}{t}$$

placed in cohomological degrees 0 and 1 and it is clear that we have the isomorphism

$$C^{-1}\colon k[t]\oplus k[t]\frac{dt}{t}\to\mathcal{H}^0\oplus\mathcal{H}^1=k[t^p]\oplus k[t^p]\frac{dt}{t},\ C^{-1}\left(u,v\frac{dt}{t}\right)=\left(u^p,v^p\frac{dt}{t}\right),$$

which satisfies the desired conditions (recall that we only need to check conditions (4.6) and (4.7) for monomials in t).

We finally consider the case when $E=0$, when $\Omega^\bullet_{\mathbf{A}^1_k}$ consists of

$$k[t]\xrightarrow{d}k[t]dt$$

placed in cohomological degrees 0 and 1. We then have the isomorphism

$$C^{-1}\colon k[t]\oplus k[t]dt\to\mathcal{H}^0\oplus\mathcal{H}^1=k[t^p]\oplus k[t^p]t^{p-1}dt,\ C^{-1}(u,v\cdot dt)=(u^p,v^pt^{p-1}\cdot dt)$$

which satisfies the desired conditions. This completes the proof of the theorem. □

In particular, given a smooth, irreducible n-dimensional scheme X over k and an SNC divisor on k, we get a surjective $\mathcal{O}_X$-linear map $t=t_{X,E}$ as the composition

$$F_*\big(\omega_X(E)\big)\to\mathcal{H}^n\big(F_*\Omega^\bullet_X(\log(E)\big)\xrightarrow{C}\omega_X(E).$$

We will refer to this map as the *trace map*. By iterating this map we obtain for every $e\geq 1$ the surjective $\mathcal{O}_X$-linear map $t^e=t^e_{X,E}$:

$$F^e_*\big(\omega_X(E)\big)\xrightarrow{F^{e-1}_*(t)}F^{e-1}_*\big(\omega_X(E)\big)\longrightarrow\ldots\longrightarrow F_*\big(\omega_X(E)\big)\xrightarrow{t}\omega_X(E).$$

If $E=0$, then we simply write t_X and t^e_X (or just t and t^e).

REMARK 4.15. The map $t_X\colon F_*\omega_X\to\omega_X$ plays an important role in birational geometry in positive characteristic (see for example [**PST17**]). Note that if $\mathcal{L}$ is a line bundle on X, if we tensor t^e_X with $\mathcal{L}$ and use the projection formula and Remark 4.11, we identify this map with

$$F^e_*(\omega_X)\otimes\mathcal{L}=F^e_*(\omega_X\otimes\mathcal{L}^{\otimes p^e})\to\omega_X\otimes\mathcal{L}.$$

By taking cohomology and using the fact that F is an affine morphism, we obtain induced morphisms

$$t^e\colon H^i(X,\omega_X\otimes\mathcal{L}^{p^e})\to H^i(X,\omega_X\otimes\mathcal{L}).$$

This map is p^{-e}-*linear* in the sense that $t^e(\lambda^{p^e}a)=\lambda\cdot t^e(a)$ for all $\lambda\in k$. This map is useful, for example, when X is projective and $\mathcal{L}$ is ample, since for $e\gg 0$, the line bundle $\omega_X\otimes\mathcal{L}^{p^e}$ is very ample and has no higher cohomology.

REMARK 4.16. It follows from our local description of the Cartier isomorphism that if $x_1,\ldots,x_n$ are algebraic coordinates on an open subset $U\subseteq X$ such that $E|_U$ is defined by $(x_1\cdots x_r)$ and if we put η for the corresponding generator $\mathrm{dlog}(x_1)\wedge\ldots\wedge\mathrm{dlog}(x_r)\wedge dx_{r+1}\wedge\ldots\wedge dx_n$ of $\omega_X(E)$, then

$$t_{X,E}(x_1^{a_1}\cdots x_n^{a_n}\eta)=\prod_{i\leq r}x_i^{a_i/p^e}\cdot\prod_{j>r}x_j^{(a_j-p^e+1)/p^e}\cdot\eta$$

if $p^e|a_i$ for all $i\leq r$ and $p^e|(a_j-p^e+1)$ for all $j>r$; otherwise $t_{X,E}(x_1^{a_1}\cdots x_n^{a_n}\eta)=0$. Indeed, it is enough to check this when $e=1$. On one hand, note that if the above divisibilities are not satisfied, then $x_1^{a_1}\cdots x_n^{a_n}\eta$ lies in $d\big(\Omega^{n-1}_X(\log E)\big)$. On the other hand, for the case when they are satisfied, by linearity we only need to check that

$$t_{X,E}(x_r^{p^e-1}\cdots x_n^{p^e-1}\eta)=\eta.$$

This follows from the fact that since C^{-1} is multiplicative and satisfies (4.6) and (4.7), we have $C^{-1}(\eta) = \big[x_r^{p^e-1} \cdots x_n^{p^e-1}\eta\big]$.

Using this explicit description, it is easy to check that $t^e_{X,E}$ can be described in terms of t^e_X: more precisely, it gets identified with the composition

$$F^e_*\big(\omega_X(E)\big) \hookrightarrow F^e_*\big(\omega_X(p^eE)\big) \to \omega_X(E),$$

where the first map is induced by the inclusion $\mathcal{O}_X(E) \hookrightarrow \mathcal{O}_X(p^eE)$ and the second map is $t^e_X \otimes \mathcal{O}_X(E)$ (see the previous remark).

Remark 4.17. It is more common (and better for extensions to the relative setting) to phrase the Cartier isomorphism and the trace map using the relative Frobenius morphism, instead of the absolute Frobenius. Note that if we define the scheme X' over k by the Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "F_{\mathrm{Spec}(k)}"] & \mathrm{Spec}(k), \end{tikzcd}$$

then the Frobenius morphism F_X induces the *relative Frobenius morphism* $F_{X/k} \colon X \to X'$. This is a morphism of schemes over k. Note that since k is perfect, f is an isomorphism of abstract schemes. Similarly, if E is an SNC divisor on X, then we get an SNC divisor E' on X'.

The Cartier isomorphism is usually phrased as an isomorphism of graded $\mathcal{O}_{X'}$-algebras

$$\bigoplus_{i=0}^n \Omega^i_{X'/k}(\log E') \to \bigoplus_{i=0}^n \mathcal{H}^i\big((F_{X/k})_*\Omega^\bullet_{X/k}(\log E)\big).$$

This is equivalent to our assertion via the isomorphism f. In particular, the map $t_{X,E}$ gets identified with a map

$$(F_{X/k})_*\big(\omega_X(E)\big) \to \omega_{X'}(E').$$

One can show that for $E = 0$, the map is the trace map with respect to the finite morphism $F_{X/k}$ (see [**Har77**, Exercise 7.2] for the description of the trace map, at least in the case of a finite morphism between projective varieties).

For the study of test ideals, the trace map is relevant because of the following proposition:

Proposition 4.18. Let X be a smooth irreducible scheme over a perfect field k. If $\mathfrak{a}$ is a coherent ideal in $\mathcal{O}_X$, then for every $e \geq 1$, we have

$$t_X\big(F^e_*(\mathfrak{a} \cdot \omega_X)\big) = \mathfrak{a}^{[1/p^e]} \cdot \omega_X.$$

Proof. Since ω_X is a line bundle and t_X is an $\mathcal{O}_X$-linear map, we know that we have

$$t_X\big(F^e_*(\mathfrak{a} \cdot \omega_X)\big) = \mathfrak{b} \cdot \omega_X,$$

for some ideal $\mathfrak{b}$ in $\mathcal{O}_X$. In order to prove that $\mathfrak{b} = \mathfrak{a}^{[1/p^e]}$, we argue locally. Therefore we may assume that X is affine and we have an algebraic system of coordinates $x_1, \ldots, x_n \in \mathcal{O}_X(X)$.

By Corollary 4.10, a basis of $\mathcal{O}_X(X)$ over $\mathcal{O}_X(X)^{p^e}$ is given by the monomials $x^u = x_1^{u_1} \cdots x_n^{u_n}$, with u in

$$\Lambda = \{u \in \mathbf{Z}_{\geq 0} \mid 0 \leq u_i \leq p^e - 1 \text{ for } 1 \leq i \leq n\}.$$

Note that if for any $f \in \mathcal{O}_X(X)$ we write

$$f = \sum_{u \in \Lambda} c_u(f)^{p^e} x^u,$$

then it follows from Remark 4.16 that

$$t_X(f dx_1 \wedge \ldots \wedge dx_n) = c_w(f) dx_1 \wedge \ldots \wedge dx_n,$$

where $w = (p^e - 1, \ldots, p^e - 1)$. Therefore we have

$$\mathfrak{b} = \{c_w(f) \mid f \in \mathfrak{a}\}.$$

On the other hand, it follows from Proposition 3.15 that

$$\mathfrak{a}^{[1/p^e]} = (c_u(f) \mid f \in \mathfrak{a}).$$

It is then clear that $\mathfrak{b} \subseteq \mathfrak{a}^{[1/p^e]}$ and the opposite inclusion follows from the fact that for every $f \in \mathfrak{a}$ and every $u \in \Lambda$, we have $c_u(f) = c_w(x^{w-u} f)$. This completes the proof. □

4.3. The Deligne-Illusie theorem

Let $p > 0$ be a prime integer. Recall that the functor from the category of complete DVRs with maximal ideals generated by p and perfect residue fields to the category of perfect fields of characteristic p, which maps a ring to its residue field, is an equivalence of categories (see [**Mat89**, Theorems 29.1 and 29.2]). In particular, for every perfect field k of characteristic p, there is a complete DVR $W(k)$, with maximal ideal $p \cdot W(k)$, and with residue field $W(k)/(p) \simeq k$; moreover, this is unique up to a canonical isomorphism that induces the identity on k. This is the *ring of Witt vectors* of k. Moreover, we see that there is a unique ring isomorphism $F \colon W(k) \to W(k)$ that induces the Frobenius morphism on the residue field. For example, if $k = \mathbf{F}_p$, then $W(k)$ is the ring $\mathbf{Z}_p$ of p-adic integers and the Frobenius morphism on $\mathbf{Z}_p$ is just the identity.

Remark 4.19. If $(R, \mathfrak{m}, k)$ is a DVR with k perfect and $\mathfrak{m} = (p)$, then $R/(p^2) \simeq W_2(k)$. Indeed, we have $\widehat{R} \simeq W(k)$ and thus $R/\mathfrak{m}^2 \simeq \widehat{R}/\mathfrak{m}^2\widehat{R} \simeq W_2(k)$.

In what follows we fix a perfect field k of characteristic 0 and put $W_2(k) = W(k)/(p^2)$. Given a scheme X over k, a *lift* of X to $W_2(k)$ is a flat scheme $\widetilde{X}$ over $\mathrm{Spec}\big(W_2(k)\big)$ and an isomorphism $\nu \colon \widetilde{X} \times_{\mathrm{Spec}(W_2(k))} \mathrm{Spec}(k) \to X$. Note that if X is smooth over k, then $\widetilde{X}$ is automatically smooth over $W_2(k)$. Suppose now that X is smooth over k and $E = \sum_{i=1}^N E_i$ is a reduced SNC divisor on X. A *lift* of the pair (X, E) to $W_2(k)$ consists of a lift $(\widetilde{X}, \nu)$ of X, together with effective Cartier divisors $\widetilde{E_i}$ on $\widetilde{X}$, flat over $\mathrm{Spec}\big(W_2(k)\big)$, such that ν identifies $\widetilde{E_i} \times_{\mathrm{Spec}(W_2(k))} \mathrm{Spec}(k)$ with E_i for all i.

For any Noetherian scheme Y, we denote by $D^b_{\mathrm{coh}}(Y)$ the derived category of coherent sheaves on Y. Note that the simplest objects in this derived category are those that are direct sums of twists of coherent sheaves: $u = \bigoplus_i \mathcal{F}_i[-i]$, for suitable coherent sheaves $\mathcal{F}_i$. Of course, in this case we have $\mathcal{F}_i = \mathcal{H}^i(u)$ for all i. The following result of Deligne and Illusie [**DI87**] is a fundamental tool in arithmetic geometry in positive characteristic.

THEOREM 4.20. *Let X be a smooth, irreducible scheme over a perfect field k of characteristic $p>0$, and E a reduced SNC divisor on X. If $p>\dim(X)$ and the pair (X,E) has a lift to $W_2(k)$, then we have an isomorphism in $D^b_{\mathrm{coh}}(X)$:*

$$F_*\Omega_X^\bullet(\log E)\simeq\bigoplus_i\Omega_X^i(\log E)[-i].$$

SKETCH OF PROOF. We only outline the argument and, for simplicity, we only treat the case $E=0$. It is enough to construct a morphism

$$\alpha\colon \bigoplus_i\Omega_X^i[-i]\to F_*\Omega_X^\bullet \tag{4.8}$$

in $D^b_{\mathrm{coh}}(X)$ such that for every i, $\mathcal{H}^i(\alpha)$ is the Cartier isomorphism $C^{-1}|_{\Omega_X^i}$. Of course, giving such α is equivalent to giving for every i a morphism

$$\alpha_i\colon \Omega_X^i[-i]\to F_*\Omega_X^\bullet \tag{4.9}$$

such that $\mathcal{H}^i(\alpha_i)=C^{-1}|_{\Omega_X^i}$.

For $i=0$, this is easy: we simply take α_0 to be the composition

$$\mathcal{O}_X\xrightarrow{C^{-1}}\mathcal{H}^0(F_*\Omega_X^\bullet)\hookrightarrow F_*\Omega_X^\bullet,$$

which clearly satisfies the required property. For $i\geq 1$, the crucial step is the construction of α_1. Indeed, if α_1 is constructed, then we get α_i for $2\leq i\leq n=\dim(X)$, as follows. Since $p>n$, we have a section of the canonical projection $\Omega_X^{\otimes i}\to\Omega_X^i$ given by

$$\sigma_i\colon \Omega_X^i\to(\Omega_X^1)^{\otimes i},\quad \sigma_i(\eta_1\wedge\ldots\wedge\eta_i)=\sum_{\tau\in S_i}\frac{1}{i!}\epsilon(\tau)\eta_{\tau(1)}\otimes\ldots\otimes\eta_{\tau(i)}.$$

Note that the multiplication on $\Omega_X^\bullet$ induces a morphism of complexes (and thus a morphism in $D^b(X)$)

$$\varphi_i\colon F_*(\Omega_X^\bullet)^{\otimes i}\to F_*(\Omega_X^\bullet).$$

We define α_i to be the composition

$$\Omega_X^i[-i]\xrightarrow{\sigma_i[-i]}(\Omega_X^1)^{\otimes i}[-i]\xrightarrow{\alpha_1^{\otimes i}}F_*(\Omega_X^\bullet)^{\otimes i}\xrightarrow{\varphi_i}F_*(\Omega_X^\bullet).$$

Since $\mathcal{H}^1(\alpha_1)=C^{-1}|_{\Omega_X^1}$ and since C^{-1} is multiplicative, it follows easily that $\mathcal{H}^i(\alpha_i)=C^{-1}|_{\Omega_X^i}$. Therefore the key point is the construction of α_1.

We only prove this in the special case when there is a morphism $\widetilde{F}\colon\widetilde{X}\to\widetilde{X}$ compatible with the Frobenius isomorphism on $W_2(k)$ and which induces F on X. In this case we will see that we can define α_1 as a morphism of complexes. Arguing locally, let us assume that $X=\mathrm{Spec}(R)$ and $\widetilde{X}=\mathrm{Spec}(\widetilde{R})$. Let us write $\Omega_R=\Omega_{R/k}$ and $\Omega_{\widetilde{R}}=\Omega_{\widetilde{R}/W_2(k)}$. Note that we have a surjective homomorphism $u\colon\widetilde{R}\to R$, whose kernel is $p\widetilde{R}$. Moreover, since $\widetilde{R}$ is flat over $W_2(k)$, it follows that the kernel of multiplication by p on $\widetilde{R}$ is $p\widetilde{R}$, which implies that we have an isomorphism of R-modules $v\colon R\simeq p\widetilde{R}$ such that $v\big(u(a)\big)=pa$ for all $a\in\widetilde{R}$. Similarly, we have

$$\Omega_R\simeq\Omega_{\widetilde{R}}\otimes_{\widetilde{R}}R\simeq\Omega_{\widetilde{R}}/p\cdot\Omega_{\widetilde{R}}.$$

We write u_1 for the surjective map $\Omega_{\widetilde{R}} \to \Omega_R$. Since $\widetilde{X}$ is smooth over $W_2(k)$, it follows that $\Omega_{\widetilde{R}}$ is a free $\widetilde{R}$-module and we have an isomorphism of R-modules

$$v_1 \colon \Omega_R \to p \cdot \Omega_{\widetilde{R}}$$

such that $v_1\big(u_1(\widetilde{\eta})\big) = p\widetilde{\eta}$ for all $\widetilde{\eta} \in \Omega_{\widetilde{R}}$.

Given $\eta \in \Omega_{R/k}$, let us choose $\widetilde{\eta} \in \Omega_{\widetilde{R}}$ such that $u_1(\widetilde{\eta}) = \eta$. Since $F^*(\eta) = 0$, it follows that $\widetilde{F}^*(\eta) = v_1(\beta)$ for some β in Ω_R. We put $\alpha_1(\eta) = \beta$. This is independent of the choice of $\widetilde{\eta}$: if $u(\widetilde{\eta}') = \eta$, then $\widetilde{\eta}-\widetilde{\eta}' \in p\cdot\Omega_{\widetilde{R}}$, hence $\widetilde{F}^*(\widetilde{\eta}-\widetilde{\eta}') = 0$.

It is clear that this gives a morphism of $\mathcal{O}_X$-modules $\Omega_X^1 \to F_*\Omega_X^1$. Let us compute $\alpha_1(dh)$ for $h \in R$. If we choose $\widetilde{h} \in \widetilde{R}$ such that $u(\widetilde{h}) = u$, we have $u_1(d\widetilde{h}) = dh$. Note that since $\widetilde{F}$ induces F on R, it follows that $\widetilde{F}^*(\widetilde{h}) = \widetilde{h}^p + p\widetilde{g}$ for some $\widetilde{g} \in \widetilde{R}$. In this case we have

$$\widetilde{F}^*(d\widetilde{h}) = p(\widetilde{h}^{p-1}d\widetilde{h} + d\widetilde{g}),$$

hence $\alpha_1(dh) = h^{p-1}dh + dg$. First, this shows that the image of α_1 is contained in the set of exact 1-forms, hence α_1 gives, indeed, a morphism of complexes $\Omega_X^1[-1] \to F_*\Omega_X^\bullet$. Second, we see that the induced map $\Omega_X^1 \to \mathcal{H}^1(F_*\Omega_X^\bullet)$ agrees with C^{-1}. This completes the proof in this case.

Over every affine open subset U of X there is a lift $\widetilde{F_U}$ of F_U as above, but this is certainly not unique. As a result, the corresponding morphisms of complexes do not glue. However, one can show (and this is the most technical part of the proof) that the morphisms glue to a morphism $\alpha_1 \colon \Omega_X^1[-1] \to F_*\Omega_X^\bullet$ in $D^b_{\mathrm{coh}}(X)$. We end the discussion of the proof here. $\square$

In the remaining part of this section we discuss applications of the theorem of Deligne-Illusie to vanishing theorems.

THEOREM 4.21. *Let X be a smooth, irreducible, n-dimensional projective scheme over a perfect field k of characteristic $p > 0$. If X has a lifting to $W_2(k)$ and $p > \dim(X)$, then for every ample line bundle $\mathcal{L}$ on X, we have*

$$H^i(X, \omega_X^j \otimes \mathcal{L}) = 0 \quad \textit{for} \quad p + q > n.$$

PROOF. Note that since $(\Omega_X^j)^\vee \simeq \omega_X^{-1} \otimes \Omega_X^{n-j}$, Serre duality gives

$$H^i(X, \omega_X^j \otimes \mathcal{L}) \simeq H^{n-i}(X, \Omega_X^{n-j} \otimes \mathcal{L}^{-1})^\vee.$$

Therefore the assertion in the theorem is equivalent to

$$H^i(X, \Omega_X^j \otimes \mathcal{L}^{-1}) = 0 \quad \text{for} \quad i + j < n. \tag{4.10}$$

The key point will be showing that for *any* line bundle $\mathcal{L}$ on X and for any r, the following holds:

$$\text{if} \quad H^i(X, \Omega_X^j \otimes \mathcal{L}^{-p}) = 0 \quad \text{for all} \quad i + j < r, \tag{4.11}$$

$$\text{then} \quad H^i(X, \Omega_X^j \otimes \mathcal{L}^{-1}) = 0 \quad \text{for all} \quad i + j < r.$$

Indeed, once we know this, it follows by induction on $e \geq 1$ that if we have $H^i(X, \Omega_X^j \otimes \mathcal{L}^{-p^e}) = 0$ for all $i + j < r$, then $H^i(X, \Omega_X^j \otimes \mathcal{L}^{-1}) = 0$ for all $i + j < r$. By taking $r = n$, we see that if $\mathcal{L}$ is ample, then for $e \gg 0$ we have $H^i(X, \Omega_X^j \otimes \mathcal{L}^{-p^e}) = 0$ for all $i + j < n$, by asymptotic Serre vanishing and Serre duality (note that if $i = n$, then $j < 0$ and the assertion is trivially satisfied). We thus obtain (4.10).

We now prove (4.11). The hypothesis gives via Theorem 4.20 (with $E = 0$) an isomorphism

$$F_*\Omega_X^\bullet \simeq \bigoplus_{i=0}^{n} \Omega_X^i[-i].$$

Tensoring this with $\mathcal{L}$, taking (hyper)cohomology, and using the projection formula and Remark 4.11 for the left-hand side, we obtain an isomorphism

$$H^m(X, \Omega_X^\bullet \otimes \mathcal{L}^{-p}) \simeq \bigoplus_{\ell=0}^{n} H^{m-\ell}(X, \Omega_X^\ell \otimes \mathcal{L}^{-1}). \tag{4.12}$$

On the other hand, we have the hypercohomology spectral sequence for the complex $F_*(\Omega_X^\bullet \otimes \mathcal{L}^{-p})$:

$$E_1^{j,i} = H^i(X, \Omega_X^j \otimes \mathcal{L}^{-p}) \Rightarrow H^{i+j}(X, \Omega_X^\bullet \otimes \mathcal{L}^{-p}).$$

The assumption that $H^i(X, \Omega_X^j \otimes \mathcal{L}^{-p}) = 0$ for $i + j < r$ says that $E_1^{i,j} = 0$ for $i + j < r$. In this case, we conclude that in the spectral sequence we have that $E_\infty^{i,j} = 0$ for $i+j < r$ and thus $H^m(X, \Omega_X^\bullet \otimes \mathcal{L}^{-p}) = 0$ for $m < r$. The decomposition in (4.12) then implies that $H^i(X, \Omega_X^j \otimes \mathcal{L}^{-1}) = 0$ for $i+j < r$, completing the proof of (4.11) and thus the proof of the theorem. $\square$

Corollary 4.22 (Kodaira-Akizuki-Nakano). If X is a smooth, irreducible, n-dimensional projective scheme over a field k of characteristic 0 and $\mathcal{L}$ is an ample line bundle on X, then

$$H^i(X, \Omega_X^j \otimes \mathcal{L}) = 0 \quad \text{for} \quad i + j > n.$$

Proof. As we have seen in Section 4.1, we can find a finitely generated $\mathbf{Z}$-subalgebra A of k and a projective smooth morphism $f\colon X_A \to S = \mathrm{Spec}(A)$, of relative dimension n, and an ample line bundle $\mathcal{L}_A$ on X_A such that there is an isomorphism $X_A \times_S \mathrm{Spec}(k) \simeq X$ that identifies the pull-back of $\mathcal{L}_A$ with $\mathcal{L}$. Moreover, we have seen in Remark 4.4 that we may assume that for every $s \in S$ and every $i, j \geq 0$, we have

$$H^i(X_A, \Omega^j_{X_A/A} \otimes \mathcal{L}_A) \otimes_A k(s) \simeq H^i(X_s, \Omega^j_{X_s/k(s)} \otimes \mathcal{L}_s) \tag{4.13}$$

(note that we only need to consider finitely many i and j).

We need to find a closed point $s \in S$ such that we can apply the theorem for X_s and $\mathcal{L}_s$. Let $\mathfrak{m}$ be a prime ideal in A that corresponds to a closed point of $A \otimes_{\mathbf{Z}} \mathbf{Q}$ and let $W = \mathrm{Spec}(A/\mathfrak{m}) \hookrightarrow S$. Since $A/\mathfrak{m}$ is a domain, the morphism $\mathbf{Z} \to R = A/\mathfrak{m}$ is injective by the assumption on $\mathfrak{m}$, and $(A/\mathfrak{m}) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a field, it follows that the morphism $g\colon W \to \mathrm{Spec}(\mathbf{Z})$ is flat, of relative dimension 0. Since its generic fiber is smooth, there is a nonempty open subset U of W on which g is étale. Let us choose a closed point $s \in S$ that lies in U and such that $p := \mathrm{char}\big(k(s)\big) > n$. Since $\mathcal{O}_{W,s}$ is étale over $\mathbf{Z}_{(p)}$, it is a DVR with maximal ideal $p \cdot \mathcal{O}_{W,s}$. Therefore $R := \mathcal{O}_{W,s}/p^2 \cdot \mathcal{O}_{W,s} \simeq W_2\big(k(s)\big)$ (see Remark 4.19). Since $X_A \times_S \mathrm{Spec}(R)$ gives a lift of $X_s = X_A \times_S \mathrm{Spec}\big(k(s)\big)$ to $W_2\big(k(s)\big)$, it follows that we may apply the theorem to each connected component of X_s to conclude that

$$H^i(X_s, \Omega^j_{X_s/k(s)} \otimes \mathcal{L}_s) = 0 \quad \text{for} \quad i + j < n.$$

Using (4.13), we conclude that if η is the generic point of S, we have

$$H^i(X_\eta, \Omega^j_{X_\eta/k(\eta)} \otimes \mathcal{L}_\eta) = 0 \quad \text{for} \quad i + j < n.$$

Since $(X_\eta)_k = X$ and $(\mathcal{L}_\eta)_k = \mathcal{L}$, base-change from $\mathrm{Spec}\big(k(\eta)\big)$ to $\mathrm{Spec}(k)$ gives the assertion in the corollary. $\square$

REMARK 4.23. Note that if we take $j = n$ in Corollary 4.22, then we recover the statement of Kodaira's Vanishing Theorem that we discussed in Chapter 2.4.

For the proof of the comparison between multiplier ideals and test ideals, we will also need the following generalization of the Kodaira-Akizuki-Nakano vanishing statement, due to Hara [**Ha98**, Corollary 3.8].

THEOREM 4.24. *Let X be a smooth, n-dimensional scheme over a field k of characteristic 0, and $E = \sum_{\ell=1}^N E_\ell$ a reduced SNC divisor on X. If $f\colon X \to S$ is a proper morphism and D is a $\mathbf{Q}$-divisor on X that is f-ample, with $\mathrm{Supp}\big(D - \lfloor D\rfloor\big) \subseteq \mathrm{Supp}(E)$, then*

$$R^i f_* \Omega_X^j(\log E)(-E + \lceil D\rceil) = 0 \quad \textit{for} \quad i + j > n.$$

REMARK 4.25. By taking $j = n$ in the above theorem, we obtain a relative version of the Kawamata-Viehweg vanishing theorem for ample $\mathbf{Q}$-divisors (cf. Theorem 2.68). We also note that the vanishing results that we discussed do not admit extensions where the divisor is allowed to be big and nef, like in the Kawamata-Viehweg theorem: for an example where the statement of Corollary 4.22 fails when $\mathcal{L}$ is only required to be big and nef, see [**Laz04**, Example 4.3.4].

The proof of Theorem 4.24 is similar to the proof of Corollary 4.22, but we need one more ingredient, that we discuss next, which will allow us to replace $p\lceil D\rceil$ by $\lceil pD\rceil$. At the same time, this will allow us to avoid the use of Serre duality in the proof, so we can get the result in the relative setting, as stated.

Suppose that X is a smooth scheme over a field k of characteristic $p > 0$ and $E = \sum_{i=1}^N E_i$ is a reduced SNC divisor on X. We first note that if B is any divisor on X supported on E, then we have a subcomplex of the de Rham complex of rational differential forms on X given by $\Omega_X^\bullet(\log E)(B)$. Indeed, suppose that $x_1, \ldots, x_n$ are algebraic coordinates in an affine open subset U of X such that $E|_U = \sum_{i \leq r} E_i|_U$ and $E_i|_U$ is defined by (x_i) for $i \leq r$. If $B|_U = \sum_{i=1}^r b_i E_i|_U$, then $\Omega_U^j(\log E)(B)$ is generated over $\mathcal{O}(U)$ by forms of the form $\frac{1}{x_1^{b_1}\cdots x_r^{b_r}}\eta$, where $\eta \in \Omega_U^j(\log E)$. Since

$$d\left(\frac{1}{x_1^{b_1}\cdots x_r^{b_r}}\eta\right) = \frac{1}{x_1^{b_1}\cdots x_r^{b_r}} \cdot d\eta - \sum_{i=1}^r \frac{b_i}{x_1^{b_1}\cdots x_r^{b_r}} \cdot \frac{dx_i}{x_i} \wedge \eta,$$

we see that d preserves $\Omega_X^\bullet(\log E)(B)$.

LEMMA 4.26. *Let X be a smooth scheme over a perfect field k of characteristic $p > 0$ and $E = \sum_{i=1}^N$ an SNC divisor on X. If $B = \sum_{i=1}^N b_i E_i$ is a divisor on X such that $0 \leq b_i \leq p - 1$ for all i, then the inclusion of complexes*

$$F_* \Omega_X^\bullet(\log E) \hookrightarrow F_* \Omega_X^\bullet(\log E)(B)$$

is a quasi-isomorphism[3].

[3]Recall that a morphism of complexes is a quasi-isomorphism if it induces isomorphisms in cohomology.

PROOF. A morphism is a quasi-isomorphism if and only if this is the case locally. Therefore we may assume that X is affine and we have a system of algebraic coordinates $x_1, \ldots, x_n$ on X such that $E = \mathrm{div}_X(x_1 \cdots x_r)$. Let us write $B = \sum_{i=1}^r b_i E_i$, where E_i is defined by (x_i). Since the de Rham complexes in the statement are obtained by pulling-back corresponding de Rham complexes on $\mathbf{A}_k^n$ (we use here Lemma 4.9), and φ is flat, it follows that we may and will assume that $X = \mathbf{A}_k^n$. In this case we have

$$\Omega_X^\bullet(\log E) \simeq \otimes_{i=1}^r \mathrm{pr}_i^*\big(\Omega_{\mathbf{A}_k^1}^\bullet(\log\{0\})\big) \otimes \otimes_{i=r+1}^n \mathrm{pr}_i^*(\Omega_{\mathbf{A}_k^1}^\bullet) \quad \text{and}$$

$$\Omega_X^\bullet(\log E)(B) \simeq \otimes_{i=1}^r \mathrm{pr}_i^*\big(\Omega_{\mathbf{A}_k^1}^\bullet(\log\{0\})(B_i)\big) \otimes \otimes_{i=r+1}^n \mathrm{pr}_i^*(\Omega_{\mathbf{A}_k^1}^\bullet)$$

where $\mathrm{pr_i} \colon \mathbf{A}_k^n \to \mathbf{A}_k^1$ is the projection onto the i-th component and B_i is the divisor on $\mathbf{A}^1$ defined by (t^{b_i}). Since the cohomology of a tensor product of complexes of k-vector spaces is computed by the Künneth theorem, it follows that it is enough to consider the case when $X = \mathbf{A}^1 = \mathrm{Spec}\big(k[t]\big)$, E is the divisor defined by (t), and $B = bE$. In this case the complex $\Omega_{\mathbf{A}^1}^\bullet(\log E)(B)$ is the complex

$$0 \to k[t]\frac{1}{t^b} \xrightarrow{d} k[t]\frac{1}{t^{b+1}}dt,$$

placed in cohomological degrees 0 and 1. Since $0 \le b \le p-1$, it is then clear that

$$\mathcal{H}^0\big(\Omega_{\mathbf{A}^1}^\bullet(\log E)(B)\big) = k[t^p] \quad \text{and} \quad \mathcal{H}^1\big(\Omega_{\mathbf{A}^1}^\bullet(\log E)(B)\big) = k[t^p]\frac{1}{t}dt,$$

and we see that the inclusion

$$F_*\Omega_{\mathbf{A}^1}^\bullet(\log E) \hookrightarrow F_*\Omega_{\mathbf{A}^1}^\bullet(\log E)(B)$$

is a quasi-isomorphism. □

COROLLARY 4.27. Let X be a smooth scheme over a perfect field k of characteristic $p > 0$ and $E = \sum_{i=1}^N E_i$ an SNC divisor on X. If D is a $\mathbf{Q}$-divisor on X such that $\mathrm{Supp}\big(D - \lfloor D \rfloor\big) \subseteq \mathrm{Supp}(E)$, then the inclusion of complexes of $\mathcal{O}_X$-modules

$$F_*\big(\Omega_X^\bullet(\log E)(-pE + p\lceil D \rceil)\big) \hookrightarrow F_*\big(\Omega_X^\bullet(\log E)(-E + \lceil pD \rceil)\big)$$

is a quasi-isomorphism.

PROOF. By hypothesis, if $B = (p-1)E - p\lceil D \rceil + \lceil pD \rceil$, then $\mathrm{Supp}(B) \subseteq \mathrm{Supp}(E)$. Note that for every $u \in \mathbf{R}$, we have $u \le \lceil u \rceil < u + 1$, hence $pu \le p\lceil u \rceil < pu + p$, and thus

$$p\lceil u \rceil - \lceil pu \rceil \in \{0, \ldots, p-1\}$$

and consequently

$$(p-1) - p\lceil u \rceil + \lceil pu \rceil \in \{0, \ldots, p-1\}.$$

It follows that if we write $B = \sum_{i=1}^N b_i E_i$, then $0 \le b_i \le p-1$ for all i. Therefore we can apply the lemma to deduce that the inclusion of complexes of $\mathcal{O}_X$-modules

$$F_*\big(\Omega_X^\bullet(\log E)\big) \hookrightarrow F_*\big(\Omega_X^\bullet(\log E)(B)\big)$$

is a quasi-isomorphism. Tensoring by $\mathcal{O}_X\big(-E + \lceil D \rceil\big)$ and using the projection formula and Remark 4.11, we obtain the assertion in the corollary. □

Proof of Theorem 4.24. Arguing as in the proof of Corollary 4.22, we see that it is enough to prove the same vanishings in the case when k is a perfect field of characteristic $p > \dim(X)$ and (X, E) has a lift to $W_2(k)$. From now on, we assume that we are in this setting, hence Theorem 4.20 gives an isomorphism in $D^b_{\mathrm{coh}}(X)$:

$$F_*\Omega_X^\bullet(\log E) \simeq \bigoplus_{j=0}^{n} \Omega_X^j(\log E)[-j]. \tag{4.14}$$

After covering S by affine open subsets, we may assume that S is affine, in which case we need to show that

$$H^i\big(X, \Omega_X^j(\log E)(-E + \lceil D \rceil)\big) = 0 \quad \text{for} \quad i + j > n.$$

As in the proof of Theorem 4.21, the key point is to show that

$$\text{if} \quad H^i\big(X, \Omega_X^j(\log E)(-E + \lceil pD \rceil)\big) = 0 \quad \text{for all} \quad i + j > n, \tag{4.15}$$

$$\text{then} \quad H^i\big(X, \Omega_X^j(\log E)(-E + \lceil D \rceil)\big) = 0 \quad \text{for all} \quad i + j > n.$$

For this, we consider the complex

$$C^\bullet := F_*\big(\Omega_X^\bullet(\log E)(-E + \lceil pD \rceil)\big).$$

Under the assumption in (4.15), the spectral sequence of hypercohomology

$$E_1^{j,i} = H^i\big(X, \Omega_X^j(\log E)(-E + \lceil pD \rceil)\big) \Rightarrow H^{i+j}(X, C^\bullet)$$

associated to $C^\bullet$ has the property that $E_1^{j,i} = 0$ for $i + j > n$. This implies that $H^m(C^\bullet) = 0$ for $m > n$. On the other hand, it follows from Corollary 4.27 that in $D^b_{\mathrm{coh}}(X)$ we have an isomorphism

$$C^\bullet \simeq F_*\big(\Omega_X^\bullet(\log E)(-pE + p\lceil D \rceil)\big) \simeq \big(F_*\Omega_X^\bullet(\log E)\big) \otimes \mathcal{O}_X(-E + \lceil D \rceil),$$

hence (4.14) gives an ismomorphism in $D^b_{\mathrm{coh}}(X)$

$$C^\bullet \simeq \bigoplus_j \Omega_X^j(\log E)(-E + \lceil D \rceil)[-j].$$

Therefore

$$\bigoplus_j H^{m-j}\big(X, \Omega_X^j(\log E)(-E + \lceil D \rceil)\big) = H^m(C^\bullet) = 0 \quad \text{for} \quad m > n,$$

and we obtain the conclusion in (4.15).

Applying (4.15) for D replaced by $pD, \ldots, p^e D$, we conclude that for every $e \geq 1$,

$$\text{if} \quad H^i\big(X, \Omega_X^j(\log E)(-E + \lceil p^e D \rceil)\big) = 0 \quad \text{for all} \quad i + j > n, \tag{4.16}$$

$$\text{then} \quad H^i\big(X, \Omega_X^j(\log E)(-E + \lceil D \rceil)\big) = 0 \quad \text{for all} \quad i + j > n.$$

Finally, let us fix a positive integer r such that rD is an integral divisor. Note that if $\ell \in \{0, 1, \ldots, r-1\}$ and $e \geq 1$ is such that $p^e \equiv \ell \pmod{r}$, then

$$\lceil p^e D \rceil = \frac{p^e - \ell}{r}(rD) + \lceil \ell D \rceil.$$

Since there are only finitely many such divisors $\lceil \ell D \rceil$, using the fact that D is ample and asymptotic Serre vanishing, we conclude that for $e \gg 0$, we have

$$H^i\big(X, \Omega_X^j(\log E)(-E + \lceil p^e D \rceil)\big) = 0 \quad \text{for all} \quad i + j > n.$$

By (4.16), this completes the proof of the theorem. □

4.4. Multiplier and test ideals: comparison via reduction mod p

Our goal in this chapter is to compare, when starting with an ideal $\mathfrak{a}$ on a smooth variety X in characteristic 0, the reductions to positive characteristic of the multiplier ideals of $\mathfrak{a}$ with the test ideals associated to the reduction of $\mathfrak{a}$ to positive characteristic. In particular, since our situation will be coming from characteristic 0, we are going to have a log resolution for the reduction of $(X, \mathfrak{a})$. The main result here is due to Hara and Yoshida [**HY03**], but the idea of using the de Rham complex goes back to the work showing that rational singularities are of dense F-rational type, due independently to Hara [**Ha98**] and Mehta and Srinivas [**MS97**]. However, we follow the presentation in [**BHLM12**], which avoids the use of local cohomology and tight closure.

We first begin by comparing in one fixed positive characteristic the test ideals and the multiplier ideals. We will see that the multiplier ideal always contains the test ideal and we will give a criterion for when they are equal. We will work in the following setting.

Assumption 4.28. We assume that k is a perfect field of characteristic $p > 0$ and X is a smooth, irreducible scheme over k, of dimension n. Let $\mathfrak{a}$ be a nonzero ideal on X and we assume that we have a projective log resolution $f\colon Y \to X$ of $(X, \mathfrak{a})$. We write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z)$ and

$$Z = \sum_{i=1}^{N} a_i E_i \quad \text{and} \quad K_{Y/X} = \sum_{i=1}^{N} k_i E_i.$$

By assumption, the divisor $E = \sum_{i=1}^{N} E_i$ has simple normal crossings. Note that in this case we have

$$\mathcal{J}(\mathfrak{a}^\lambda) = f_*\mathcal{O}_Y(K_{Y/X} - \lfloor \lambda F \rfloor) \quad \text{for every} \quad \lambda \in \mathbf{R}_{\geq 0}.$$

We begin with an easy lemma that shows that for SNC divisors we have equality.

Lemma 4.29. *With the notation in Assumption 4.28, if $\mathfrak{b} = \mathcal{O}_Y(-Z)$, we have*

$$\tau(\mathfrak{b}^\lambda) = \mathcal{J}(\mathfrak{b}^\lambda) \quad \textit{for every} \quad \lambda \in \mathbf{R}_{\geq 0}.$$

Proof. Since it is enough to check this locally, we may assume that Y is affine and we have algebraic coordinates $x_1, \ldots, x_n$ such that $\mathfrak{b} = (x_1^{a_1} \cdots x_r^{a_r})$. Note that in this case, $\mathcal{J}(\mathfrak{b}^\lambda)$ is generated by $x_1^{\lfloor \lambda a_1 \rfloor} \cdots x_r^{\lfloor \lambda a_r \rfloor}$.

In order to compute $\tau(\mathfrak{b}^\lambda)$, let us compute $(\mathfrak{b}^{\lceil \lambda p^e \rceil})^{[1/p^e]}$ for $e \geq 1$. By Corollary 4.10, a basis of $\mathcal{O}_Y(Y)$ over $\mathcal{O}_Y(Y)^{p^e}$ is given by $x^u = x_1^{u_1} \cdots x_n^{a_n}$, where $0 \leq a_i \leq p^e - 1$ for all i. It then follows from Proposition 3.15 that if $b_i = \lceil \lambda a_i p^e \rceil$, then

$$(\mathfrak{b}^{\lceil \lambda p^e \rceil})^{[1/p^e]} = \left(x_1^{b_1} \cdots x_r^{b_r}\right)^{[1/p^e]} = \left(x_1^{\lfloor b_1/p^e \rfloor} \cdots x_r^{\lfloor b_r/p^e \rfloor}\right).$$

For every i, we have

$$\lambda a_i \leq \frac{b_i}{p^e} < \lambda a_i + \frac{1}{p^e},$$

hence for $e \gg 0$, we have $\lfloor b_i/p^e \rfloor = \lfloor \lambda a_i \rfloor$. By definition of the test ideal, it follows that $\tau(\mathfrak{b}^\lambda) = \mathcal{J}(\mathfrak{b}^\lambda)$. □

A key tool for the comparison between multiplier and test ideals is provided by a commutative diagram relating the trace maps on X and on Y. Note first that we have a canonical isomorphism $\rho\colon f_*\omega_Y \to \omega_X$: the map

$$f^*\omega_X \to \omega_Y = f^*\omega_X(K_{Y/X})$$

induces after applying f_* a map

$$\omega_X = f_*f^*(\omega_X) \to f_*\omega_Y = f_*f^*(\omega_X) \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y(K_{Y/X}),$$

which is an isomorphism since $K_{Y/X}$ is an exceptional effective divisor (see Lemma 2.31). We denote by ρ the inverse isomorphism. Note that if $\mathfrak{b} \subseteq \mathcal{O}_Y$ is an ideal, then it follows from the above description of ρ that

$$\rho\big(f_*(\mathfrak{b}\cdot\omega_Y)\big) = f_*\big(\mathfrak{b}\cdot\mathcal{O}_X(K_{Y/X})\big). \tag{4.17}$$

We have the following commutative diagram relating the trace maps on X and Y via ρ for every $e \geq 1$:

$$\begin{tikzcd} f_*(F_Y^e)_*\omega_Y \arrow[r, "f_*t_Y^e"] \arrow[d, "(F_X^e)_*\rho"'] & f_*\omega_Y \arrow[d, "\rho"] \\ (F_X^e)_*\omega_X \arrow[r, "t_X^e"] & \omega_X, \end{tikzcd} \tag{4.18}$$

where we note that $f_*(F_Y^e)_*\omega_Y = (F_X^e)_*f_*\omega_Y$. In order to check the commutativity of the diagram, it is enough to check[4] it on the open subset U of X over which f is an isomorphism, but there commutativity is clear.

The following is the first general result concerning the relation between multiplier ideals and test ideals.

Proposition 4.30. With the notation in Assumption 4.28, we have

$$\tau(\mathfrak{a}^\lambda) \subseteq \mathcal{J}(\mathfrak{a}^\lambda) \quad \text{for all} \quad \lambda \in \mathbf{R}_{\geq 0}.$$

Proof. It is enough to show that

$$(\mathfrak{a}^m)^{[1/p^e]} \subseteq \mathcal{J}(\mathfrak{a}^{m/p^e}) \quad \text{for all} \quad m \geq 0,\, e \geq 1. \tag{4.19}$$

Indeed, if this holds, then for every $\lambda \in \mathbf{R}_{\geq 0}$, by taking $e \gg 0$, we obtain

$$\tau(\mathfrak{a}^\lambda) = \big(\mathfrak{a}^{\lceil \lambda p^e \rceil}\big)^{[1/p^e]} \subseteq \mathcal{J}\big(\mathfrak{a}^{\lceil \lambda p^e\rceil/p^e}\big) = \mathcal{J}(\mathfrak{a}^\lambda),$$

where the last equality follows from the fact that $\frac{\lceil \lambda p^e \rceil}{p^e} - \lambda$ is nonnegative and converges to 0 when e goes to infinity.

In order to prove (4.19), we compute the image of

$$\mathcal{M} := f_*(F_Y^e)_*\omega_Y(-mZ) \subseteq f_*(F_Y^e)_*\omega_Y$$

via the two compositions in the commutative diagram (4.18). On Y we have a surjective map

$$(F_Y^e)_*\omega_Y(-mZ) \xrightarrow{t_Y^e} \omega_Y\big(-\lfloor m/p^e Z\rfloor\big)$$

[4]We use the fact that if $\mathcal{E}$ is a torsion-free sheaf on an integral scheme X, and $j\colon U \hookrightarrow X$ is an open immersion, then the canonical map $\mathcal{E} \hookrightarrow j_*(\mathcal{E}|_U)$ is an isomorphism. It follows that for every coherent sheaf $\mathcal{F}$ and any two morphisms $\varphi, \psi\colon \mathcal{F} \to \mathcal{E}$, we have $\varphi = \psi$ if and only if $\varphi|_U = \psi|_U$.

by Lemma 4.29 (see also Proposition 4.18). We deduce that the image of $\mathcal{M}$ by the top horizontal map in the diagram (4.18) lies inside $f_*\omega_Y\big(-\lfloor m/p^e Z\rfloor\big)$, and thus its further image via ρ lies inside

$$f_*\mathcal{O}_Y\big(K_{Y/X} - \lfloor m/p^e Z\rfloor\big)\cdot\omega_X = \mathcal{J}(\mathfrak{a}^{m/p^e})\omega_X.$$

On the other hand, we have $\mathfrak{a}^m \subseteq f_*\mathcal{O}_Y(K_{Y/X} - mZ)$ since $K_{Y/X}$ is effective, hence $(F_X^e)_*(\mathfrak{a}^m\omega_X)$ is contained in the image of $\mathcal{M}$ via the left vertical map in (4.18). This implies that

$$\begin{aligned}(\mathfrak{a}^m)^{[1/p^e]}\omega_X = t_X^e\big((F_X^e)_*(\mathfrak{a}^m\omega_X)\big) &\subseteq (t_X^e\circ(F_X^e)_*\rho)(\mathcal{M}) = (\rho\circ f_*t_Y^e)(\mathcal{M})\\ &\subseteq \mathcal{J}(\mathfrak{a}^{m/p^e})\omega_X,\end{aligned}$$

where the first equality follows from Proposition 4.18. This completes the proof of (4.19) and thus the proof of the proposition. □

From now on, we fix $\lambda > 0$.

ASSUMPTION 4.31. Let us choose $\mu > \lambda$ such that $\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^\mu)$ (it is enough to take μ such that $\lambda < \mu < \frac{\lfloor\lambda a_i\rfloor+1}{a_i}$ for all i with $a_i > 0$). We choose[5] a $\mathbf{Q}$-divisor D on Y such that D is f-ample and $-D$ is effective and supported on $\mathrm{Supp}(E)$. We put $G = \mu(D - Z)$. Since $-D$ is effective, after possibly replacing D by ϵD, with $0 < \epsilon \ll 1$, we may and will assume that

$$\lceil G\rceil = \lceil -\mu Z\rceil. \tag{4.20}$$

Note that by construction $G \leq 0$ and $\mathrm{Supp}(G) \subseteq \mathrm{Supp}(E)$.

PROPOSITION 4.32. With the above notation, if the canonical map

$$f_*(F_Y^e)_*\omega_Y\big(\lceil p^e G\rceil\big) \to f_*\omega_Y\big(\lceil G\rceil\big)$$

induced by t_Y^e is surjective for every $e \geq 1$, then $\tau(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)$.

We first give a weak version of Skoda's theorem for multiplier ideals in positive characteristic.

LEMMA 4.33. *With the notation in Assumption 4.28, there is a positive integer r such that*

$$\mathcal{J}(\mathfrak{a}^m) \subseteq \mathfrak{a}^{m-r} \quad \textit{for every integer} \quad m \geq r.$$

PROOF. Let $\mathcal{R} = \oplus_{m\geq 0}\mathfrak{a}^m$. If we show that

$$\mathcal{M} := \bigoplus_{m\geq 0}\mathcal{J}(\mathfrak{a}^m) = \bigoplus_{m\geq 0} f_*\mathcal{O}_Y(K_{Y/X} - mZ)$$

is a finitely generated $\mathcal{R}$-module, then the assertion in the lemma follows easily: we can simply choose r such that $\mathcal{M}$ is locally generated in degrees $\leq r$. In fact, we will show more generally that for every coherent sheaf $\mathcal{F}$ on Y, the $\mathcal{R}$-module $\oplus_{m\geq 0}f_*\mathcal{F}(-mZ)$ is finitely generated.

---

[5]For our purpose, by working locally on X, we may assume that X is affine. In this case there is an effective divisor H on Y which is f-ample and does not contain any f-exceptional divisor in its support, so that $D = H - f^*\big(f_*(H)\big)$ is clearly f-ample, supported on $\mathrm{Exc}(f)$, and $-D$ is effective. However, such D exists in general: if G is a fixed f-ample divisor on X and $D = G - f^*\big(f_*(G)\big)$, then D is f-ample and supported on $\mathrm{Exc}(f)$, hence $-D$ is effective by the Negativity Lemma, see [**KM98**, Lemma 3.39].

By assumption, f factors as $Y \xrightarrow{h} W \xrightarrow{g} X$, where $W = \mathrm{Bl}_{\mathfrak{a}}(X) = \mathcal{P}roj(\mathcal{R})$. If we write $\mathfrak{a} \cdot \mathcal{O}_W = \mathcal{O}_W(-T)$, for an effective Cartier divisor T on W, then $\mathcal{O}_Y(-Z) = h^*\mathcal{O}_W(-T)$ and $\mathcal{O}_W(-T) \simeq \mathcal{O}_W(1)$, hence using standard properties of the Proj construction, we see that the graded $\mathcal{R}$-module associated to the coherent sheaf $h_*(\mathcal{F})$ on W:

$$\bigoplus_{m\geq 0} f_*\mathcal{F}(-mZ) = \bigoplus_{m\geq 0} g_*\big(h_*(\mathcal{F}) \otimes \mathcal{O}_W(m)\big)$$

is a finitely generated $\mathcal{R}$-module. This completes the proof. □

Proof of Proposition 4.32. Note first that t_Y^e induces indeed a map as in the statement: this follows by the projection formula and Remark 4.11, since $p^e\lceil G\rceil \geq \lceil p^e G\rceil$. We argue as in the proof of Proposition 4.30, by describing the image of

$$\mathcal{M} := (F_Y^e)_* f_*\omega_Y\big(\lceil p^e G\rceil\big) \subseteq f_*(F_Y^e)_*\omega_Y$$

via the two compositions in the commutative diagram (4.18). Note first that by assumption, the image of $\mathcal{M}$ via the top horizontal map in the diagram is

$$f_*\omega_Y\big(\lceil G\rceil\big) = f_*\omega_Z\big(-\lfloor \mu Z\rfloor\big),$$

where the equality follows from our assumption (4.30). In turn, the image of this by ρ is $\mathcal{J}(\mathfrak{a}^\mu) \cdot \omega_X$.

On the other hand, the image of $\mathcal{M}$ by the left vertical map in the diagram is $(F_X^e)_*(J_e \cdot \omega_X)$, where

$$J_e = f_*\mathcal{O}_Y(K_{Y/X} + \lceil p^e G\rceil).$$

Using the commutativity of the diagram and Proposition 4.18, we thus conclude that

$$\mathcal{J}(\mathfrak{a}^\mu) = J_e^{[1/p^e]} \quad \text{for every} \quad e \geq 1.$$

Let r be as in Lemma 4.33. Since $D \leq 0$, we see that

$$J_e = f_*\mathcal{O}_Y(K_{Y/X} - \lfloor \mu p^e(Z-D)\rfloor) \subseteq f_*\mathcal{O}_Y(K_{Y/X} - \lfloor \mu p^e Z\rfloor) = \mathcal{J}(\mathfrak{a}^{\mu p^e}).$$

By our assumption on μ, we thus get

$$\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^\mu) = J_e^{[1/p^e]} \subseteq \mathcal{J}(\mathfrak{a}^{\mu p^e})^{[1/p^e]} \subseteq \big(\mathfrak{a}^{\lfloor \mu p^e\rfloor - r}\big)^{[1/p^e]} \subseteq \tau(\mathfrak{a}^{\alpha_e}),$$

where $\alpha_e = (\lfloor \mu p^e\rfloor - r)/p^e$. Since

$$\lim_{e\to\infty} \alpha_e = \mu > \lambda,$$

we see that for $e \gg 0$, we have $\tau(\mathfrak{a}^{\alpha_e}) \subseteq \tau(\mathfrak{a}^\lambda)$, and thus $\mathcal{J}(\mathfrak{a}^\lambda) \subseteq \tau(\mathfrak{a}^\lambda)$. Since the opposite inclusion follows from Proposition 4.30, this completes the proof. □

Proposition 4.34. Let $f\colon Y \to S$ be a morphism of schemes of finite type over k, with Y smooth and irreducible, of dimension n. If E is an SNC divisor on Y and G is a $\mathbf{Q}$-divisor on Y supported on E, then the canonical morphism

$$f_*(F_Y)_*^e\big(\omega_Y(\lceil p^e G\rceil)\big) \to f_*\big(\omega_Y(\lceil G\rceil)\big)$$

is surjective for every $e \geq 1$, provided the the following conditions hold:

$$R^i f_*\big(\Omega_Y^{n-i}(\log E)(-E + \lceil p^\ell G\rceil)\big) = 0 \quad \text{for all} \quad i \geq 1 \quad \text{and} \quad \ell \geq 1, \quad \text{and} \tag{4.21}$$

$$R^{i+1} f_*\big(\Omega_Y^{n-i}(\log E)(-E + \lceil p^\ell G\rceil)\big) = 0 \quad \text{for all} \quad i \geq 1 \quad \text{and} \quad \ell \geq 0. \tag{4.22}$$

PROOF. It is enough to treat the case $e=1$: indeed, if $e>1$, then the map in the statement is the composition

$$f_*(F_Y)_*^e\big(\omega_Y(\lceil p^eG\rceil)\big)\to f_*(F_Y)_*^{e-1}\big(\omega_Y(\lceil p^{e-1}G\rceil)\big)\to\cdots\to f_*\big(\omega_Y(\lceil G\rceil)\big)$$

and we see that this is surjective by applying the case $e=1$ for the divisors $G, pG,\ldots,p^{e-1}G$. Therefore we now assume that $e=1$ (in which case we will see that we only need (4.21) for $\ell=1$ and (4.22) for $\ell=0$).

We consider the complex of $\mathcal{O}_Y$-modules

$$C_* := (F_Y)_*\big(\Omega_Y^\bullet(\log E)(-E+\lceil pG\rceil)\big).$$

Note that Corollary 4.27 implies that the inclusion

$$(F_Y)_*\Omega_Y^\bullet(\log E)\otimes\mathcal{O}_Y(-E-\lceil G\rceil)\simeq(F_Y)_*\big(\Omega_Y^\bullet(\log E)(-pE+p\lceil G\rceil)\big)\hookrightarrow C^\bullet$$

is a quasi-isomorphism, where the isomorphism follows from the projection formula and Remark 4.11. The Cartier isomorphism (see Theorem 4.12) thus implies that for every i, with $0\le i\le n$, we have an isomorphism

$$\mathcal{H}^i(C^\bullet)\simeq\Omega_Y^i(\log E)\otimes\mathcal{O}_Y\big(-E+\lceil G\rceil\big). \tag{4.23}$$

It is then straightforward to check that, using this isomorphism for $i=n$, the map in the statement of the proposition is identified with the map obtained by applying f_* to the canonical surjection $C^n\to\mathcal{H}^n(C^\bullet)$. For every i, with $0\le i\le n$, we put

$$\mathcal{B}^i:=\operatorname{Im}\big(C^{i-1}\xrightarrow{d}C^i\big)\quad\text{and}\quad\mathcal{Z}^i:=\operatorname{Ker}\big(C^i\xrightarrow{d}C^{i+1}\big),$$

so that for $0\le i\le n$ we have exact sequences

$$0\to\mathcal{B}^i\to\mathcal{Z}^i\to\mathcal{H}^i(C^\bullet)\to 0\quad\text{and} \tag{4.24}$$

$$0\to\mathcal{Z}^i\to C^i\to\mathcal{B}^{i+1}\to 0. \tag{4.25}$$

Note that condition (4.21) for $\ell=1$ says that $R^if_*C^{n-i}=0$ for $i\ge1$ and condition (4.22) says, using the isomorphism (4.23), that $R^{i+1}f_*\big(\mathcal{H}^{n-i}(C^\bullet)\big)=0$ for $i\ge1$. By taking the long exact sequence for higher direct images corresponding to (4.24), we get the exact sequence

$$R^{i+1}f_*(\mathcal{B}^{n-i})\to R^{i+1}f_*(\mathcal{Z}^{n-i})\to R^{i+1}f_*\big(\mathcal{H}^{n-i}(C^\bullet)\big)=0 \tag{4.26}$$

and by taking that corresponding to (4.25), we get the exact sequence

$$0=R^if_*(C^{n-i})\to R^if_*(\mathcal{B}^{n-i+1})\to R^{i+1}f_*(\mathcal{Z}^{n-i}). \tag{4.27}$$

We prove by decreasing induction on i, with $0\le i\le n$, that

$$R^{i+1}f_*(\mathcal{B}^{n-i})=0. \tag{4.28}$$

This clearly holds for $i=n$ since $R^{n+1}f_*=0$ (recall that $\dim(Y)=n$). Suppose now that (4.28) holds for $i\ge1$. In this case it follows from (4.26) that $R^{i+1}f_*(\mathcal{Z}^{n-i})=0$ and then we deduce from (4.27) that $R^if_*(\mathcal{B}^{n-i+1})=0$. This completes the proof of the induction step. For $i=0$, we get from (4.28) that $R^1f_*(\mathcal{B}^n)=0$. In this case, the long exact sequence for higher direct images associated to (4.24) for $i=n$ implies that the induced map $f_*(C^n)\to f_*\big(\mathcal{H}^n(C^\bullet)\big)$ is surjective. This gives the assertion in the proposition. $\square$

REMARK 4.35. We note that in the setting of reduction to positive characteristic and under suitable positivity conditions on G, the vanishings in (4.22) will be guaranteed by the corresponding vanishings in characteristic 0 (see Theorem 4.24). On the other hand, the vanishings in (4.21) will be guaranteed by asymptotic Serre vanishing when taking $p \gg 0$.

We next turn to the comparison between multiplier ideals in characteristic 0 and test ideals in positive characteristic. From now on, until the end of this section, we work in the following setting. Let X be a smooth, irreducible n-dimensional variety over an algebraically closed field k of characteristic 0 and let $\mathfrak{a}$ be a nonzero ideal in $\mathcal{O}_X$. We fix a projective log resolution $f\colon Y \to X$ of $(X, \mathfrak{a})$ and write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z)$ and let $E = \sum_i E_i$ be the reduced SNC divisor whose support is $\mathrm{Supp}(Z) \cup \mathrm{Exc}(f)$. If d is the least common multiple of the coefficients of Z, then it is clear that if $\lambda \in \left[\frac{i}{d}, \frac{i+1}{d}\right)$, then $\mathcal{J}(\mathfrak{a}^\lambda) = \tau(\mathfrak{a}^{i/d})$.

We choose $A \in \mathrm{FG}_{\mathbf{Z}}(k)$ such that we have models $f_A\colon Y_A \to X_A$ and $\mathfrak{a}_A \subseteq \mathcal{O}_{X_A}$ and $(E_i)_A$ for f, $\mathfrak{a}$, and E_i, respectively, over A. We may and will assume that for every $\lambda \leq n$ we have a model $\mathcal{J}(\mathfrak{a}^\lambda)_A$ of the multiplier $\mathcal{J}(\mathfrak{a}^\lambda)$ over A, as follows: we choose models $\mathcal{J}(\mathfrak{a}^\lambda)_A$ for $\lambda \leq n$ with $d\lambda \in \mathbf{Z}$ and then put $\mathcal{J}(\mathfrak{a}^\lambda)_A = \mathcal{J}(\mathfrak{a}^{i/d})$ if $\lambda \in \left[\frac{i}{d}, \frac{i+1}{d}\right)$ with $\lambda \leq n$; for $\lambda > n$, because of Skoda's theorem for multiplier ideals (see Corollary 2.111), we may and will take

$$\mathcal{J}(\mathfrak{a}^\lambda)_A := \mathfrak{a}_A^{\lceil \lambda \rceil - r} \cdot \mathcal{J}(\mathfrak{a}^{\lambda - \lceil \lambda \rceil + r})_A.$$

After possibly inverting a suitable nonzero element $a \in A$, we may and will assume that for all closed point $s \in S$, X_s is a smooth, irreducible, n-dimensional variety over the finite field $k(s)$ (see Remark 4.5). Moreover, we may and will assume that the induced morphism $f_s\colon Y_s \to X_s$ gives a log resolution of $(X_s, \mathfrak{a}_s)$, with corresponding SNC divisor E_s. Furthermore, it follows from Remark 4.4 that we may and will assume that

$$\mathcal{J}(\mathfrak{a}^\lambda)_s = \mathcal{J}(\mathfrak{a}_s^\lambda) \quad \text{for all} \quad \lambda \leq n \tag{4.29}$$

(indeed, it is enough to guarantee this equality for those λ with $d\lambda \in \mathbf{Z}$ and there are only finitely many such λ that are $\leq n$).

The following are the main results, due to Hara and Yoshida [**HY03**], concerning the comparison between multiplier ideals and test ideals.

PROPOSITION 4.36. With the above notation, for every closed point $s \in U$, we have

$$\tau(\mathfrak{a}_s^\lambda) \subseteq \mathcal{J}(\mathfrak{a}^\lambda)_s \quad \text{for all} \quad \lambda \geq 0.$$

PROOF. By Skoda's theorem for test ideals (see Theorem 3.34) and the way we defined $\mathcal{J}(\mathfrak{a}^\lambda)$ for $\lambda > n$, it is clear that it is enough to guarantee the inclusion in the proposition for $\lambda \leq n$. This follows from (4.29) and Proposition 4.30. $\square$

THEOREM 4.37. *With the above notation, for every $\lambda \geq 0$ there is a nonempty open subset $U_\lambda \subseteq \mathrm{Spec}(A)$ such that*

$$\tau(\mathfrak{a}_s^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)_s \quad \textit{for all} \quad s \in U_\lambda.$$

REMARK 4.38. The assertion in the theorem is independent of the model. This follows from Remark 4.6 and the fact that if we have a smooth, irreducible scheme W over a finite field k, and a nonzero ideal $\mathfrak{a}$ on W such that $(W, \mathfrak{a})$ has a log resolution, then for every finite extension K/k, if we put X_K and $\mathfrak{a}_K$ for the base-change of X and $\mathfrak{a}$ to $\mathrm{Spec}(K)$, then $\mathcal{J}(\mathfrak{a}^\lambda) = \tau(\mathfrak{a}^\lambda)$ if and only if $\mathcal{J}(\mathfrak{a}_K^\lambda) = \tau(\mathfrak{a}_K^\lambda)$. Indeed, we have $\mathcal{J}(\mathfrak{a}_K^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)\cdot\mathcal{O}_{X_K}$ by flat base-change and $\tau(\mathfrak{a}_K^\lambda) = \tau(\mathfrak{a}^\lambda)\cdot\mathcal{O}_{X_K}$ by Proposition 3.31, since the morphism $X_K \to X$ is smooth.

Proof of Theorem 4.37. Again, by Skoda's theorem for test ideals (see Theorem 3.34) and the way we defined $\mathcal{J}(\mathfrak{a}^\lambda)$ for $\lambda > n$, it is clear that it is enough to treat the case when $\lambda \leq n$. We proceed as in Assumption 4.31: we first choose $\mu > \lambda$ such that $\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^\mu)$. We next choose a $\mathbf{Q}$-divisor D on Y such that D is f-ample and $-D$ is effective and supported on $\mathrm{Supp}(E)$; moreover, if $G = \mu(D - Z)$, then

$$\lceil G \rceil = \lceil -\mu Z \rceil. \tag{4.30}$$

Note that G is again f-ample: this assertion is local on X, hence we may assume that X is affine; in this case $\mathfrak{a}$ is globally generated on X, hence $\mathcal{O}_Y(-Z)$ is globally generated on Y, which easily implies our assertion.

After possibly inverting a nonzero element in A, we may assume that for every closed point $s \in \mathrm{Spec}(A)$, the divisor G_s is f_s-ample. Therefore μ and G_s satisfy the conditions in Assumption 4.31 with respect to $\mathfrak{a}_s$ and f_s. By Propositions 4.32 and 4.34, it is enough to find a nonempty open subset U_λ of $\mathrm{Spec}(A)$ such that for every closed point $s \in U_\lambda$, if $p = \mathrm{char}\big(k(s)\big)$, then we have

$$R^i(f_s)_*\big(\Omega_{Y_s}^{n-i}(\log E_s)(-E_s + \lceil p^\ell G_s \rceil)\big) = 0 \quad \text{for all} \quad i \geq 1 \quad \text{and} \quad \ell \geq 1, \quad \text{and} \tag{4.31}$$

$$R^{i+1}(f_s)_*\big(\Omega_{Y_s}^{n-i}(\log E_s)(-E_s + \lceil p^\ell G_s \rceil)\big) = 0 \quad \text{for all} \quad i \geq 1 \quad \text{and} \quad \ell \geq 0. \tag{4.32}$$

We first use Theorem 4.24 and choose U_λ such that for every closed point $s \in \mathrm{Spec}(A)$, the vanishings in (4.32) hold for $\ell = 0$. Let us now choose d such that dG is an integral divisor. We use Lemma 4.8 in order to choose m_0 and U_λ such that, in addition, we have

$$R^i(f_s)_*\big(\Omega_{Y_s}^{j}(\log E_s)(-E_s + \lceil m G_s \rceil)\big) = 0 \quad \text{for all} \quad i \geq 1,\ j \geq 0 \quad \text{and} \quad m \geq m_0 \tag{4.33}$$

for all closed points $s \in U_\lambda$; indeed, it is enough to apply the lemma for the line bundle $\mathcal{L} = \mathcal{O}_Y(dG)$ and the sheaves

$$\Omega_Y^j(\log E)(-E + \lceil tG \rceil), \quad \text{with} \quad 0 \leq t \leq d-1.$$

If we shrink U_λ such that for every closed point $s \in U_\lambda$, we have $\mathrm{char}\big(k(s)\big) \geq m$, then the vanishings in (4.33) imply the ones in (4.31) as well as those in (4.32) with $\ell \geq 1$. This completes the proof of the theorem. □

We obtain the following consequence relating the log canonical threshold of an ideal in characteristic 0 to the F-pure thresholds of its reductions mod p.

Corollary 4.39. Let A be a domain, finitely generated over $\mathbf{Z}$, and let K the algebraic closure of $\mathrm{Frac}(A)$. If X_A is a smooth scheme over $\mathrm{Spec}(A)$, with $X_K := X_A \times_A K$ connected, and $\mathfrak{a}$ is a nonzero ideal on X_A and $\mathfrak{a}_K$ denotes the corresponding ideal on $X_K := X \times_A K$, then the following hold:

i) There is an open subset U in $\mathrm{Spec}(A)$ such that for every closed point $s \in U$, we have
$$\mathrm{lct}(\mathfrak{a}_K) \geq \mathrm{fpt}(\mathfrak{a}_s).$$

ii) For every $\epsilon > 0$, there is an open subset $V_\epsilon \subseteq \mathrm{Spec}(A)$ such that for every closed point $s \in V_\epsilon$, we have

$$\mathrm{fpt}(\mathfrak{a}_s) > \mathrm{lct}(\mathfrak{a}_K) - \epsilon.$$

PROOF. Arguing as in Remark 4.38, we see that in order to prove the assertion, we can choose a different model of X_K and $\mathfrak{a}_K$. Therefore we may and will assume that we have a morphism $f\colon Y_A \to X_A$ that gives a model for a log resolution of $(X_K, \mathfrak{a}_K)$. In this case the first assertion follows from Proposition 4.36 and the second assertion follows from Theorem 4.37 applied for some $\lambda \in \big(\mathrm{lct}(\mathfrak{a}) - \epsilon, \mathrm{lct}(\mathfrak{a})\big)$. $\square$

REMARK 4.40. It is typical that in the situation in the corollary there is no open subset V of $\mathrm{Spec}(A)$ such that $\mathrm{lct}(\mathfrak{a}_K) = \mathrm{fpt}(\mathfrak{a}_s)$ for all closed points $s \in \mathrm{Spec}(A)$, see Example 3.69. In particular, this shows that the subsets U_λ in Theorem 4.37 can not be taken independently of λ.

4.5. Reduction mod p and the Weak Ordinarity Conjecture

In this section we discuss one more connection between multiplier ideals and test ideals, which is still conjectural, and its relation with a conjecture of arithmetic flavor.

4.5.1. A conjectural relation between multiplier ideals and test ideals. Suppose that we are in the same setting as in Theorem 4.37: let X be a smooth, irreducible n-dimensional variety over an algebraically closed field k of characteristic 0 and let $\mathfrak{a}$ be a nonzero ideal in $\mathcal{O}_X$. After fixing a log resolution $f\colon Y \to X$ of $(X, \mathfrak{a})$, we choose models X_A, $\mathfrak{a}_A$, and f_A over $\mathrm{Spec}(A)$, for some $A \in \mathrm{FG}_{\mathbf{Z}}(k)$, as well as models $\mathcal{J}(\mathfrak{a}^\lambda)_A$ for the multiplier ideals $\mathcal{J}(\mathfrak{a}^\lambda)$, with $\lambda \geq 0$.

CONJECTURE 4.41. With the above notation, there is a dense subset of closed points $S \subseteq \mathrm{Spec}(A)$ such that for every $s \in S$, we have

$$\tau(\mathfrak{a}_s^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)_s \quad \text{for all} \quad \lambda \geq 0.$$

A special case of the above conjecture predicts the following connection between log canonical and F-pure thresholds:

CONJECTURE 4.42. Let A be a domain, finitely generated over $\mathbf{Z}$, and let K the algebraic closure of $\mathrm{Frac}(A)$. If X_A is a smooth scheme over $\mathrm{Spec}(A)$, with $X_K := X \times_A K$ connected, $\mathfrak{a}$ is a nonzero ideal on X_A, and $\mathfrak{a}_K$ denotes the corresponding ideal on $X_K := X \times_A K$, then there is a dense open subset of closed points $S \subseteq \mathrm{Spec}(A)$ such that

$$\mathrm{fpt}(\mathfrak{a}_s) = \mathrm{lct}(\mathfrak{a}_K) \quad \text{for all} \quad s \in S.$$

EXAMPLE 4.43. If $A = \mathbf{Z}$ and $X = \mathrm{Spec}\big(\mathbf{Z}[x, y]\big)$ and $\mathfrak{a} = (x^2 + y^3)$, then it follows from Example 3.69 that the set of prime ideals $p\mathbf{Z}$, with $p \equiv 1 \pmod 3$ satisfies the conclusion of Conjecture 4.42.

EXAMPLE 4.44. If $A = \mathbf{Z}$ and $X = \mathrm{Spec}\big(\mathbf{Z}[x, y, z]\big)$ and $\mathfrak{a}$ is generated by a homogeneous polynomial of degree 3 such that the corresponding curve Y in $\mathbf{P}^2_{\mathbf{Q}}$ is smooth (hence an elliptic curve), then we have seen in Example 3.72 that if p is a prime such that the curve Y_p is smooth, then the equality in Conjecture 4.42 holds at p if and only if the elliptic curve Y_p is ordinary. We will see in Example 4.63

below that the conjecture is satisfied in this case. In fact, one can say more about the set of primes that satisfy the conjecture. The behavior depends on whether Y has complex multiplication. If this is the case and if K is the associated quadratic extension of $\mathbf{Q}$, then the reduction Y_p is ordinary if and only if p splits completely in K; this is the case for a set of primes of density 1/2. On the other hand, if Y does not have complex multiplication (which is the"generic case"), then Serre [**Ser72**] showed that the set of primes p for which the reduction mod p is ordinary has density 1; on the other hand, Elkies showed in [**Elk87**] that there are infinitely many primes p for which the reduction Y_p is supersingular.

4.5.2. Some p-linear algebra. Before discussing a related conjecture with an arithmetic flavor, we give a brief introduction to some basic facts of p-linear algebra, following [**CL98**]. In what follows, k is a perfect field of characteristic $p > 0$.

DEFINITION 4.45. If V is a finite-dimensional vector space over k, a *p-linear map* $\varphi\colon V \to V$ is a group homomorphism that satisfies

$$\varphi(au) = a^p\varphi(u) \quad \text{for all} \quad a \in k,\ u \in V.$$

EXAMPLE 4.46. For us, the main example arises as the Frobenius action on cohomology. Recall from Example 3.72 that if X is a proper scheme over k, then the Frobenius morphism $F\colon X \to X$ induces for every i a map

$$F\colon H^i(X, \mathcal{O}_X) \to H^i(X, F_*\mathcal{O}_X) = H^i(X, \mathcal{O}_X).$$

This is a p-linear map.

REMARK 4.47. If $\varphi\colon V \to V$ is a p-linear map, then φ is a linear map $V \to W$, where W is induced from V by restriction of scalars via Frobenius. Since k is perfect, we have $\dim_k(W) = \dim_k(V)$. We thus see that φ is bijective if and only if it is surjective (injective).

PROPOSITION 4.48. If $\varphi\colon V \to V$ is a p-linear map, then there are vector subspaces V_{nil} and V_{ss} of V preserved by φ such that $\varphi|_{V_{\mathrm{nil}}}$ is nilpotent, $\varphi|_{V_{\mathrm{ss}}}$ is bijective, and $V = V_{\mathrm{nil}} \oplus V_{\mathrm{ss}}$. Moreover, such a decomposition is unique.

PROOF. It is clear that $\mathrm{Ker}(\varphi)$ is a vector subspace of V and we also see that $\mathrm{Im}(\varphi)$ is a vector subspace using the fact that k is perfect. We put

$$V_{\mathrm{nil}} := \bigcup_{m\geq 1} \mathrm{Ker}(\varphi^m) \quad \text{and} \quad V_{\mathrm{ss}} := \bigcap_{m\geq 1} \mathrm{Im}(\varphi^m).$$

It is straightforward to see that $\varphi(V_{\mathrm{nil}}) \subseteq V_{\mathrm{nil}}$ and $\varphi(V_{\mathrm{ss}}) \subseteq V_{\mathrm{ss}}$. Since V is finite-dimensional, it follows that there is $m \geq 1$ such that

$$V_{\mathrm{nil}} = \mathrm{Ker}(\varphi^m) \quad \text{and} \quad V_{\mathrm{ss}} = \mathrm{Im}(\varphi^m).$$

If $u \in V^{\mathrm{ss}}$, then

$$u \in \mathrm{Im}(\varphi^{m+1}) \subseteq \varphi\big(\mathrm{Im}(\varphi^m)\big) = \varphi(V_{\mathrm{ss}}),$$

hence $\varphi|_{V_{\mathrm{ss}}}$ is surjective, and thus bijective by Remark 4.47. Since $\varphi^m(V_{\mathrm{nil}}) = 0$, it follows that $\varphi|_{V_{\mathrm{nil}}}$ is nilpotent.

We next show that $V = V_{\mathrm{nil}} \oplus V_{\mathrm{ss}}$. If $u \in V_{\mathrm{nil}} \cap V_{\mathrm{ss}}$, then $u = \varphi^m(v)$ for some $v \in V$ and $\varphi^m(u) = 0$. Therefore $\varphi^{2m}(v) = 0$, hence $u = \varphi^m(v) = 0$. Therefore $V_{\mathrm{nil}} \cap V_{\mathrm{ss}} = 0$.

Suppose now that $u \in V$. Since $\varphi^m(u) \in V_{\rm ss}$, we can also write $\varphi^m(u) = \varphi^{2m}(w)$, for some $w \in V$. We can thus write $u = \big(u - \varphi^m(w)\big) + \varphi^m(w)$ and $\big(u - \varphi^m(w)\big) \in \mathrm{Ker}(\varphi^m) = V_{\rm nil}$ and $\varphi^m(w) \in \mathrm{Im}(\varphi^m) = V_{\rm ss}$.

In order to prove uniqueness, suppose that we have another decomposition $V = V'_{\rm nil} \oplus V'_{\rm ss}$ with the same properties. Since $\varphi|_{V'_{\rm nil}}$ is nilpotent, it follows that $V'_{\rm nil} \subseteq V_{\rm nil}$ and since $\varphi|_{V'_{\rm ss}}$ is surjective gives $V'_{\rm ss} \subseteq V_{\rm ss}$. It is then clear, by considering dimensions, that $V_{\rm nil} = V'_{\rm nil}$ and $V_{\rm ss} = V'_{\rm ss}$. □

REMARK 4.49. Note that if k is finite, with $|k| = p^e$, then $\varphi^e \colon V \to V$ is a k-linear map and the decomposition in the above proposition is the same if we do it with respect to this linear map.

DEFINITION 4.50. If $\varphi\colon V \to V$ is a p-linear map, we say that φ is *nilpotent* (or *semisimple*) if $V = V_{\rm nil}$ (resp. $V = V_{\rm ss}$).

EXAMPLE 4.51. If $\dim_k(V) = 1$ and $\varphi\colon V \to V$ is p-linear, then if we choose a nonzero $u \in V$, we can write $\varphi(u) = au$ for some $a \in k$. If $a = 0$, then φ is nilpotent and if $a \neq 0$, then φ is semisimple.

REMARK 4.52. If $\varphi\colon V \to V$ is a p-linear map and $V_K = K \otimes_k V$, where K/k is a field extension, then we get a p-linear map $\varphi_K \colon V_K \to V_K$ given by $\varphi(a \otimes u) = a^p \otimes \varphi(u)$. Since φ_K is clearly nilpotent on $(V_{\rm nil})_K$ and it is clearly surjective on $(V_{\rm ss})_K$, it follows that

$$(V_K)_{\rm nil} = (V_{\rm nil})_K \quad \text{and} \quad (V_K)_{\rm ss} = (V_{\rm ss})_K.$$

REMARK 4.53. If $\varphi\colon V \to V$ is a p-linear map and $W \subseteq V$ is a vector subspace that is preserved by V, then φ induces p-linear maps on both W and V/W. It is clear that φ is nilpotent on $W \cap V_{\rm nil}$ and injective (hence bijective by Remark 4.47) on $W \cap V_{\rm ss}$. The argument in the proof of Proposition 4.48 for showing that $V = V_{\rm nil} \oplus V_{\rm ss}$ implies that $W = (W \cap V_{\rm nil}) \oplus (W \cap V_{\rm ss})$. We thus conclude that

$$W_{\rm nil} = W \cap V_{\rm nil} \quad \text{and} \quad W_{\rm ss} = W \cap V_{\rm ss}.$$

We can similarly see that

$$(V/W)_{\rm nil} = (V_{\rm nil} + W)/W \quad \text{and} \quad (V/W)_{\rm ss} = (V_{\rm ss} + W)/W.$$

We leave the details as an exercise for the reader.

REMARK 4.54. Suppose that $\varphi\colon V \to V$ and $\varphi'\colon V' \to V'$ are two p-linear maps, where V and V' are finite-dimensional vector spaces over the perfect field k. We then have a p-linear map

$$\varphi \otimes \varphi' \colon W = V \otimes_k V' \to W, \quad u \otimes u' \to \varphi(u) \otimes \varphi'(u').$$

It is easy to see that

$$W_{\rm nil} = (V_{\rm nil} \otimes_k V'_{\rm nil}) \oplus (V_{\rm ss} \otimes_k V'_{\rm nil}) \oplus (V_{\rm nil} \otimes_k V'_{\rm ss}) \quad \text{and} \quad W_{\rm ss} = V_{\rm ss} \otimes_k V'_{\rm ss}.$$

In particular, ψ is semisimple if and only if both φ and φ' are semisimple.

REMARK 4.55. Let $\varphi\colon V \to V$ be a p-linear map and let $n = \dim_k(V)$. We have an induced p-linear map $\wedge^n\varphi \colon \wedge^n V \to \wedge^n V$. It is straightforward to see that $\wedge^n\varphi$ is semisimple if and only if φ is semisimple.

PROPOSITION 4.56. If k is an algebraically closed field, V is a finite-dimensional vector space over k, and $\varphi\colon V \to V$ is a p-linear map, then

$$V^{\varphi=1} := \{u \in V \mid \varphi(u) = u\}$$

is an $\mathbf{F}_p$-vector subspace of V and we have an isomorphism

$$g\colon k \otimes_{\mathbf{F}_p} V^{\varphi=1} \to V_{\rm ss}, \quad a \otimes v \to av.$$

In particular, the following hold:

i) We have $\dim_{\mathbf{F}_p} V^{\varphi=1} = \dim_k(V_{\rm ss})$, and
ii) The $\mathbf{F}_p$-linear map $\varphi - 1_V\colon V \to V$ is bijective.

PROOF. The fact that $V^{\varphi=1}$ is an $\mathbf{F}_p$-vector subspace of V is clear since $a^p = a$ for all $a \in \mathbf{F}_p$. Note also that $V^{\varphi=1} \subseteq V_{\rm ss}$: this is clear if we recall that $V_{\rm ss} = \cap_{m\geq 1}{\rm Im}(\varphi^m)$ (see the proof of Proposition 4.48). We thus have a k-linear map g as in the proposition. In order to see that g is injective, we need to show that if $u_1, \ldots, u_n \in V^{\varphi=1}$ are linearly independent over $\mathbf{F}_p$, then they are independent also over k. Suppose that this is not the case and n is minimal with this property. Of course, we need to have $n \geq 2$. Note that if $u_1 + a_2u_2 + \ldots + a_nu_n = 0$, then by applying φ we get $u_1 + a_2^pu_2 + \ldots + a_n^pu_n = 0$. By subtracting the two relations and using the minimality of n, we see that $a_i^p - a_i = 0$ for all i, hence $a_i \in \mathbf{F}_p$ for all i, a contradiction.

Let $W = {\rm Im}(g)$. We next show that $\varphi - 1_W$ is surjective on W. Indeed, suppose that $u_1, \ldots, u_n$ give a basis of $V^{\varphi=1}$ and $u = a_1u_1 + \ldots + a_nu_n \in W$. If $v = b_1u_1 + \ldots + b_nu_n$, then $\varphi(v) - v = u$ if and only if $b_i^p - b_i = a_i$ for all i. Since k is algebraically closed, we can find such $b_1, \ldots, b_n$, which proves our claim.

In order to show that φ is surjective, after replacing V by $V_{\rm ss}$, we may assume that φ is semisimple. Since it is clear that $\varphi(W)$ is preserved by φ, we see that φ induces a semisimple p-linear map $\overline{\varphi}$ on V/W (see Remark 4.53). Note that in V/W there is no nonzero element fixed by $\overline{\varphi}$: indeed, otherwise there is $u \in V \smallsetminus W$ such that $\varphi(u) - u \in W$. As we have seen in the previous paragraph, we can find $w \in W$ such that $\varphi(u) - u = \varphi(w) - w$. We thus conclude that $u - w \in V^{\varphi=1} \smallsetminus W$, a contradiction.

We thus see that in order to prove the surjectivity of g it is enough to show that whenever $V = V_{\rm ss}$ is nonzero, there is a nonzero $v \in V$ such that $\varphi(v) = v$. Since $V \neq 0$, there is a nonzero $u \in V$ and a relation

$$a_0u + a_1\varphi(u) + \ldots + a_n\varphi^n(u) = 0,$$

such that not all a_i are 0. We choose such a relation with n minimal. Note that since φ is injective, we have $a_0 \neq 0$, hence we may assume that $a_0 = 1$.

If $n = 1$, then $u + a_1\varphi(u) = 0$ and we must have $a_1 \neq 0$. Since k is algebraically closed, there is $\lambda \in k$ such that $\lambda^{p-1} + a_1 = 0$ (note that $\lambda \neq 0$) We then have $\varphi(\lambda u) = \lambda^p\varphi(u) = (-a_1\lambda)(-a_1^{-1}u) = \lambda u$. We are thus done in this case.

If $n > 1$, then we look for $a, b_1, \ldots, b_n \in k$ such that

$$\big(u + b_1\varphi(u) + \ldots + b_{n-1}\varphi^{n-1}(u)\big) + a\varphi\big(u + b_1\varphi(u) + \ldots + b_{n-1}\varphi^{n-1}(u)\big) = 0. \tag{4.34}$$

This holds if the following equalities are satisfied:

$$b_1 + a = a_1,\ b_2 + ab_1^p = a_2, \ldots, b_{n-1} + ab_{n-2}^p = a_{n-1}, \quad \text{and}$$

$$ab_{n-1}^p = a_n.$$

Note that the first equations uniquely determine the b_i in terms of a and then the last equation is of the form $Q(a) = 0$, for some monic polynomial Q of degree $1 + p + \ldots + p^{n-1}$. Since k is algebraically closed, P has a root, hence we can find $a, b_1, \ldots, b_{n-1}$ such that (4.34) is satisfied.

In this case, the minimality in our choice of n implies that $u + b_1\varphi(u) + \ldots + b_{n-1}\varphi^{n-1}(u) \neq 0$. On the other hand, in this case it follows from (4.34) that we can apply the case $n = 1$ to conclude the existence of a nonzero $u \in V^{\varphi=1}$. This completes the proof of the fact that g is an isomorphism.

The assertion in i) is clear. The one in ii) follows if we show that $\varphi - 1_V$ is surjective separately on V_{ss} and V_{nil}. On $V_{\mathrm{ss}} = \mathrm{Im}(g)$ we have already seen this. On the other hand, this is clear on V_{nil}, since $\varphi|_{V_{\mathrm{nil}}}$ being nilpotent implies that $\varphi - 1_V$ is invertible on V_{nil}. This completes the proof of the proposition. □

The following example involves étale cohomology. While not needed in what follows, it provides a nice translation for when the Frobenius action on cohomology is semisimple or nilpotent in terms of étale cohomology.

Example 4.57. Let X be a proper scheme over a perfect field k of characteristic $p > 0$. Let $\overline{k}$ be an algebraic closure of k and $X_{\overline{k}}$ the base-change of X to $\overline{k}$. We consider the p-linear map $F\colon H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_X)$ induced by the Frobenius morphism, as well as its extension to $\overline{k}$:

$$F\colon H^i(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) \to H^i(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}).$$

On $X_{\overline{k}}$ we have the Artin-Schreier sequence, which is exact in the étale topology:

$$0 \longrightarrow \mathbf{F}_p \longrightarrow \mathcal{O}_{X_{\overline{k}}} \xrightarrow{F-1} \mathcal{O}_{X_{\overline{k}}} \longrightarrow 0$$

(surjectivity of $F - 1$ comes from the fact that for every k-algebra A and every $a \in A$, the A-algebra $A[x]/(x^p - x - a)$ is étale). Since $F - 1$ is surjective on $H^i(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ by Proposition 4.56, the long exact sequence in cohomology for the above exact sequence breaks into short exact sequences of $\mathbf{F}_p$-vector spaces

$$0 \longrightarrow H^i_{\text{ét}}(X, \mathbf{F}_p) \longrightarrow H^i(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) \xrightarrow{F-1} H^i(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) \longrightarrow 0.$$

Using again Proposition 4.56, we conclude that

$$\dim_{\mathbf{F}_p} H^i_{\text{ét}}(X, \mathbf{F}_p) = \dim_k H^i(X, \mathcal{O}_X)_{\mathrm{ss}}.$$

4.5.3. The Weak Ordinarity Conjecture. The following conjecture is a special case of a folklore conjecture which predicts that smooth projective varieties have a dense set of reductions to positive characteristic that are ordinary in the sense of Bloch and Kato [**BK86**] (whose precise meaning is not relevant for us).

Conjecture 4.58 (Weak Ordinarity Conjecture). *Let X be a smooth, connected, n-dimensional variety over an algebraically closed field k of characteristic 0. Given $A \in \mathrm{FG}_{\mathbf{Z}}(k)$ and a model X_A of X over A, there is a dense set of closed points $S \subseteq \mathrm{Spec}(A)$ such that for every $s \in S$, the p-linear map induced by Frobenius*

$$F\colon H^n(X_s, \mathcal{O}_{X_s}) \to H^n(X_s, \mathcal{O}_{X_s})$$

is semisimple.

Remark 4.59. The assertion in the conjecture is independent of the choice of $\mathbf{Z}$-algebra A and model X_A. Indeed, this follows from Remark 4.6 and the fact that if we have a proper scheme W over a finite field k and $W' = W \times_k k'$, where k'/k is a finite extension, then $F\colon H^i(X,\mathcal{O}_X)\to H^i(X,\mathcal{O}_X)$ is semisimple if and only if $F\colon H^i(X',\mathcal{O}_{X'})\to H^i(X',\mathcal{O}_{X'})$ is semisimple. Indeed, we have $H^i(X',\mathcal{O}_{X'})\simeq k'\otimes_k H^i(X,\mathcal{O}_X)$ and the Frobenius action is obtained by extending scalars (see Remark 4.52).

Remark 4.60. In order to prove Conjecture 4.58 for X, it is clearly enough to show that for every model X_A, there is a closed point in $\mathrm{Spec}(A)$ such that $F\colon H^n(X_s,\mathcal{O}_{X_s})\to H^n(X_s,\mathcal{O}_{X_s})$ is semisimple.

Remark 4.61. In order to prove the general case of Conjecture 4.58, it is enough to prove the case when $k=\overline{\mathbf{Q}}$. Indeed, suppose that X, A, and X_A are as in the conjecture. Let $\mathfrak{m}$ be a prime ideal in A that corresponds to a maximal ideal $\mathfrak{m}_{\mathbf{Q}}$ in $A_{\mathbf{Q}}:=A\otimes_{\mathbf{Z}}\mathbf{Q}$. The field $K=A_{\mathbf{Q}}/\mathfrak{m}_{\mathbf{Q}}$ is a finite extension of $\mathbf{Q}$ by Nullstellensatz, and let O_K be the ring of integers of K. Since A is finitely generated over $\mathbf{Z}$, it follows that there is a nonzero $g\in O_K$ such that the composition $A\to A_{\mathbf{Q}}\to K$ has image inside $B:=(O_K)_g$. We put $X_B:=X_A\times_A B$ and $X_{\overline{\mathbf{Q}}}=X_A\times_A\overline{\mathbf{Q}}$. Since the geometric generic fiber of $X\to\mathrm{Spec}(A)$ is smooth, connected, of dimension n, it follows that $X_{\overline{\mathbf{Q}}}$ is a smooth, irreducible n-dimensional variety over $\overline{\mathbf{Q}}$. Moreover, X_B gives a model of $X_{\overline{\mathbf{Q}}}$ over B. If we know Conjecture 4.58 for $X_{\overline{\mathbf{Q}}}$, then we have a closed point $t\in\mathrm{Spec}(B)$ such that the action of Frobenius on $H^n\big((X_{\overline{\mathbf{Q}}})_t,\mathcal{O}_{(X_{\overline{\mathbf{Q}}})_t}\big)$ is semisimple. If s is the image of t in $\mathrm{Spec}(A)$, then we have a finite extension $k(s)\hookrightarrow k(t)$ and $X_s\times_{k(s)}k(t)=(X_{\overline{\mathbf{Q}}})_t$. We conclude that the action of the Frobenius on $H^n(X_s,\mathcal{O}_{X_s})$ is semisimple as in Remark 4.59. Since we can find one such s for every model, this implies Conjecture 4.58 in general.

Remark 4.62. If Y is a g-dimensional Abelian variety over a field k_0, then there is an isomorphism

$$H^g(Y,\mathcal{O}_Y)\simeq\wedge^g H^1(Y,\mathcal{O}_Y),$$

and if k_0 is a perfect field of characteristic $p>0$, then the Frobenius action on the left-hand side is the determinant of the Frobenius action on $H^1(Y,\mathcal{O}_Y)$. In particular, it follows from Remark 4.55 that $F\colon H^g(Y,\mathcal{O}_Y)\to H^g(Y,\mathcal{O}_Y)$ is semisimple if and only if $F\colon H^1(Y,\mathcal{O}_Y)\to H^1(Y,\mathcal{O}_Y)$ is semisimple. We note that this is equivalent to saying that Y is *ordinary* in the sense that the number of p-torsion points on $Y\times_{k_0}\overline{k_0}$ is p^g (the largest possible).

Suppose now that X is a smooth, geometrically connected, projective curve over k_0 and $Y=\mathrm{Pic}^0(X)$ is the corresponding Picard variety, parametrizing line bundles of degree 0 on X. In this case we have an isomorphism

$$H^1(X,\mathcal{O}_X)\simeq H^1(Y,\mathcal{O}_Y)$$

which is compatible with the Frobenius action if k_0 is a perfect field of characteristic $p>0$. We thus see that Conjecture 4.58 holds for X if and only if it holds for $\mathrm{Pic}^0(X)$.

Arguing as in Remark 4.61, we see that in order to prove Conjecture 4.58 for all Abelian varieties of dimension n, it is enough to treat n-dimensional Abelian varieties over $\overline{\mathbf{Q}}$. For $n=1$ (that is, for elliptic curves), this is not too complicated: in fact, we prove a more general statement in Example 4.63 below. The case of Abelian surfaces is known, due to Ogus [**Og82**, Proposition 2.7] (see also [**CL98**, Théorème 6.3]), while the case $g\geq 3$ is open. As we have already discussed, knowing the conjecture for Abelian surfaces gives the case of curves of genus 2. The case of curves of genus ≥ 3 is wide open.

Essentially the only other case in which Conjecture 4.58 is known is that of $K3$ surfaces, for which a proof can be given following Ogus' approach, see [**JR03**] and [**BZ09**].

EXAMPLE 4.63. Let X be a smooth, irreducible, projective curve defined over $\overline{\mathbf{Q}}$, of genus $g \geq 1$. Let $J = \mathrm{Pic}^0(X)$ be the Picard variety parametrizing line bundles of degree 0 on X. Let ℓ be a prime integer that does not divide $2g$ and let K be a finite extension of $\mathbf{Q}$ such that X (hence also J) is defined over K and all ℓ-torsion points of J are K-rational (that is, the subscheme $J[\ell]$ of J given by the kernel of multiplication by ℓ on J consists of ℓ^{2g} copies of $\mathrm{Spec}(K)$). Suppose that we have a model X_A of X over a suitable localization A of the ring of integers O_K of K. We claim that if $s \in \mathrm{Spec}(A)$ is a closed point such that $p = \mathrm{char}\big(k(s)\big) > (2g)^2$ and p splits completely in K (that is, there are $[K : \mathbf{Q}]$ different primes in O_K that lie over $p\mathbf{Z}$), then the p-linear map induced by Frobenius $F\colon H^1(X_s, \mathcal{O}_{X_s}) \to H^1(X_s, \mathcal{O}_{X_s})$ is not nilpotent. Note that there are infinitely many such closed points $s \in \mathrm{Spec}(A)$ by Čebotarev's density theorem. In particular, we see that for $g = 1$, this gives a positive answer to Conjecture 4.58

The argument that follows makes use of several results not covered in these notes, but we include it nevertheless. Arguing by contradiction, suppose that F is a nilpotent p-map. Note first that since p splits completely in k, we have $k(s) = \mathbf{F}_p$. In particular, F is a linear map. Second, recall that Fulton's trace formula [**Ful78**] says that

$$\#X_s(\mathbf{F}_p) \equiv \sum_{i\geq 0} (-1)^i \mathrm{trace}\big(H^i(X_s, \mathcal{O}_{X_s})|F\big) \ (\mathrm{mod}\, p)$$

(this formula holds for arbitrary proper schemes over $\mathbf{F}_p$). Since F is the identity on $H^0(X_s, \mathbf{F}_p)$ and since we assume it is nilpotent on $H^1(X_s, \mathbf{F}_p)$, we conclude that

$$\#X_s(\mathbf{F}_p) \equiv 1 \ (\mathrm{mod}\, p). \tag{4.35}$$

On the other hand, we have the Hasse-Weil bound (equivalent to the Riemann hypothesis for curves over finite fields, see [**Har77**, Exercise V.1.10]):

$$|\#X_s(\mathbf{F}_p) - (p+1)| \leq 2g\sqrt{p}. \tag{4.36}$$

By combining (4.35) and (4.37) with our assumption that $p > (2g)^2$, we conclude that

$$\#X_s(\mathbf{F}_p) = p + 1. \tag{4.37}$$

Finally, we use the fact that we can compute $\#X_s(\mathbf{F}_p)$ via the trace formula for the ℓ-adic cohomology of X_s. More precisely, we have

$$\#X_s(\mathbf{F}_p) = 1 - a + p,$$

where

$$a = \mathrm{trace}\big(H^1_{\text{ét}}(X_{\overline{k(s)}}, \mathbf{Z}_\ell)|F\big).$$

By (4.37), we have $a = 0$. On the other hand, we have

$$H^1_{\text{ét}}(X_{\overline{k(s)}}, \mathbf{Z}_\ell) \simeq H^1_{\text{ét}}(J_{\overline{k(s)}}, \mathbf{Z}_\ell) \simeq \left(\varprojlim_{n\geq 1} J_{\overline{k(s)}}[\ell^n]\right)^\vee.$$

Our choice of ℓ and K implies that $J_{\overline{k(s)}}[\ell] \simeq (\mathbf{Z}/\ell\mathbf{Z})^{2g}$, with the action of the Frobenius being trivial. This implies that $a \equiv 2g \pmod{\ell}$, which implies that ℓ divides $2g$, a contradiction. This completes the proof.

The following result relates the two conjectures that we discussed in this section.

THEOREM 4.64. *Conjecture 4.41 holds for all varieties if and only if Conjecture 4.58 holds for all varieties.*

We do not give the proof in these notes (though hopefully this might be included at some point). The fact that Conjecture 4.41 implies Conjecture 4.58 is rather elementary, see [**Mus12**]. Given a smooth, irreducible n-dimensional projective variety X over k, one considers an embedding in some $\mathbf{P}_k^N$. By composing with a suitable Veronese embedding, one may assume that $r := N - n \geq n+1$ and the ideal defining X in $\mathbf{P}_k^N$ is generated by quadrics. If $\mathfrak{a} \subseteq k[x_0, \ldots, x_N]$ is the ideal generated by the product h of r general quadrics in the ideal of X, then it is easy to see that $(x_0, \ldots, x_N)^{2r-N-1} \subseteq \mathcal{J}(h^\lambda)$ for every $\lambda < 1$. Finally, one can show that if $(x_0, \ldots, x_N)^{2r-N-1} \subseteq \tau(h_s^\lambda)$ for every $\lambda < 1$, then the Frobenius action on $H^n(X_s, \mathcal{O}_{X_s})$ is semisimple.

The proof of the fact that Conjecture 4.58 implies Conjecture 4.41 is more involved, see [**MS11**].

CHAPTER 5

Arcs, jets, and singularities

In this chapter we give an introduction to jet schemes and arc schemes and relate these to the study of singularities. As a first application, we give a description of multiplier ideals and log canonical thresholds in arbitrary characteristic in terms of the codimension of certain subspaces in the space of arcs. After a brief discussion of the Grothendieck ring of algebraic varieties, we give a second application, the construction of the Denef-Loeser motivic zeta function associated to a hypersurface in a smooth variety and a proof of its rationality. An important result in this setting concerns the behavior of spaces of arcs under birational transformations. We avoid the general result, by only using it for smooth blow-ups, in which case it is an easy exercise.

5.1. Jet schemes and arc schemes

While we will mostly be interested in the case of schemes of finite type over a field, in order to treat families of such schemes, it is convenient to give the definition of jet schemes in the relative setting, for schemes over a fixed commutative ring A. We write $\mathcal{S}ch/A$ for the category of schemes over A. To simplify the notation, if X is a scheme over A and B is an A-algebra, we write $X \times_A B$ for the fiber product $X \times_{\operatorname{Spec} A} \operatorname{Spec} B$.

Theorem 5.1. *For every nonnegative integer m, the functor*

$$\mathcal{S}ch/A \to \mathcal{S}ch/A, \quad Y \rightsquigarrow Y \times_A A[t]/(t^{m+1})$$

has a right-adjoint.

A scheme X over A is taken by this right-adjoint functor to $(X/A)_m$, the m^{th} *jet scheme* of X (over A). Most of the time A is understood from the context, in which case we simply write X_m. If $f\colon X \to Y$ is a morphism of schemes over A, then the corresponding morphism $X_m \to Y_m$ is denoted f_m.

We prove Theorem 5.1 in a few steps. We first note that by general nonsense about the existence of adjoint functors, it is enough to show that for every scheme $X \in \mathcal{S}ch/A$, there is $X_m \in \mathcal{S}ch/A$ such that for every $Y \in \mathcal{S}ch/A$, we have a functorial bijection

$$\operatorname{Hom}_A(Y, X_m) \simeq \operatorname{Hom}_A\big(Y \times_A A[t]/(t^{m+1}), X\big). \tag{5.1}$$

Furthermore, it is standard to see that it is enough to have such an isomorphism for all affine schemes over A, that is, for every A-algebra B we have a functorial bijection

$$\operatorname{Hom}_A(\operatorname{Spec} B, X_m) \simeq \operatorname{Hom}_A\big(\operatorname{Spec} B[t]/(t^{m+1}), X\big). \tag{5.2}$$

We first treat the case of affine schemes.

LEMMA 5.2. *If X is an affine scheme over A, then the jet scheme $(X/A)_m$ exists.*

PROOF. Let $R = \mathcal{O}_X(X)$ and let's choose a surjective homomorphism of A-algebras

$$\varphi\colon A[x_i \mid i \in I] \to R,$$

for a suitable set I, and let $(f_j)_{j\in J}$ be a system of generators of $\mathrm{Ker}(\varphi)$. For an A-algebra B, giving a morphism $\operatorname{Spec} B[t]/(t^{m+1}) \to X$ is equivalent to giving an A-algebra homomorphism

$$\alpha\colon A[x_i \mid i \in I] \to B[t]/(t^{m+1})$$

such that $\alpha(f_j) = 0$ for all $j \in J$. Such a homomorphism α is uniquely determined by elements $b_{i,\ell} \in B$ for $i \in I$ and $0 \le \ell \le m$ such that

$$\alpha(x_i) = \sum_{\ell=0}^{m} b_{i,\ell} t^{\ell}.$$

The key point is that for every $f \in A[x_i \mid i \in I]$ and every ℓ with $0 \le \ell \le m$ there is $f^{(\ell)} \in A[x_{i,0}, x_{i,1}, \ldots, x_{i,\ell}]$ such that for every α as above we have

$$\alpha(f) = \sum_{\ell=0}^{m} f^{(\ell)}(b_{i,0}, \ldots, b_{i,\ell} \mid i \in I) t^{\ell}.$$

We thus conclude that in this case the affine scheme over A corresponding to the A-algebra

$$A[x_i \mid i \in I]/(f_j^{(\ell)} \mid j \in J, 0 \le \ell \le m)$$

satisfies (5.2), which completes the proof of the lemma. $\square$

EXAMPLE 5.3. Let's give one explicit example. Suppose that X is the closed subscheme of $\operatorname{Spec} A[x,y]$ defined by $(x^2 + y^3)$. In this case the jet scheme X_2 is the closed subscheme of $\operatorname{Spec} A[x, x', x'', y, y', y'']$ defined by the coefficients of $(x + x't + x''t^2)^2 + (y + y't + y''t^2)^3 \bmod t^3$. Therefore the ideal of X_2 is generated by

$$x^2 + y^3, 2xx' + 3y^2y', \text{ and } (x')^2 + 2xx'' + 3y^2y'' + 3y(y')^2.$$

REMARK 5.4. If X is a scheme over A and m is a nonnegative integer such that X_m exists, then the canonical A-algebra homomorphism $A[t]/(t^{m+1}) \to A$ with kernel $(t)/(t^{m+1})$ induces a morphism of schemes $\operatorname{Spec} A \to \operatorname{Spec} A[t]/(t^{m+1})$ and thus a functorial map

$$\mathrm{Hom}_A(Y, X_m) \simeq \mathrm{Hom}_A\big(Y \times_A A[t]/(t^{m+1}), X\big) \to \mathrm{Hom}_A(Y, X) \tag{5.3}$$

for every scheme $Y \in \mathcal{S}ch/A$. It follows that there is a unique morphism $\varphi_m^X\colon X_m \to X$ that induces (5.3) for every Y. If X is understood, we simply write φ_m for φ_m^X.

REMARK 5.5. Note that if $m = 0$, then $(X/A)_m = X$ and the morphism φ_0 is the identity.

LEMMA 5.6. *Let X be a scheme over A and $j\colon U \to X$ an open immersion. If $(X/A)_m$ exists, then $(U/A)_m$ exists and we have a Cartesian diagram*

$$\begin{tikzcd} (U/A)_m \arrow[r, "j_m"] \arrow[d, "\varphi_m^U"'] & (X/A)_m \arrow[d, "\varphi_m^X"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

PROOF. Since $i\colon Y \hookrightarrow Y \times_A A[t]/(t^{m+1})$ is a closed immersion, with the two schemes having the same underlying topological space, a morphism $\gamma\colon Y \times_A A[t]/(t^{m+1}) \to X$ factors through U if and only if this is the case for the composition $\gamma \circ i$. Note also that if $\widetilde{\gamma}\colon Y \to X_m$ is the morphism corresponding to γ via the definition of $(X/A)_m$, then $\varphi_m^X \circ \widetilde{\gamma}\colon Y \to X$ is $\gamma \circ j$. This easily implies that $(\varphi_m^X)^{-1}(U)$ satisfies the defining property of $(U/A)_m$ and we get the assertion in the lemma. $\square$

We can now prove the existence of jet schemes in general.

PROOF OF THEOREM 5.1. Let $X = \bigcup_{i \in I} U^{(i)}$ be an affine open cover of X. By Lemma 5.2, for every $i \in I$, the jet scheme $(U^{(i)}/A)_m$ exists and we have a canonical morphism $\varphi_m^{(i)}\colon \big(U^{(i)}/A\big)_m \to U^{(i)}$ given by Remark 5.4. For every $i, j \in I$, it follows from Lemma 5.6 that $\big(\varphi_m^{(i)}\big)^{-1}\big(U^{(i)} \cap U^{(j)}\big)$ and $\big(\varphi_m^{(j)}\big)^{-1}\big(U^{(i)} \cap U^{(j)}\big)$ both satisfy the defining property of $\big(U^{(i)} \cap U^{(j)}/A\big)_m$, and thus we get a canonical isomorphism

$$\alpha_{i,j}\colon \big(\varphi_m^{(i)}\big)^{-1}\big(U^{(i)} \cap U^{(j)}\big) \to \big(\varphi_m^{(j)}\big)^{-1}\big(U^{(i)} \cap U^{(j)}\big).$$

The fact that they are canonical implies that they satisfy the cocycle condition. We can thus glue the schemes $U^{(i)}$ using the glueing isomorphisms $\alpha_{i,j}$ to a scheme in $\mathcal{S}ch/A$. It is now straightforward to see that this satisfies the defining property of $(X/A)_m$. $\square$

REMARK 5.7. It follows from Lemma 5.6 and the proof of Lemma 5.2 that for every scheme X over A and every nonnegative integer m, the canonical morphism $\varphi_m^X\colon (X/A)_m \to X$ is affine.

REMARK 5.8. If A is a Noetherian ring and X is a scheme of finite type over A, then $(X/A)_m$ is a scheme of finite type over A for every nonnegative integer m. Indeed, this follows from Lemma 5.6 and the description of $(X/A)_m$ when X is affine in the proof of Lemma 5.2 (indeed, under our assumptions, the set I in that proof can be taken to be finite).

REMARK 5.9. If $i\colon X \hookrightarrow Y$ is a closed immersion of schemes over A, then the induced morphism $i_m\colon (X/A)_m \to (Y/A)_m$ is a closed immersion for every nonnegative integer m. Indeed, after covering Y by affine open subsets U_i and using Lemma 5.6, we reduce to the case when Y (hence also X) is affine. In this case the assertion follows from the description of $(X/A)_m$ and $(Y/A)_m$ in the proof of Lemma 5.2.

REMARK 5.10. The maps introduced in Remark 5.4 have the following more general version. For every nonnegative integers $m \geq p$, the A-algebra homomorphism $A[t]/(t^{m+1}) \to A[t]/(t^{p+1})$ that maps t to t induces for every scheme Y in

$\mathcal{S}ch/A$ a closed immersion

$$\iota_{m,p}\colon Y \times_A A[t]/(t^{p+1}) \hookrightarrow Y \times_A A[t]/(t^{m+1}).$$

For every scheme $X \in \mathcal{S}ch/A$ we obtain a functorial map

$$\mathrm{Hom}_A\bigl(Y, (X/A)_m\bigr) \to \mathrm{Hom}_A\bigl(Y, (X/A)_p\bigr)$$

that is identified with

$$\mathrm{Hom}_A\bigl(Y \times_A A[t]/(t^{m+1}), X\bigr) \to \mathrm{Hom}_A\bigl(Y \times_A A[t]/(t^{p+1}), X\bigr), \quad \gamma \rightsquigarrow \gamma \circ \iota_{m,p}.$$

This is induced by a unique morphism $\varphi^X_{m,p}\colon (X/A)_m \to (X/A)_p$ of schemes over A. When the scheme X is understood, we simply write $\varphi_{m,p}$ for $\varphi^X_{m,p}$. Note that $\varphi^X_{m,0} = \varphi^X_m$. Also, it is clear that if $q \leq p \leq m$, then $\varphi^X_{p,q} \circ \varphi^X_{m,p} = \varphi^X_{m,q}$. In particular, since both φ^X_m and φ^X_p are affine (see Remark 5.7), it follows that all maps $\varphi^X_{m,p}$ are affine.

Example 5.11. It follows from the proof of Lemma 5.2 that if $X = \mathbf{A}^n_A$, then $(X/A)_m \simeq \mathbf{A}^{(m+1)n}_A$. Moreover, if $m \geq p$, then $(X/A)_m$ is isomorphic to $(X/A)_p \times_A \mathbf{A}^{(m-p)n}_A$ as schemes over $(X/A)_p$.

The following proposition shows that taking jet schemes commutes with base-change.

Proposition 5.12. If X is a scheme over A and $Y = X \times_A B$, where B is an A-algebra, then for every nonnegative integer m, we have an isomorphism

$$(Y/B)_m \simeq (X/A)_m \times_A B.$$

Moreover, the map $\varphi^Y_{m,p}$ is obtained by base-change from $\varphi^X_{m,p}$ for every $m \geq p$.

Proof. The assertions follow from the definition of the jet schemes and the fact that given a scheme Z over B, we have functorial bijections

$$\mathrm{Hom}_B\bigl(Z, (Y/B)_m\bigr) \simeq \mathrm{Hom}_B\bigl(Z \times_B B[t]/(t^{m+1}), X \times_A B\bigr)$$

$$\simeq \mathrm{Hom}_A\bigl(Z \times_A A[t]/(t^{m+1}), X\bigr) \simeq \mathrm{Hom}_A\bigl(Z, (X/A)_m\bigr) \simeq \mathrm{Hom}_B\bigl(Z, (X/A)_m \times_A B\bigr),$$

where the third bijection uses the fact that we have an isomorphism

$$Z \times_A A[t]/(t^{m+1}) \simeq Z \times_B B[t]/(t^{m+1})$$

of schemes over A. $\square$

The following proposition extends the assertion in Lemma 5.6 to the case of étale morphisms.

Proposition 5.13. If A is a Noetherian ring and $f\colon Y \to X$ is an étale morphism of schemes of finite type over A, then for every nonnegative integer m, the following diagram is Cartesian:

$$\begin{tikzcd}
(Y/A)_m \arrow[r, "f_m"] \arrow[d, "\varphi^Y_m"'] & (X/A)_m \arrow[d, "\varphi^X_m"] \\
Y \arrow[r, "f"] & X.
\end{tikzcd}$$

In particular, the morphism $f_m\colon (Y/A)_m \to (X/A)_m$ is étale.

Proof. Let Z be a scheme over A and $g\colon Z \to Y$ and $h\colon Z \to (X/A)_m$ be such that $\varphi_m^X \circ h = f \circ g$. If $\gamma\colon Z \times_A A[t]/(t^{m+1}) \to X$ corresponds to h, then we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "g"'] & Z \times_A A[t]/(t^{m+1}) \arrow[d, "\gamma"] \\ Y \arrow[r, "f"] & X, \end{tikzcd}$$

in which the top horizontal map i is a closed immersion defined by a nilpotent ideal. Since f is étale, hence formally étale, it follows that there is a unique morphism $\delta\colon Z \times_A A[t]/(t^{m+1}) \to Y$ such that $\delta \circ i = g$ and $f \circ \delta = \gamma$. If $\widetilde{h}\colon Z \to (Y/A)_m$ corresponds to δ, these conditions are equivalent to $\varphi_m^Y \circ \widetilde{h} = h$ and $f_m \circ \widetilde{h} = h$. This gives the first assertion in the proposition and the second one follows from the fact that the property of being étale is preserved by base-change. $\square$

We next turn to the definition of the arc scheme.

Definition 5.14. If X is a scheme over A, we have seen in Remark 5.10 that for every $m \geq p \geq 0$ we have affine morphisms $\varphi_{m,p}^X\colon (X/A)_m \to (X/A)_p$ and these are compatible with composition. In this case the inverse limit $(X/A)_\infty := \varprojlim_m (X/A)_m$ is a well-defined scheme over A, that comes with affine morphisms $\psi_m^X\colon (X/A)_\infty \to (X/A)_m$ for all m. By definition of the inverse limit, for every affine open subset $U \subseteq X$, we have

$$\mathcal{O}\big(\psi_0^{-1}(U)\big) = \varinjlim_m \mathcal{O}\big(\varphi_m^{-1}(U)\big) = \varinjlim_m \big(\mathcal{O}(U/A)_m\big).$$

The scheme $(X/A)_\infty$ is the *arc scheme* of X over A.

If $f\colon Y \to X$ is a morphism of schemes over A, then by passing to inverse limit, the morphisms $f_m\colon (Y/A)_m \to (X/A)_m$ induce a morphism $f_\infty\colon (Y/A)_\infty \to (X/A)_\infty$. In this way, by mapping X to $(X/A)_\infty$ and f to f_∞, we get a functor $\mathcal{S}ch/A \to \mathcal{S}ch/A$.

Example 5.15. It follows from Example 5.11 that if $X = \mathbf{A}^n$, with $n \geq 1$, then $(X/A)_\infty$ is isomorphic to the infinite-dimensional affine space $\operatorname{Spec} A[x_i \mid i \geq 1]$ over A. In particular, we see that typically $(X/A)_\infty$ is not of finite type over A.

Remark 5.16. Several of the properties discussed so far for jet schemes have analogues for the arc scheme. We only mention two of these:

i) If $f\colon U \hookrightarrow X$ is an open immersion of schemes over A, then the morphism $f_\infty\colon (U/A)_\infty \hookrightarrow (X/A)_\infty$ is an open immersion. In fact this is an isomorphism onto $\psi_0^{-1}(U)$: this follows from the definition of the arc schemes and Lemma 5.6.
ii) If $f\colon Y \hookrightarrow X$ is a closed immersion of schemes over A, then $f_\infty\colon (Y/A)_\infty \hookrightarrow (X/A)_\infty$ is a closed immersion. In order to check this, it is enough to consider the case when X is affine, in which case it follows from Remark 5.9 and the fact that a direct limit of surjective homomorphisms is surjective.

Remark 5.17. We work over a fixed ring A. If $X = \operatorname{Spec} S$ is affine, then it follows from the definition of the arc and jet schemes that for every A-algebra R, we have functorial bijections

$$\begin{aligned}\mathrm{Hom}_A(\mathrm{Spec}\, R, X_\infty) &\simeq \mathrm{Hom}_{A-\mathrm{alg}}\big(\mathcal{O}(X_\infty), R\big) \simeq \mathrm{Hom}_{A-\mathrm{alg}}\big(\varinjlim \mathcal{O}(X_m), R\big)\\ &\simeq \varprojlim_m \mathrm{Hom}_{A-\mathrm{alg}}\big(\mathcal{O}(X_m), R\big) \simeq \varprojlim_m \mathrm{Hom}_{A-\mathrm{alg}}\big(S, R[t]/(t^{m+1})\big)\\ &\simeq \mathrm{Hom}_{A-\mathrm{alg}}\big(S, R[\![t]\!]\big) \simeq \mathrm{Hom}_A\big(\mathrm{Spec}\, R[\![t]\!], X\big).\end{aligned}$$

It is the case that if R is a local ring, then for every X, we still have a functorial bijection

$$\mathrm{Hom}_A(\mathrm{Spec}\, R, X_\infty) \simeq \mathrm{Hom}_A\big(\mathrm{Spec}\, R[\![t]\!], X\big). \tag{5.4}$$

Indeed, we have $X = \bigcup_{U\subseteq X} U$ and $X_\infty = \bigcup_{U\subseteq X} U_\infty$, where the unions are over the affine open subsets U of X, and since both R and $R[\![t]\!]$ are local rings, we deduce[1] that

$$\mathrm{Hom}_A(\mathrm{Spec}\, R, X_\infty) = \bigcup_{U\subseteq X} \mathrm{Hom}_A(\mathrm{Spec}\, R, U_\infty) \quad \text{and}$$

$$\mathrm{Hom}_A\big(\mathrm{Spec}\, R[\![t]\!], X\big) = \bigcup_{U\subseteq X} \mathrm{Hom}_A\big(\mathrm{Spec} R[\![t]\!], U\big),$$

and we apply the affine case. We will only need this in the case when R is a field. In fact, it is known that as long as X is a Noetherian scheme, for every ring R the canonical map

$$\mathrm{Hom}_A\big(\mathrm{Spec}\, R[\![t]\!], X\big) \to \varprojlim_m \mathrm{Hom}_A\big(\mathrm{Spec}\, R[t]/(t^{m+1}), X\big)$$

is bijective and thus we have the functorial bijection (5.4). However, we will not need this fact, whose proof is much more subtle (see [**Bha16**]).

PROPOSITION 5.18. If Z is a scheme over A and X and Y are schemes over Z and $p\colon X\times_Z Y \to X$ and $q \times X\times_Z Y\to Y$ are the two projections, then for every $m\geq 0$, the morphism

$$(p_m, q_m)\colon (X\times_Z Y/A)_m \to (X/A)_m \times_{(Z/A)_m} (Y/A)_m$$

is an isomorphism. The corresponding assertion also holds for the arc schemes.

PROOF. For jet schemes, the assertion follows from the fact that the functor $X \rightsquigarrow X_m$ has a left adjoint and thus commutes with fiber products. The assertion for arc schemes follows by taking the inverse limit: note that inverse limits of affine morphisms of schemes commute with fiber products since direct limits of A-algebras commute with tensor products. □

From now on we assume that $A = k$ is an algebraically closed field of arbitrary characteristic and the schemes we consider are of finite type over k. As usual, for such a scheme X, unless explicitly mentioned otherwise, a *point* of X is a closed point. By definition, a point of the jet scheme X_m corresponds to an *m-jet* on X, that is, a morphism $\mathrm{Spec}\, k[t]/(t^{m+1}) \to X$ and the image of the unique point in the domain, typically denoted by 0, is $\gamma(0) = \varphi_m(\gamma)$. Note that for $m = 1$, this is

[1] We use here the easy fact that if R is a local ring and $\gamma\colon \mathrm{Spec}\, R \to Y$ is a morphism to a scheme Y and V is an open subset of Y such that γ maps the closed point of $\mathrm{Spec}\, R$ to V, then γ factors through V.

precisely a tangent vector at $\gamma(0)$. In general, for $m \geq p$, the morphism $\varphi_{m,p}$ maps $\gamma\colon \operatorname{Spec} k[t]/(t^{m+1}) \to X$ to the composition

$$\operatorname{Spec} k[t]/(t^{p+1}) \hookrightarrow \operatorname{Spec} k[t]/(t^{m+1}) \to X.$$

If X is a scheme as above, we write $X_\infty(k)$ for the set of k-valued points of the arc scheme X_∞, with its Zariski topology (and sometimes refer to this as the *space of arcs* of X). Note that by Remark 5.17, an element of $X_\infty(k)$ can be identified with an *arc* on X, that is, a morphism $\operatorname{Spec} k[\![t]\!] \to X$. Again, we denote by 0 the closed point in $\operatorname{Spec} k[\![t]\!]$, so $\psi_0(\gamma) = \gamma(0)$. More generally, the m-jet $\psi_m(\gamma)$ is given by the composition

$$\operatorname{Spec} k[t]/(t^{m+1}) \hookrightarrow \operatorname{Spec} k[\![t]\!] \to X.$$

REMARK 5.19. Since $k[\![t]\!]$ is a domain, it follows that if Y and Y' are closed subschemes of X with the same support, then $Y_\infty(k) = Y'_\infty(k)$ as subsetes of $X_\infty(k)$. Therefore we sometimes write $W_\infty(k)$ also when W is a closed subset of X.

For every X and every nonnegative integer m, we have a section $\sigma_m\colon X \to X_m$ of the morphism φ_m. This corresponds to the morphism $X \times \operatorname{Spec} k[t]/(t^{m+1}) \to X$ given by the projection onto the first component. Note that for every $x \in X$, the m-jet $\sigma_m(x)$ is the *constant m-jet* at x, given by the composition

$$\operatorname{Spec} k[t]/(t^{m+1}) \to \operatorname{Spec}(k) \xrightarrow{x} X.$$

It is clear that if $m \geq p$, then $\varphi_{m,p} \circ \sigma_m = \sigma_p$. Similarly, we have a section $\sigma_\infty\colon X \to X_\infty$ of the morphism ψ_0 that maps x to the *constant arc* at x, and such that $\psi_m \circ \sigma_\infty = \sigma_m$ for all m.

For every X and every nonnegative integer m, we also have a morphism

$$\beta_m\colon \mathbf{A}^1 \times X_m \to X_m$$

given as follows. Note first that by the defining property of X_m, we have a morphism $\eta\colon \operatorname{Spec} k[t]/(t^{m+1}) \times X_m \to X$ that corresponds to the identity map on X_m. We also have a morphism

$$\alpha_m\colon \mathbf{A}^1 \times \operatorname{Spec} k[t]/(t^{m+1}) \to \operatorname{Spec} k[t]/(t^{m+1})$$

induced by the k-algebra homomorphism

$$k[t]/(t^{m+1}) \to k[s] \otimes_k k[t]/(t^{m+1}), \quad t \rightsquigarrow s \otimes st.$$

The morphism β_m then corresponds via the defining property of X_m to the composition

$$\mathbf{A}^1 \times \operatorname{Spec} k[t]/(t^{m+1}) \times X_m \xrightarrow{\alpha_m \times 1_{X_m}} \operatorname{Spec} k[t]/(t^{m+1}) \times X_m \xrightarrow{\eta} X.$$

On points, we see that if $\gamma\colon \operatorname{Spec} k[t]/(t^{m+1}) \to X$ is an m-jet on X and if $\lambda \in k$, then $\beta_m(\lambda, \gamma)$ is given by the composition

$$\operatorname{Spec} k[t]/(t^{m+1}) \to \operatorname{Spec} k[t]/(t^{m+1}) \xrightarrow{\gamma} X,$$

where the first map is induced by $t \rightsquigarrow \lambda t$. In particular, we see that $\beta_m(0, \gamma) = \sigma_m\big(\gamma(0)\big)$. It is clear that the morphisms β_m are compatible with the projections $\varphi_{m,p}$, in the obvious sense.

Note that by restricting β_m to $k^* \times X_m$, we obtain an action of the 1-dimensional torus k^* on X_m. Moreover, if $m \geq p$, then the projection $\varphi_{m,p}$ is k^*-equivariant. We similarly have a morphism $\mathbf{A}^1 \times X_\infty \to X_\infty$ that induces an action of k^* on X_∞. Moreover, each projection $\psi_m\colon X_\infty \to X_m$ is k^*-equivariant.

Remark 5.20. If W is an irreducible component of X_m, since we have $W \subseteq \beta_m(\mathbf{A}^1 \times W)$, which is irreducible, it follows that $\beta_m(\mathbf{A}^1 \times W) = W$. In particular, it follows that for every $\gamma \in W$, if $x = \gamma(0)$, then $\sigma_m(x) \in W$. We thus see that $\varphi_m(W) = \sigma_m^{-1}(W)$ is closed in X.

Remark 5.21. If X is connected, then for every nonnegative integer m, the scheme X_m is connected. Indeed, if this is not the case, then we can label the irreducible components of X_m as $Z'_1, \ldots, Z'_r, Z''_1, \ldots, Z''_s$ such that $Z' = \cup_{i=1}^r Z'_i$ and $Z'' = \cup_{j=1}^s Z''_j$ are disjoint. It follows from the previous remark that $\varphi_m(Z')$ and $\varphi_m(Z'')$ are disjoint closed subsets of X (if $x \in X$ lies in the intersection, then $\sigma_m(x) \in Z' \cap Z''$) whose union is X, contradicting the fact that X is connected.

Definition 5.22. Let X, Y, and F be schemes of finite type over k and let $f \colon Y \to X$ be a morphism.

i) We say that f is *locally trivial*, with fiber F, if X has an open cover $X = \cup_{i \in I} U_i$ such that for every $i \in I$, we have an isomorphism $f^{-1}(U_i) \simeq U_i \times F$ of schemes over U_i.
ii) Suppose now that X, Y, and F are reduced. We say that f is *piecewise trivial*, with fiber F, if there is a cover $X = \cup_{i \in I} X_i$, with each X_i locally closed in X, such that for every i, we have an isomorphism $f^{-1}(X_i)_{\mathrm{red}} \simeq X_i \times F$ of schemes over X_i.

Proposition 5.23. If X is a smooth n-dimensional variety[2], then for every $m \geq p$, the morphism $\varphi_{m,p} \colon X_m \to X_p$ is locally trivial, with fiber $\mathbf{A}^{(m-p)n}$. In particular, for every m, the jet scheme X_m is a smooth variety, of dimension $(m+1)n$.

Proof. In order to prove the first assertion, after covering X by suitable affine open subsets, we may assume that we have an étale morphism $f \colon X \to Y = \mathbf{A}^n$. In this case, it follows from Proposition 5.13 that for every $m \geq p$ we have a commutative diagram

$$\begin{array}{ccccc} X_m & \xrightarrow{\varphi^X_{m,p}} & X_p & \xrightarrow{\varphi^X_p} & X \\ \downarrow{\scriptstyle f_m} & & \downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle f} \\ Y_m & \xrightarrow{\varphi^Y_{m,p}} & Y_p & \xrightarrow{\varphi^Y_p} & Y \end{array}$$

in which the right square and the big one are Cartesian; therefore the left square is Cartesian as well. Since $Y_m \simeq Y_p \times \mathbf{A}^{(m-p)n}$ as schemes over Y_p (see Example 5.11), it follows that $X_m \simeq X_p \times \mathbf{A}^{(m-p)n}$ as schemes over X_p. This gives the first assertion in the proposition. We then deduce that X is smooth and irreducible (being connected by Remark 5.21), of dimension $n + mn = (m+1)n$. $\square$

Corollary 5.24. If X is a smooth, irreducible variety, then $\psi_m \colon X_\infty(k) \to X_m$ is open and surjective for every $m \geq 0$.

Proof. Surjectivity of ψ_m is a consequence of the fact that $X_\infty(k) = \varprojlim_m X_m$ and of the surjectivity of the maps $X_q \to X_m$ for $q \geq m$, which follows from Proposition 5.23. In order to see that ψ_m is open, note first that by definition of the inverse limit, the Zariski topology on $X_\infty(k)$ is the inverse limit topology

[2] Recall that a variety is automatically assumed to be irreducible.

(this is the coarsest topology that makes all maps $X_\infty(k) \to X_m$ continuous). By Proposition 5.23, all maps $\varphi_{m,p}\colon X_m \to X_p$, for $m \geq p$, flat, hence open. This implies that ψ_m is open (we use the fact that every open subset of $X_\infty(k)$ is a union of subsets of the form $\psi_p^{-1}(U)$, for some p and some open subset $U \subseteq X_p$). $\square$

Remark 5.25. The argument in the proof of the Proposition 5.23 shows that if X is smooth and $x_1, \ldots, x_n$ give a regular system of parameters at a point $P \in X$, then we have an isomorphism $\psi_0^{-1}(P) \simeq \big(tk[\![t]\!]\big)^n$ that associates to an arc $\gamma\colon \operatorname{Spec} k[\![t]\!] \to X$, with $\gamma(0) = P$ and associated local homomorphism $\gamma^*\colon \mathcal{O}_{X,P} \to k[\![t]\!]$ the n-tuple $\big(\gamma^*(x_1), \ldots, \gamma^*(x_n)\big)$.

5.2. Cylinders in the space of arcs

Let X be a fixed smooth variety of dimension $n \geq 1$, over an algebraically closed field k. We begin by introducing certain subsets of $X_\infty(k)$.

Definition 5.26. A subset $C \subseteq X_\infty(k)$ is a *cylinder* if it can be written as $C = \psi_m^{-1}(S)$ for some $m \geq 0$ and some constructible subset $S \subseteq X_m$.

Remark 5.27. It is clear that the cyclinders form an algebra of subsets of $X_\infty(k)$.

Remark 5.28. If $C \subseteq X_\infty(k)$ is a cylinder, then for every $p \geq 0$, its image $\psi_p(C) \subseteq X_p$ is a constructible subset. Indeed, we may assume that $C = \psi_m^{-1}(S)$, for some constructible subset $S \subseteq X_m$ and some $m \geq p$. In this case, it follows from the surjectivity of ψ_m that $\psi_p(C) = \varphi_{m,p}(S)$, which is constructible by Chevalley's constructibility theorem.

The next proposition shows that for a subset of some X_m, certain topological properties are equivalent for the subset in X_m and for its inverse image in $X_\infty(k)$.

Proposition 5.29. Let $S \subseteq X_m$ be any subset, for some $m \geq 1$, and let $C = \psi_m^{-1}(S)$.

- i) The set C is open (closed, locally closed) in $X_\infty(k)$ if and only if S is open (respectively closed, locally closed) in X_m.
- ii) We have $\overline{C} = \psi_m^{-1}(\overline{S})$.
- iii) If S is a locally closed subset, then C is irreducible if and only if S is irreducible. Moreover, if $S = S_1 \cup \ldots \cup S_r$ is the decomposition in irreducible components, then $C = \psi_m^{-1}(S_1) \cup \ldots \cup \psi_m^{-1}(S_r)$ is the decomposition of C in irreducible components.

Proof. In order to prove i), we only need to prove the "only if" part, since the converse follows from the fact that ψ_m is continuous. Recall that by Corollary 5.24, each map $\psi_m\colon X_\infty(k) \to X_m$ is surjective and open. This implies that if C is open, then S is open. If C is closed, then $\psi_m^{-1}(X_m \smallsetminus S)$ is open, and by what we have already seen, we deduce that $X_m \smallsetminus S$ is open, hence S is closed.

We next prove the assertion in ii). Since $C \subseteq \psi_m^{-1}(\overline{S})$, which is closed, we clearly have $\overline{C} \subseteq \psi_m^{-1}(\overline{S})$. For the reverse inclusion, suppose that $\gamma \in \psi_m^{-1}(\overline{S})$. If $\gamma \not\in \overline{C}$, it follows that there is an open subset U of $X_\infty(k)$ that contains γ and such that $U \cap \psi_m^{-1}(S) = \emptyset$. In this case $\psi_m(\gamma) \in \psi_m(U)$, which is open, $\psi_m(\gamma)$ also lies in $\overline{S}$, but $\psi_m(U) \cap S = \emptyset$, a contradiction.

We can now prove the remaining assertion in i). If C is locally closed, then there is an open subset U of $X_\infty(k)$ such that $C = U \cap \overline{C} = U \cap \psi_m^{-1}(\overline{S})$. Applying

ψ_m, we get $S = \psi_m(C) = \psi_m(U) \cap \overline{S}$, which is locally closed since $\psi_m(U)$ is open in X_m.

If C is irreducible, then S is irreducible since the image of an irreducible topological space by a continuous map is irreducible. Suppose now that S is irreducible. If C is reducible, then there are open subsets $U_1, U_2 \subseteq X_\infty(k)$ such that $U_1 \cap C$ and $U_2 \cap C$ are nonempty, but $U_1 \cap U_2 \cap C = \emptyset$. Since $\psi_m(U_1) \cap S$ and $\psi_m(U_2) \cap S$ are open in S and nonempty, and since S is irreducible, it follows that $\psi_m(U_1) \cap \psi_m(U_2) \cap S$ is nonempty, hence it contains some $u \in X_m$. In this case $\psi_m^{-1}(u) \subseteq C$ and $\psi_m^{-1}(u) \cap U_1$ and $\psi_m^{-1}(u) \cap U_2$ are nonempty, but their intersection is empty. This contradicts the fact that $\psi_m^{-1}(u) \simeq \operatorname{Spec} k[x_1, x_2, \ldots]$ is irreducible (the isomorphism follows, for example, from the description of the fibers of ψ_0 in Remark 5.25). This gives the first assertion in iii) and the second one is an immediate consequence. $\square$

REMARK 5.30. Arguing as in the proof of the above proposition, we see that if $f\colon X \to Y$ is a morphism of algebraic varieties which is locally trivial, with fiber F which is irreducible, and if $C \subseteq X$ is a subset that is a union of fibers of f, then C is open, closed, or locally closed if and only if $f(C)$ has the same property. Moreover, $\overline{C} = f^{-1}\big(\overline{f(C)}\big)$, and if C is locally closed, then C is irreducible if and only $f(C)$ has this property.

The most important examples of cylinders arise by imposing order conditions along closed subschemes of X, as follows. Suppose that Y is a closed subscheme of X, defined by the ideal $\mathcal{I}_Y$. If $\gamma\colon \operatorname{Spec} k[\![t]\!] \to X$ is an arc on X, then by pulling back $\mathcal{I}_Y$, we get an ideal $\gamma^{-1}(\mathcal{I}_Y) \subseteq k[\![t]\!]$. We put $\operatorname{ord}_Y(\gamma) = m$ if this ideal is equal to (t^m) (we put $\operatorname{ord}_Y(\gamma) = \infty$ if the ideal is 0). For every nonnegative integer m, we define the *contact loci*

$$\operatorname{Cont}^m(Y) := \{\gamma \in X_\infty(k) \mid \operatorname{ord}_Y(\gamma) = m\} \quad \text{and}$$
$$\operatorname{Cont}^{\geq m}(Y) := \{\gamma \in X_\infty(k) \mid \operatorname{ord}_Y(\gamma) \geq m\}.$$

LEMMA 5.31. *For every nonnegative integer m, the contact locus $\operatorname{Cont}^{\geq m}(Y)$ is a closed cyclinder and the contact locus $\operatorname{Cont}^m(Y)$ is a locally closed cylinder in $X_\infty(k)$.*

PROOF. The second assertion follows from the first one since

$$\operatorname{Cont}^m(Y) = \operatorname{Cont}^{\geq m}(Y) \smallsetminus \operatorname{Cont}^{\geq m+1}(Y).$$

The first assertion is clear if $m = 0$. On the other hand, for $m \geq 1$, we have

$$\operatorname{Cont}^{\geq m}(Y) = \psi_{m-1}^{-1}(Y_{m-1}),$$

and thus it is a closed cylinder by Remark 5.9. $\square$

REMARK 5.32. Let $f\colon X' \to X$ be a morphism of smooth varieties. It is clear from the definition that if $\gamma' \in X'_\infty(k)$ and $\gamma = f_\infty(\gamma')$, then for every closed subscheme Y of X, we have $\operatorname{ord}_Y(\gamma) = \operatorname{ord}_{\gamma'}\big(f^{-1}(Y)\big)$. In particular, we have

$$\operatorname{Cont}^m\big(f^{-1}(Y)\big) = f_\infty^{-1}\big(\operatorname{Cont}^m(Y)\big) \quad \text{for all} \quad m \geq 0.$$

DEFINITION 5.33. If C is a closed cylinder in $X_\infty(k)$ and if we write $C = \psi_m^{-1}(S)$ with $S \subseteq X_m$ (so S is closed in X_m by Proposition 5.29), we define the *codimension* of C to be

$$\operatorname{codim}(C) := \operatorname{codim}_{X_m}(S) = (m+1)n - \dim(S).$$

Note that this is independent of our choice of m: this follows from the fact that if $m' > m$, then $\varphi_{m',m}$ is locally trivial, with fiber $\mathbf{A}^{(m'-m)n}$ by Proposition 5.23.

REMARK 5.34. It is clear from the definition that if $C \subseteq C'$ are closed cylinders in $X_\infty(k)$, then $\operatorname{codim}(C) \geq \operatorname{codim}(C')$. It is also straightforward to see that if $C_1, \ldots, C_r$ are closed cylinders in $X_\infty(k)$, then

$$\operatorname{codim}(C_1 \cup \ldots \cup C_r) = \min_i \operatorname{codim}(C_i).$$

The following result shows a certain incompatibility between cylinders and arc spaces of proper subschemes. As a consequence of this proposition, when dealing with cyclinders in $X_\infty(k)$, we will be able to ignore subsets of the form $Y_\infty(k)$, where Y is a proper closed subscheme of X.

PROPOSITION 5.35. If $C \subseteq X_\infty(k)$ is a nonempty cylinder and Y is a proper closed subscheme of X, then $C \not\subseteq Y_\infty(k)$.

PROOF. Let us suppose that $C \subseteq Y_\infty(k)$. This implies that there is $m \geq 0$ and $\gamma \in X_\infty(k)$ such that $\psi_m^{-1}\big(\psi_m(\gamma)\big) \subseteq Y_\infty(k)$. Let us choose a system of parameters $x_1, \ldots, x_n$ of $\mathcal{O}_{X,P}$, where $P = \psi_0(\gamma)$. This gives an isomorphism $\widehat{\mathcal{O}_{X,P}} \simeq k[\![y_1, \ldots, y_n]\!]$ that maps each x_i to y_i and an isomorphism $\psi_0^{-1}(P) \simeq \big(tk[\![t]\!]\big)^n$ (see Remark 5.25). If $f \in k[\![y_1, \ldots, y_n]\!]$ corresponds to a nonzero local section of the ideal defining Y in X, and if γ corresponds to $(u_1, \ldots, u_n) \in \big(tk[\![t]\!]\big)^n$, then our hypothesis says that

$$f(u_1 + t^{m+1}v_1, \ldots, u_n + t^{m+1}v_n) = 0 \quad \text{for all} \quad v_1, \ldots, v_n \in tk[\![t]\!]. \tag{5.5}$$

We have a formal power series in $(n+1)$ variables $g \in k[\![y, y_1, \ldots, y_n]\!]$ given by

$$g(t, y_1, \ldots, y_n) = f(u_1 + t^m y_1, \ldots, u_n + t^m y_n).$$

It is straightforward to see that since $f \neq 0$, we also have $g \neq 0$.

Our assumption says that $g(t, w_1, \ldots, w_n) = 0$ for all $w_1, \ldots, w_n \in tk[\![t]\!]$. We argue by induction on $n \geq 1$ to show that $g = 0$, and thus we have a contradiction. Suppose first that $n = 1$ and let us write $g(t, y) = \sum_{m \geq 0} a_m(t) y^m$. If $g \neq 0$, let m_0 be smallest such that $a_{m_0} \neq 0$. Our assumption implies that for every $w \in tk[\![t]\!] \smallsetminus \{0\}$, we have $\sum_{m \geq m_0} a_m(t) w^{m-m_0} = 0$. It is clear that if we write $w = \sum_{j \geq 1} b_j t^j$, then $\sum_{m \geq m_0} a_m(t) w^{m-m_0} = \sum_{j \geq 0} P_j(b) t^j$ for some polynomials $P_j \in k[z_1, z_2, \ldots]$. Since $P_j(b) = 0$ for all $b \neq (0, 0, \ldots)$ and all $j \geq 0$ and since k is infinite, it follows that $P_j = 0$ for all j. In particular, $\sum_{m \geq m_0} a_m(t) w^{m-m_0} = 0$ also when $w = 0$, hence $a_{m_0} = 0$, a contradiction.

Suppose now that $n \geq 2$ and we know the assertion for $n-1$. Let us write $g = \sum_{j \geq 0} g_j(t, y_1, \ldots, y_{n-1}) y_n^j$. Since $g(t, w_1, \ldots, w_{n-1}, w_n) = 0$ for all $w_1, \ldots, w_n \in tk[\![t]\!]$, applying the case $n = 1$, we conclude that $g_j(t, w_1, \ldots, w_{n-1}) = 0$ for all j and all $w_1, \ldots, w_{n-1} \in tk[\![t]\!]$. The induction hypothesis thus implies that $g_j = 0$ for all j, hence $g = 0$. This completes the proof of the proposition. $\square$

The next result shows that proper birational morphisms induce maps between the corresponding spaces of arcs that are bijective outside small subsets (more precisely outside spaces of arcs of suitable proper closed subschemes).

PROPOSITION 5.36. Let $f\colon Y \to X$ be a proper birational morphism between varieties over k. If $Z \subset X$ is a proper closed subset such that f is an isomorphism over $X \smallsetminus Z$ and $W = f^{-1}(Z)$, then f_∞ induces a bijection

$$Y_\infty(k) \smallsetminus W_\infty(k) \to X_\infty(k) \smallsetminus Z_\infty(k).$$

PROOF. It is clear that $f_\infty^{-1}\big(Z_\infty(k)\big) = W_\infty(k)$, hence we only need to show that for every $\gamma \in X_\infty(k) \smallsetminus Z_\infty(k)$, the fiber $f_\infty^{-1}(\gamma)$ has precisely one element. By assumption, we have $\gamma(\eta) \in X \smallsetminus Z$, where η is the generic point of $\operatorname{Spec} k[\![t]\!]$. Since $f^{-1}(X \smallsetminus Z) \to X \smallsetminus Z$ is an isomorphism, it follows that there is a unique morphism $\widetilde{\delta}\colon \operatorname{Spec} k(\!(t)\!) \to Y$ such that the following diagram

$$\begin{tikzcd}
\operatorname{Spec} k(\!(t)\!) \arrow[r, "\widetilde{\delta}"] \arrow[d, "j"'] & Y \arrow[d, "f"] \\
\operatorname{Spec} k[\![t]\!] \arrow[r, "\gamma"'] & X
\end{tikzcd}$$

is commutative. Since f is proper, it follows from the valuative criterion for properness that there is a unique morphism $\delta\colon \operatorname{Spec} k[\![t]\!] \to Y$ such that $f \circ \delta = \gamma$ and $\delta \circ j = \widetilde{\delta}$. We see that δ is the unique element of $Y_\infty(k)$ such that $f_\infty(\delta) = \gamma$. $\square$

We next consider in more detail the behavior of the map f_∞ in the case when f is a smooth blow-up.

PROPOSITION 5.37. Let X be a smooth variety and $Z \subset X$ a smooth subvariety of codimension $r \geq 2$. If $f\colon X' \to X$ is the blow-up along Z, with exceptional divisor E, and if for every $a \geq 0$, we put $C'_a = \operatorname{Cont}^a(E) \subseteq X'_\infty(k)$ and $C_a = \operatorname{Cont}^a(Z) \subseteq X_\infty(k)$, then for every $m \geq e$, the morphism f_m induces a map

$$\psi_m^{X'}(C'_a) = f_m^{-1}\big(\psi_m^X(C_a)\big) \to \psi_m^X(C_a)$$

that is locally trivial, with fiber $\mathbf{A}^{(r-1)a}$. Moreover, if $\gamma, \delta \in \psi_m^{X'}(C'_a)$ are such that $f_m(\gamma) = f_m(\delta)$, then $\varphi_{m,m-a}^{X'}(\gamma) = \varphi_{m,m-a}^{X'}(\delta)$.

PROOF. The assertion is local on X, hence we may assume that we have an étale morphism $X \to \mathbf{A}^n$ such that Z is the inverse image of a linear subspace of codimension r. Using Proposition 5.13, we see that it is enough to prove the assertion in the proposition when $X = \mathbf{A}^n$, with coordinates $x_1, \ldots, x_n$, and Z is defined by the ideal $(x_1, \ldots, x_r)$, which we assume to be the case from now on.

In order to simplify the notation, we write $S_a = \psi_m^X(C_a)$ and $S'_a = \psi_m^{X'}(C'_a)$. It follows from definition that S_a consists of those m-jets $\gamma\colon \operatorname{Spec} k[t]/(t^{m+1}) \to X$ such that the inverse image of the ideal $\mathcal{I}_Z$ defining Z is (t^a), while S'_a consists of those m-jets $\delta\colon \operatorname{Spec} k[t]/(t^{m+1}) \to X'$ such that the inverse image of the ideal $\mathcal{O}_Y(-E)$ defining E is (t^a). Since $\mathcal{I}_Z \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$, it follows that $S'_a = f_m^{-1}(S_a)$ (this is just a set-theoretic statement).

For a nonzero $u \in k[t]/(t^{m+1})$ we put $\operatorname{ord}_t(u) = q \leq m$ if $u = t^q u'$ for an invertible u' and we put $\operatorname{ord}_t(u) = \infty$ if $u = 0$. It is clear that

$$S_a = \big\{(u_1, \ldots, u_n) \in \big(k[t]/(t^{m+1})\big)^{\oplus n} \mid \min\{\operatorname{ord}_t(u_1), \ldots, \operatorname{ord}_t(u_r)\} = a\big\}$$

and this is a locally closed subset of $X_m = \big(k[t]/(t^{m+1})\big)^{\oplus n}$. For $1 \leq j \leq r$, we put

$$U_j = \big\{(u_1, \ldots, u_n) \in S_a \mid \operatorname{ord}_t(u_j) = a\big\}.$$

We clearly have an open cover $S_a = U_1 \cup \ldots \cup U_r$.

As usual, we can cover Y by affine charts $W^{(1)}, \ldots, W^{(r)}$, where $W^{(j)} \simeq \mathbf{A}^n$ has coordinates $y_1, \ldots, y_n$ such that $x_i = y_i$ if $i = j$ or $i > r$ and $x_i = y_i y_j$, otherwise. Note that we have

$$V_j := W_m^{(j)} \cap S'_a = \big\{(v_1, \ldots, v_n) \in \big(k[t]/(t^{m+1})\big)^{\oplus n} \mid \mathrm{ord}_t(v_j) = a\big\}$$

and that f_m induces a morphism

$$\alpha_j \colon V_j \to X_m, \quad \alpha_j(v_1, \ldots, v_n) = (v_1 v_j \ldots, v_j, \ldots, v_r v_j, v_{r+1}, \ldots, v_n).$$

It is clear that $\alpha_j(V_j) \subseteq U_j$. Moreover, for every $u = (u_1, \ldots, u_n) \in U_j$, there is $v = (v_1, \ldots, v_n) \in V_j$ that maps to u. In fact, $v_j, v_{r+1}, \ldots, v_n$ are uniquely determined and the classes of $v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_r$ in $k[t]/(t^{m+1-e})$ are uniquely determined, but these are the only constraints on these elements. In other words, we have an isomorphism $V_j \simeq U_j \times \mathbf{A}^{(r-1)e}$ that identifies α_j with the projection onto the first component followed by the embedding in X_m.

We thus obtain the first assertion in the proposition once we note that $V_\ell = f_m^{-1}(U_\ell)$ for $1 \leq \ell \leq r$. This follows by noting that if $v = (v_1, \ldots, v_n) \in V_j$ is such that $\alpha_j(v) \in U_\ell$, for some $\ell \neq j$, then $\mathrm{ord}_t(v_\ell) = 0$, hence $v \in W_m^{(\ell)} \cap S'_a = V_\ell$.

Suppose now that $\gamma, \delta \in \psi_m^{X'}(C'_a)$ are such that $f_m(\gamma) = f_m(\delta)$. First, this implies that there is j such that $\gamma, \delta \in V_j$. Second, it follows from the above description of the fibers of $f_m|_{V_j}$ that γ and δ have the same image in X'_{m-a}, and we thus get the last assertion in the proposition. □

Corollary 5.38. Let X be a smooth variety and $Z \subset X$ a smooth subvariety of codimension $r \geq 2$. If $f \colon X' \to X$ is the blow-up along Z, with exceptional divisor E, then for every $a \geq 0$, we have a bijection between the locally closed cylinders $C' \subseteq \mathrm{Cont}^a(E)$ and the locally closed cylinders $C \subseteq \mathrm{Cont}^a(Z)$ that maps C' to $f_\infty(C')$ and C to $f_\infty^{-1}(C)$. This bijection preserves irreducibility and if C corresponds to C', then

$$\mathrm{codim}(\overline{C}) = (r-1)a + \mathrm{codim}(\overline{C'}).$$

Proof. Note that $\mathrm{Cont}^a(E) = f_\infty^{-1}\big(\mathrm{Cont}^a(Z)\big)$ (see Remark 5.32) and f_∞ induces a bijective map $\mathrm{Cont}^a(E) \to \mathrm{Cont}^a(Z)$ by Proposition 5.36. It follows that we get a bijective map between the subsets C' of $\mathrm{Cont}^a(E)$ and the subsets C of $\mathrm{Cont}^a(Z)$ that maps C' to $f_\infty(C')$ and whose inverse maps C to $f_\infty^{-1}(C)$.

If $C \subseteq \mathrm{Cont}^a(Z)$ is a locally closed cylinder and if we write $C = (\psi_m^X)^{-1}(S)$, then $C' := f_\infty^{-1}(C) = (\psi_m^{X'})^{-1}(S')$, where $S' = f_m^{-1}(S)$. Therefore C' is a locally closed cylinder. We may clearly assume that $m \geq e$, in which case the induced morphism $S' \to S$ is locally trivial, with fiber $\mathbf{A}^{(r-1)a}$ by Proposition 5.37. It follows from Remark 5.30 that S is irreducible if and only if S' is irreducible and, using also Proposition 5.29, we conclude that C is irreducible if and only if C' is irreducible. In general, we have $\dim(S') = \dim(S) + (r-1)a$, hence $\mathrm{codim}(\overline{C}) = \mathrm{codim}(\overline{C'}) + (r-1)a$.

In order to complete the proof of the corollary, it is enough to show that if $C' \subseteq \mathrm{Cont}^a(E)$ is a locally closed cylinder, then $C := f_\infty(C')$ is a locally closed cylinder. Let q be such that $C' = (\psi_q^{X'})^{-1}(A)$ for some locally closed subset A in X'_q and let $m = q + a$, so $C' = (\psi_m^{X'})^{-1}(S')$, where $S' = (\varphi_{m,q}^{X'})^{-1}(A)$. We will show that S' is a union of fibers of f_m, its image $f_m(S')$ is locally closed in X_m, and $C = (\psi_m^X)^{-1}\big(f_m(S')\big)$. In particular, C is a locally closed cylinder.

Indeed, if $\gamma \in S'$ and $\delta \in X'_m$ are such that $f_m(\gamma) = f_m(\delta)$, then it follows from Proposition 5.37 that $\varphi^{X'}_{m,q}(\delta) = \varphi^{X'}_{m,q}(\gamma) \in A$, hence $\delta \in S'$. This shows that S' is a union of fibers of f_m. In this case, since the morphism $\psi^{X'}_m\big(\mathrm{Cont}^a(E)\big) \to \psi^X_m\big(\mathrm{Cont}^a(Z)\big)$ is locally trivial, with irreducible fiber, and S' is locally closed, we conclude that $f_m(S')$ is locally closed (see Remark 5.30).

The inclusion $C \subseteq (\psi^X_m)^{-1}\big(f_m(S')\big)$ is clear. In order to prove the reverse inclusion, suppose that $\alpha \in (\psi^X_m)^{-1}\big(f_m(S')\big)$. In particular, we have $\alpha \in \mathrm{Cont}^a(Z)$, hence by Proposition 5.36 we know there is $\beta \in X'_\infty(k)$ such that $f_\infty(\beta) = \alpha$. Since $f_m\big(\psi^{X'}_m(\beta)\big) \in f_m(S')$, it follows from what we have already shown that $\psi^{X'}_m(\beta) \in S'$, and thus $\beta \in C'$. This implies that $\alpha \in C$. □

COROLLARY 5.39. Let X be a smooth variety and $Z \subset X$ a smooth subvariety of codimension $r \geq 2$. If $f\colon X' \to X$ is the blow-up along Z, with exceptional divisor E, then the following hold:

i) For every closed cylinder $C' \subseteq X'_\infty(k)$, the closure $\overline{f_\infty(C')}$ of its image in X_∞ is a closed cylinder. Moreover, if C' is irreducible, then $\overline{f_\infty(C')}$ is irreducible, and if $a = \min\{i | C' \cap \mathrm{Cont}^i(E) \neq \emptyset\}$, then
$$\mathrm{codim}\big(\overline{f_\infty(C)}\big) = \mathrm{codim}(C) + (r-1)a.$$
ii) If $C \subseteq X_\infty(k)$ is an irreducible, closed cylinder, then there is a unique irreducible closed cylinder $C' \subseteq X'_\infty(k)$ such that $C = \overline{f_\infty(C')}$.

PROOF. We first prove i). By Proposition 5.29, we can write C' as a finite union of irreducible closed cylinders: $C' = C'_1 \cup \ldots \cup C'_q$. Since $\overline{f_\infty(C')} = \overline{f_\infty(C'_1)} \cup \ldots \cup \overline{f_\infty(C'_q)}$, it follows that it is enough to prove the first assertion in i) when C' is irreducible, hence we make this assumption. Since the closure of an irreducible set is irreducible and the image of an irreducible topological space by a continuous map is irreducible, we see that $\overline{f_\infty(C')}$ is closed and irreducible.

In order to show that it is a cyclinder, let a be as i) (note that a is a nonnegative integer, since $C' \not\subseteq E_\infty$ by Proposition 5.35). In this case $C'_0 := C' \cap \mathrm{Cont}^a(E)$ is a locally closed cylinder and a nonempty open subset of C'. Since C' is irreducible, it follows that $\overline{f_\infty(C')} = \overline{f_\infty(C'_0)}$. On the other hand, $f_\infty(C'_0)$ is a cylinder by Corollary 5.38 and thus its closure is a cylinder as well by Proposition 5.29ii). We conclude that $\overline{f_\infty(C')}$ is a cylinder and one more application of Corollary 5.38 gives
$$\mathrm{codim}\big(\overline{f_\infty(C')}\big) = \mathrm{codim}(C') + (r-1)a.$$
We next prove ii). Given the irreducible closed cylinder $C \subseteq X_\infty(k)$, let $a = \min\big\{i | C \cap \mathrm{Cont}^i(Z) \neq \emptyset\big\}$ (note that this is a nonnegative integer since $C \not\subseteq Z_\infty(k)$ by Proposition 5.35). Then $C_0 := C \cap \mathrm{Cont}^a(Z)$ is a locally closed, irreducible cylinder and $C = \overline{C_0}$. By Corollary 5.38, $C'_0 = f^{-1}_\infty(C_0)$ is an irreducible locally closed cylinder such that $C_0 = f_\infty(C'_0)$. It is then clear that $\overline{C'_0}$ is a closed irreducible cylinder such that $f_\infty(\overline{C_0}) = C$. If C'' is another closed irreducible cylinder in $X_\infty(k)$ such that $\overline{f_\infty(C'')} = C$, we see that $a = \min\big\{i \mid C'' \cap \mathrm{Cont}^i(E) \neq \emptyset\big\}$. If $C''_0 = C'' \cap \mathrm{Cont}^a(E)$, then $C''_0 \subseteq C'_0$, hence by taking closures, we get $C'' \subseteq C'$. By part i), we have $\mathrm{codim}(C') = \mathrm{codim}(C'')$, and since both C' and C'' are closed irreducible cylinders, we conclude that $C' = C''$. This completes the proof of the corollary. □

Recall that in the setting of Proposition 5.37, the relative canonical divisor $K_{Y/X} = (r-1)E$ (see Example 2.19). The proposition admits the following generalization to arbitrary morphisms, which is the key ingredient in many applications of the spaces of arcs.

THEOREM 5.40. *Let $f\colon X' \to X$ be a proper, birational morphism between smooth varieties and suppose that* $\mathrm{char}(k) = 0$. *For every $e \geq 0$, if $C_e = \mathrm{Cont}^e(K_{X'/X})$, then for every $m \geq 2e$, the following hold:*

i) *If $\gamma, \delta \in X'_m$ are such that $\gamma \in \psi_m^{X'}(C_e)$ and $f_m(\gamma) = f_m(\delta)$, then $\varphi^{X'}_{m,m-e}(\gamma) = \varphi^{X'}_{m,m-e}(\delta)$. In particular, $\psi_m^{X'}(C_e)$ is a union of fibers of f_m.*
ii) *The map*
$$\psi_m^{X'}(C_e) \to f_m\big(\psi_m^{X'}(C_e)\big)$$
induced by f_m is piecewise trivial[3]*, with fiber $\mathbf{A}^e$.*

For a proof of this theorem, we refer to [**DL99**, Lemma 3.4]. In fact, the result in *loc. cit.* treats a more general case, when X is allowed to be singular. For a slightly different presentation of the proof and a statement that also works in positive characteristic, see [**EM09**, Theorem 6.2]. The proof of this result is somewhat technical, so we omit it, since for our purpose the version in Proposition 5.37 will suffice.

5.3. Invariants of valuations via the arc space

Let X be a smooth variety[4] over k. Our goal is to establish a dictionary between divisorial valuations with center on X and irreducible closed cylinders in $X_\infty(k)$. This was done in characteristic 0 in [**ELM04**] and in arbitrary characteristic in [**Zhu17**]. We follow here the latter approach. It will turn out to be convenient to also consider integer multiples of divisorial valuations.

DEFINITION 5.41. If Y is a variety over k, a *non-normalized divisorial valuation* with center on Y is a valuation of $k(Y)$ of the form $v = q \cdot \mathrm{ord}_E$, where E is a divisor over Y and q is a positive integer. If Y is smooth, then $A_Y(\mathrm{ord}_E)$ is defined and we put $A_Y(v) := q \cdot A_Y(\mathrm{ord}_E)$.

We review some general notions regarding valuations.

DEFINITION 5.42. Given a discrete valuation $v\colon K \to \mathbf{Z} \cup \{\infty\}$ on a field K, the *valuation ring* of v is the subring
$$R_v = \{a \in K \mid v(a) \geq 0\}$$
of K. It is well-known (and easy to prove) that this is a DVR, with maximal ideal
$$\mathfrak{m}_v = \{a \in K \mid v(a) > 0\}.$$

DEFINITION 5.43. Suppose that Y is a variety over k and v is a discrete valuation of the function field $k(Y)$ of Y. A *center* of v on Y is a (not-necessarily-closed) point $\xi \in Y$ such that $\mathcal{O}_{Y,\xi} \subseteq R_v$ and the inclusion is a local homomorphism (alternatively, we will say that the center is the corresponding closed irreducible subset

[3] In general, both the source and target of this map are constructible subsets, but the notion of piecewise trivial map extends in an obvious way to this setting.

[4] In this section, we require *varieties* to be separated.

$\overline{\{\xi\}}$ of Y). Note that since Y is separated, it follows from the valuative criterion for separatedness that v has at most oner center on Y.

EXAMPLE 5.44. Let Y be a variety over k and $v = q \cdot \mathrm{ord}_E$, where E is a prime divisor on a normal variety W that has a birational morphism $f\colon W \to Y$. In this case $R_v = \mathcal{O}_{W,E}$ and the center of v on Y is $\overline{f(E)}$. We thus recover the definition we gave in Chapter 2.1. In particular, we see that in this case we have $\mathrm{trdeg}_k(R_v/\mathfrak{m}_v) = \dim(Y) - 1$.

REMARK 5.45. Suppose that Y is a normal variety over k and v is a discrete valuation of $k(Y)$ that has center Z on Y of codimension 1. In this case $v = q \cdot \mathrm{ord}_Z$, where $q \in \mathbf{Z}_{>0}$ is such that v takes value q on a uniformizer of $\mathfrak{m}_v$. Indeed, since $\mathcal{O}_{Y,Z}$ is a DVR and the inclusion $\mathcal{O}_{Y,Z} \hookrightarrow R_v$ is a local homomorphism, it follows that $\mathcal{O}_{Y,Z} = R_v$: if $u \in R_v \smallsetminus \mathcal{O}_{Y,Z}$, then u^{-1} lies in the maximal ideal of $\mathcal{O}_{Y,Z}$, and thus in $\mathfrak{m}_v$, a contradiction. The description of v is now clear.

A key ingredient that will allow us to handle divisorial valuations in arbitrary characteristic (when resolution of singularities is not available) is the following result of Zariski that says that given a divisorial valuation of the function field of a variety, after successively blowing-up the center of the valuation finitely many times, we achieve a model on which the center has codimension 1. This is particularly useful when the original variety is smooth, in which case the blow-ups are easy to analyze. We give the proof following [**KM98**, Lemma 2.45]. Note that while we state and prove the result only for smooth varieties, a small modification of the argument gives the proof for an arbitrary variety.

We first set some notation. Suppose that X is a smooth variety of dimension n over k and v is a discrete valuation of $k(X)$, with center on X. We define recursively the following sequence of varieties and (not-necessarily-closed) points on them. We first put $X_0 = X$ and let ξ_0 be the center of v on X. Given $j \geq 0$, suppose that we have constructed X_j and $\xi_j \in X_j$. Let $f_j\colon X_{j+1} \to X_j$ be the blow-up of X_j along $\overline{\{\xi_j\}}$. Since f_j is proper and v has a center on X_j, it follows from the valuative criterion for properness that v also has a center on X_{j+1}. We denote this by ξ_{j+1}. Note that by definition we have $f_j(\xi_{j+1}) = \xi_j$. Note also that each ring $\mathcal{O}_{X_j,\xi_j}$ is regular. Indeed, arguing by induction on j, we may assume that X_j is smooth in a neighborhood of ξ_j. We can choose an open neighborhood U_j of ξ_j that is smooth and such that $U_j \cap \overline{\{\xi_j\}}$ is smooth, in which case $f_j^{-1}(U_j)$ is smooth. Since $\xi_{j+1} \in f_j^{-1}(U_j)$, we conclude that X_{j+1} is smooth in a neighborhood of ξ_{j+1}.

PROPOSITION 5.46. *With the above notation, if $\mathrm{trdeg}_k(R_v/\mathfrak{m}_v) = n-1$, then there is N such that $\overline{\{\xi_N\}}$ is a prime divisor E on X_N and $v = q \cdot \mathrm{ord}_E$ for some $q \in \mathbf{Z}_{>0}$.*

REMARK 5.47. Note that by Example 5.44, the proposition applies whenever v is a non-normalized divisorial valuation with center on X. However, the result also shows that a discrete valuation with center on X is a non-normalized divisorial valuation if and only if $\mathrm{trdeg}_k(R_v/\mathfrak{m}_v) = n-1$.

PROOF OF PROPOSITION 5.46. Let's describe first the process of going from $\mathcal{O}_{X_j,\xi_j}$ to $\mathcal{O}_{X_{j+1},\xi_{j+1}}$. Let $u_1, \ldots, u_d$ be generators of the maximal ideal of $\mathcal{O}_{X_j,\xi_j}$. After reordering them, we may assume that $v(u_1) \leq v(u_i)$ for all $i \geq 2$, hence $\frac{u_i}{u_1} \in R_v$ for $1 \leq i \leq d$. It is then easy to see using the description of the blow-up charts on X_{j+1} that if $R_{j+1} = \mathcal{O}_{X_j,\xi_j}[u_2/u_1, \ldots, u_d/u_1]$ and $\mathfrak{p}_{j+1} = \mathfrak{m}_v \cap R_{j+1}$, then $\mathcal{O}_{X_{j+1},\xi_{j+1}} = (R_{j+1})_{\mathfrak{p}_{j+1}}$.

The first step in the proof is to show that

$$R_v = \bigcup_{j \geq 0} \mathcal{O}_{X_j,\xi_j}.$$

Indeed, suppose that $a \in R_v$. For a given j, we can write $a = \frac{a_1}{a_2}$, with $a_1, a_2 \in \mathcal{O}_{X_j,\xi_j}$. We choose j such that when we write this, $v(a_2) \in \mathbf{Z}_{\geq 0}$ is minimal. If $v(a_2) = 0$, then a_2 is invertible in R_v and thus also in $\mathcal{O}_{X_j,\xi_j}$, hence $a \in \mathcal{O}_{X_j,\xi_j}$. We next show that if $v(a_2) \geq 1$, then we contradict the minimality of $v(a_2)$. Since $v(a_2) > 0$, we see that also $v(a_1) = v(a) + v(a_2) > 0$; therefore both a_1 and a_2 lie in the maximal ideal of $\mathcal{O}_{X_j,\xi_j}$. As above, suppose that $u_1, \ldots, u_d$ are generators of this maximal ideal, ordered such that $v(u_1) \leq v(u_i)$ for all i. We write

$$a_1 = \sum_{i=1}^{d} c_{1,i} u_i \quad \text{and} \quad a_2 = \sum_{i=1}^{d} c_{2,i} u_i,$$

for some $c_{1,i}, c_{2,i} \in \mathcal{O}_{X_j,\xi_j}$. As we have seen, in this case we have

$$a_1' := \frac{a_1}{u_1} = \sum_{i=1}^{d} c_{1,i} \frac{u_i}{u_1} \in \mathcal{O}_{X_{j+1},\xi_{j+1}}$$

and similarly $a_2' := a_2/u_1 \in \mathcal{O}_{X_{j+1},\xi_{j+1}}$. Since we can write $a = \frac{a_1'}{a_2'}$ and $v(a_2') = v(a_2) - v(u_1) < v(a_2)$, we contradict the minimality of $v(a_2)$.

The next step is to choose $w_1, \ldots, w_{n-1} \in R_v$ such that their images in $R_v/\mathfrak{m}_v$ give a transcendence basis over k. Using the first step, we can choose N such that $w_1, \ldots, w_{n-1} \in \mathcal{O}_{X_N,\xi_N}$. In particular, this implies that $E := \overline{\{\xi_N\}}$ is a prime divisor on X_N. Since we have seen that X_N is smooth in a neighborhood of ξ_N, we conclude that $v = q \cdot \mathrm{ord}_E$ for some $q \in \mathbf{Z}_{>0}$ by Remark 5.45. $\square$

From now on, we fix a smooth variety X over k. Given a closed, irreducible cylinder $C \subseteq X_\infty(k)$ that does not dominate X (that is, $\overline{\psi_0(C)} \neq X$), we define a discrete valuation ord_C of $k(X)$, as follows. Let us choose first an affine open subset $U \subseteq X$ such that $\psi_0(C) \cap U \neq \emptyset$ (therefore $C_U := C \cap \psi_0^{-1}(U)$ is a nonempty open subset of C and also a cylinder in $U_\infty(k)$). Note that for every $\gamma \in C_U$ and every $f \in \mathcal{O}_X(U)$, we can define $\mathrm{ord}_\gamma(f) := \mathrm{ord}_{V(f)}(\gamma) \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$. For every $f \in \mathcal{O}_X(U)$, we put

$$\mathrm{ord}_C(f) := \min\{\mathrm{ord}_\gamma(f) \mid \gamma \in C_U\}.$$

LEMMA 5.48. *For every closed, irreducible cylinder $C \subseteq X_\infty(k)$ that does not dominate X, we obtain in this way a discrete valuation ord_C of $k(X)$ which is independent of the choice of affine open subset U. Its center on X is equal to $\overline{\psi_0(C)}$.*

PROOF. Note first that if $f \neq 0$, then $\mathrm{ord}_C(f) < \infty$: this follows from the fact that $C_U \not\subseteq V(f)_\infty$ by Proposition 5.35. Of course, we have $\mathrm{ord}_C(0) = \infty$.

If $f, g \in \mathcal{O}_X(U)$, then it is clear that for every $\gamma \in C_U$ we have

$$\mathrm{ord}_\gamma(f + g) \geq \min\big\{\mathrm{ord}_\gamma(f), \mathrm{ord}_\gamma(g)\big\}.$$

This immediately implies

$$\operatorname{ord}_C(f+g) \geq \min\big\{\operatorname{ord}_C(f), \operatorname{ord}_C(g)\big\}.$$

Given any $f, g \in \mathcal{O}_X(U)$, we have

$$\operatorname{ord}_\gamma(fg) = \operatorname{ord}_\gamma(f) + \operatorname{ord}_\gamma(g) \quad \text{for all} \quad \gamma \in C_U.$$

Note also that for f we have $\operatorname{ord}_C(f) = \operatorname{ord}_\gamma(f)$ for all γ in a nonempty open subset of C_U and similarly for g. Since C_U is irreducible, it follows that for every $f, g \in \mathcal{O}_X(U)$, we have

$$\operatorname{ord}_C(fg) = \operatorname{ord}_C(f) + \operatorname{ord}_C(g).$$

Note that if $f \in \mathcal{O}_X(U)$, then $\operatorname{ord}_C(f) > 0$ if and only if f vanishes on $\psi_0(C_U)$; in particular, since C does not dominate X, we have $\operatorname{ord}_C(f) \in \mathbf{Z}_{>0}$ for some f. It is now straightforward to see that ord_C extends uniquely to a discrete valuation of $k(X)$ by putting $\operatorname{ord}_C(f/g) = \operatorname{ord}_C(f) - \operatorname{ord}_C(g)$ for any two $f, g \in \mathcal{O}_X(U)$ with $g \neq 0$. Moreover, we see that the center of ord_C on X is $\overline{\psi_0(C)}$.

Finally, in order to show that the definition is independent of the choice of U, it is enough to note that if $V \subseteq U$ is an affine open subset of U such that $C \cap V_\infty \neq \emptyset$, then for every $f \in \mathcal{O}_X(U)$, we have

$$\min\big\{\operatorname{ord}_\gamma(f) \mid \gamma \in C_U\big\} = \min\big\{\operatorname{ord}_\gamma(f) \mid \gamma \in C_V\big\}$$

due to the fact that the minimum on C_U is achieved on an open subset of this cylinder, which intersects $C \cap \psi_0^{-1}(V)$. It follows that if ord'_C is the valuation of $k(X)$ defined using V, we have $\operatorname{ord}_C = \operatorname{ord}'_C$ on $\mathcal{O}_X(U)$, and thus the two valuations have to agree on the fraction field $k(X)$ of $\mathcal{O}_X(U)$. $\square$

REMARK 5.49. Suppose that $f\colon X' \to X$ is the blow-up of a smooth variety X along a smooth subvariety Z. If $C' \subseteq X'_\infty(k)$ is a closed irreducible cylinder that does not dominate X', then it follows from Corollary 5.39 that $C := \overline{f_\infty(C')}$ is a closed irreducible cylinder in $X_\infty(k)$. It clearly does not dominate X since $\psi_0^X(C) \subseteq f\big(\overline{\psi_0^{X'}(C')}\big)$. Note that we have $\operatorname{ord}_C = \operatorname{ord}_{C'}$ as valuations of $k(X) = k(X')$. Indeed, it is clear that if we choose affine open subsets U' and U in X' and X, respectively, such that $U' \cap C' \neq \emptyset$, $U \cap C \neq \emptyset$, and f induces a morphism $U' \to U$, then for every $\varphi \in \mathcal{O}_X(U)$, we have a nonempty open subset $V \subseteq C$ such that $\operatorname{ord}_\gamma(\varphi) = \operatorname{ord}_C(\varphi)$, in which case for every $\delta \in C' \cap f_\infty^{-1}(V)$ we have $\operatorname{ord}_\delta(\varphi \circ f) = \operatorname{ord}_C(\varphi)$. This implies that $\operatorname{ord}_{C'} = \operatorname{ord}_C$ on $\mathcal{O}_X(U)$, and thus on $k(X)$.

We next associate a subset of $X_\infty(k)$ to every non-normalized divisorial valuation v of $k(X)$ with center on X, as follows. Let $\pi\colon Y \to X$ be a birational morphism, with Y normal, and E a prime divisor on Y such that $v = q \cdot \operatorname{ord}_E$ for some positive integer q. After possibly replacing Y by a suitable open subset, we may assume that both Y and E are smooth. We put

$$\operatorname{Cyl}(v) := \overline{\pi_\infty\big(\operatorname{Cont}^{\geq q}(E)\big)}.$$

LEMMA 5.50. *The subset* $\operatorname{Cyl}(v) \subseteq X_\infty(k)$ *is closed and irreducible and it is independent of the choice of model* (Y, E). *Moreover,* $\overline{\psi_0^X\big(\operatorname{Cyl}(v)\big)}$ *is the center of* v *on* X.

PROOF. Note that if E is a smooth prime divisor on the smooth variety Y and q is a positive integer, then $\mathrm{Cont}^{\geq q}(E)$ is an irreducible closed cylinder: indeed, it is equal to $(\psi_{q-1}^Y)(E_{q-1})$ and E_{q-1} is irreducible by Proposition 5.23. This implies that $\mathrm{Cyl}(v)$ is irreducible and it is closed by definition. Moreover, $\psi_0^Y\big(\mathrm{Cont}^{\geq q}(E)\big) = E$. This easily implies that $\overline{\psi_0^X\big(\mathrm{Cyl}(v)\big)}$ is the center of v on X.

Let us show that $\mathrm{Cyl}(v)$ does not depend on the choice of model. Using the comparison of models giving the same valuation in Remark 2.5, we see that it is enough to show that if $g\colon Z \to Y$ is a birational morphism between two smooth varieties, F is a smooth prime divisor on Z such that $E := \overline{g(F)}$ is a smooth prime divisor in Y, then for every $q \in \mathbf{Z}_{>0}$ we have

$$\mathrm{Cont}^{\geq q}(E) = \overline{g_\infty\big(\mathrm{Cont}^{\geq q}(F)\big)}.$$

Note that there are open subsets $U \subseteq Y$ and $V \subseteq Z$ such that $U \cap E \neq \emptyset$, $V \cap F \neq \emptyset$, and g induces an isomorphism $V \simeq U$. Using the fact that $\mathrm{Cont}^{\geq q}(F)$ and $\mathrm{Cont}^{\geq q}(E)$ are irreducible, we then conclude that

$$\overline{g_\infty\big(\mathrm{Cont}^{\geq q}(F)\big)} = \overline{g_\infty\big(\mathrm{Cont}^{\geq q}(F) \cap V_\infty(k)\big)} = \overline{\mathrm{Cont}^{\geq q}(E) \cap U_\infty(k)} = \mathrm{Cont}^{\geq q}(E).$$

$\square$

The following is the main result relating divisorial valuations and cylinders in the arc space.

THEOREM 5.51. *Let X be a smooth variety over k.*

i) *For every non-normalized divisorial valuation v of $k(X)$, with center on X, $\mathrm{Cyl}(v)$ is a closed irreducible cylinder that does not dominate X and $\mathrm{ord}_{\mathrm{Cyl}(v)} = v$. Moreover, we have $\mathrm{codim}\big(\mathrm{Cyl}(v)\big) = A_X(v)$.*

ii) *For every closed irreducible cylinder $C \subseteq X_\infty(k)$ that does not dominate X, the valuation ord_C is a non-normalized divisorial valuation of $k(X)$ with center on X. Moreover, we have $C \subseteq \mathrm{Cyl}(\mathrm{ord}_C)$.*

PROOF. We first prove i). We already know that $\mathrm{Cyl}(v)$ is closed and irreducible and does not dominate X by Lemma 5.50. Therefore we only need to show that it is a cylinder, its codimension is $A_X(v)$, and the corresponding valuation is v. Let us temporarily denote $\mathrm{Cyl}(v)$ by $\mathrm{Cyl}_X(v)$ in order to keep track of the variety X. We first note that if U is an open subset of X such that U intersects nontrivially the center of v on X, then is is enough to prove the assertion for $\mathrm{Cyl}_U(v)$. Indeed, we have $\mathrm{Cyl}_U(v) = \mathrm{Cyl}_X(v) \cap (\psi_0^X)^{-1}(U)$, and if this is a cylinder in $U_\infty(k)$, then $\mathrm{Cyl}_X(v) = \overline{\mathrm{Cyl}_U(v)}$ is a cylinder in $X_\infty(k)$ by Proposition 5.29. Moreover, the codimensions are equal and they determine the same valuation of $k(X)$.

By Proposition 5.46, we have a sequence of morphisms

$$X_r \xrightarrow{f_{r-1}} X_{r-1} \longrightarrow \ldots \longrightarrow X_1 \xrightarrow{f_0} X,$$

where each f_i is the blow-up of X_i along the center of v on X_i (assumed to have codimension ≥ 2), such that $v = q \cdot \mathrm{ord}_E$ for some prime divisor E on X_r and some positive integer q. We argue by induction on $r \geq 0$.

Suppose first that $r = 0$. After possibly replacing X by a suitable open subset that intersects E nontrivially, we may assume that X is affine, E is smooth and we have $x_1, \ldots, x_n \in \mathcal{O}_X(X)$ that give an algebraic system of coordinates on X such that E is defined by (x_1). In this case

$$C_X(v) = \mathrm{Cont}^{\geq q}(E) = (\psi_{q-1}^X)^{-1}(E_{q-1})$$

is an irreducible closed cylinder and its codimension is $q = A_X(v)$ by Proposition 5.24. In order to check that $\mathrm{ord}_{C_X(v)} = q \cdot \mathrm{ord}_E$, we need to show that if $h = x_1^m g$, where $g \notin (x_1)$, then $\mathrm{ord}_{C_X(v)}(h) = qm$. The fact that $\mathrm{ord}_{C_X(v)}(h) \geq qm$ is clear: if $\delta \in \mathrm{Cont}^{\geq q}(E)$, then $\mathrm{ord}_\delta(x_1) \geq q$ and thus $\mathrm{ord}_\delta(h) \geq mq$. In order to show the opposite inequality, it is enough to find $\gamma \in X_\infty(k)$ such that $\mathrm{ord}_\gamma(x_1) = q$ and $\mathrm{ord}_\gamma(g) = 0$. Since $g \notin (x_1)$, there is $P \in E$ such that $g(P) = 0$. For example, we can take $\gamma \in (\psi_0^X)^{-1}(P)$ that corresponds to $\widehat{\mathcal{O}_{X,P}} \to k[\![t]\!]$ that maps x_1 to t^q and $x_i - x_i(P)$ to 0 for $i \geq 2$.

We next suppose $r \geq 1$ and assume the assertion holds for $r-1$. After possibly replacing X by a suitable open subset that intersects nontrivially the center of v on X, we may assume that this center is smooth, hence the morphism $f = f_0 \colon X_1 \to X$ is the blow-up of a smooth irreducible subvariety of codimension $r \geq 2$, with exceptional divisor F. By the induction hypothesis, we know that $\mathrm{Cyl}_{X_1}(v)$ is an irreducible closed cylinder, $\mathrm{ord}_{\mathrm{Cyl}_{X_1}(v)} = v$, and $\mathrm{codim}\big(\mathrm{Cyl}_{X_1}(v)\big) = A_{X_1}(v)$. Note that by definition we have $\mathrm{Cyl}_X(v) = \overline{f_\infty\big(\mathrm{Cyl}_{X_1}(v)\big)}$, hence Corollary 5.39 implies that $\mathrm{Cyl}_X(v)$ is an irreducible closed cylinder and

$$\mathrm{codim}\big(\mathrm{Cyl}_X(v)\big) = \mathrm{codim}\big(\mathrm{Cyl}_{X_1}(v)\big) + (r-1)a, \tag{5.6}$$

where $a = \min\{i \mid \mathrm{Cyl}_{X_1}(v) \cap \mathrm{Cont}^i(F) \neq \emptyset\}$. In other words, $a = \mathrm{ord}_\gamma(F)$ for a general arc $\gamma \in \mathrm{Cyl}_{X_1}(v)$, that is $a = \mathrm{ord}_{\mathrm{Cyl}_{X_1}(v)}(F) = v(F) = q \cdot \mathrm{ord}_E(F)$. Since $K_{X_1/X} = (r-1)F$ by Example 2.19 and we have

$$A_X(\mathrm{ord}_E) = A_{X_1}(\mathrm{ord}_E) + \mathrm{ord}_E(K_{X_1/X})$$

by Remark 2.21, we conclude from (5.6) that

$$\mathrm{codim}\big(C_X(v)\big) = A_{X_1}(v) + (r-1)q \cdot \mathrm{ord}_E(F) = q\big(A_{X_1}(\mathrm{ord}_E) + \mathrm{ord}_E(K_{X_1/X})\big) = A_X(v).$$

Finally, since $\mathrm{Cyl}_X(v) = \overline{f_\infty\big(C_{X_1}(v)\big)}$, it follows from Remark 5.49 that $\mathrm{ord}_{C_X(v)} = \mathrm{ord}_{C_{X_1}(v)} = v$. This completes the proof of i).

Suppose now that $C \subseteq X_\infty(k)$ is a closed irreducible cylinder that does not dominate X. We have seen in Lemma 5.50 that if $Z = \overline{\psi_0^X(C)}$, then Z is the center of ord_C on X. We show by induction on $\mathrm{codim}(C)$ that ord_C is a non-normalized divisorial valuation of $k(X)$ with center on X and that $C \subseteq \mathrm{Cyl}_X(\mathrm{ord}_C)$. Note that for both assertions we may replace X by an open subset that intersects Z nontrivially. In particular, we may and will assume that Z is smooth.

Since $C \subseteq (\psi_0^X)^{-1}(Z)$, we have $\mathrm{codim}(C) \geq 1$ and if equality holds then $\mathrm{codim}_X(Z) = 1$. Note first that we are done if $\mathrm{codim}_X(Z) = 1$. Indeed, in this case it follows from Remark 5.45 that $\mathrm{ord}_C = p \cdot \mathrm{ord}_Z$ for some positive integer p. In fact, we have $p = \mathrm{ord}_Z(\gamma)$ for $\gamma \in C$ general, hence $C \subseteq \mathrm{Cont}^{\geq q}(Z) = \mathrm{Cyl}_X(q \cdot \mathrm{ord}_Z)$.

In particular, we see that we are done if $\mathrm{codim}(C) = 1$. Let us suppose now that C is arbitrary and that we know the assertion for cylinders C' in the space of arcs of a smooth variety that satisfy $\mathrm{codim}(C') < \mathrm{codim}(C)$. As we have seen, we may assume that $\mathrm{codim}_X(Z) = r \geq 2$. Let $f \colon X' \to X$ be the blow-up of X along Z (recall that we assume that Z is smooth). In this case it follows from Corollary 5.39 that there is an irreducible closed cylinder $C' \subseteq X'_\infty$ such that $\overline{f_\infty(C')} = C$. Moreover, since $C \subseteq \mathrm{Cont}^{\geq 1}(Z)$, it follows that $\mathrm{codim}(C') < \mathrm{codim} C$.

By induction, $\mathrm{ord}_{C'}$ is a non-normalized divisorial valuation and since $\mathrm{ord}_C = \mathrm{ord}_{C'}$ by Remark 5.49, it follows that ord_C has the same property. Finally, by induction we have $C' \subseteq \mathrm{Cyl}_{X'}(\mathrm{ord}_{C'})$, hence

$$C = \overline{f_\infty(C')} \subseteq \overline{\mathrm{Cyl}_{X'}(\mathrm{ord}_{C'})} = \mathrm{Cyl}_X(\mathrm{ord}_C).$$

This completes the proof of the theorem. □

Remark 5.52. Let X be a smooth variety and Y a proper closed subscheme of X. If m is a positive integer and C is an irreducible component of $\mathrm{Cont}^{\geq m}(Y)$, then there is a non-normalized divisorial valuation v of $k(X)$ with center on X such that $C = \mathrm{Cyl}(v)$. Indeed, Theorem 5.51 implies that $v = \mathrm{ord}_C$ is a non-normalized divisorial valuation and $C \subseteq \mathrm{Cyl}(v)$. Since $\mathrm{ord}_C = \mathrm{ord}_{\mathrm{Cyl}(v)}$, it follows that $\mathrm{Cyl}(v) \subseteq \mathrm{Cont}^{\geq m}(Y)$ and since C is an irreducible component of $\mathrm{Cont}^{\geq m}(Y)$, we have $C = \mathrm{Cyl}(v)$.

We can use Theorem 5.51 in order to give a characterization of log canonical thresholds in terms of jet schemes.

Corollary 5.53. Let X be a smooth variety of dimension n. If Y is a proper closed subscheme of X, defined by the ideal $\mathfrak{a}$, then

$$\mathrm{lct}(\mathfrak{a}) = n - \sup_{m\geq 0} \frac{\dim(Y_m)}{m+1} = \inf_C \frac{\mathrm{codim}(C)}{\mathrm{ord}_C(\mathfrak{a})},$$

where the infimum on the right-hand side is taken over all irreducible closed cylinders C that do not dominate X.

Proof. By definition, we have

$$\mathrm{lct}(\mathfrak{a}) = \inf_v \frac{A_X(v)}{v(\mathfrak{a})},$$

where the infimum is over all divisorial valuations v of $k(X)$ with center on X. Of course, if we replace v by a positive integer multiple, then the corresponding quotient does not change, hence we may let v run instead over all non-normalized valuations of $k(X)$ with center on X.

If $C \subseteq X_\infty(k)$ is an irreducible cylinder that does not dominate X, then it follows from Theorem 5.51 that ord_C is a non-normalized divisorial valuation v and $C \subseteq \mathrm{Cyl}(v)$ both define the same valuation v, so that

$$\frac{\mathrm{codim}(C)}{\mathrm{ord}_C(\mathfrak{a})} \geq \frac{\mathrm{codim}\big(\mathrm{Cyl}(v)\big)}{\mathrm{ord}_{\mathrm{Cyl}(v)}(\mathfrak{a})} = \frac{A_X(v)}{v(\mathfrak{a})}.$$

This gives the equality

$$\mathrm{lct}(\mathfrak{a}) = \inf_C \frac{\mathrm{codim}(C)}{\mathrm{ord}_C(\mathfrak{a})}.$$

Note now that if C is an irreducible cylinder that does not dominate X and $m = \mathrm{ord}_C(\mathfrak{a}) \geq 1$, then $C \subseteq \mathrm{Cont}^{\geq m}(Y) = \psi_{m-1}^{-1}(Y_{m-1})$, hence

$$\mathrm{codim}(C) \geq \mathrm{codim}\big(\mathrm{Cont}^{\geq m}(Y)\big) = mn - \dim(Y_{m-1}).$$

We thus obtain

$$c := \inf_C \frac{\mathrm{codim}(C)}{\mathrm{ord}_C(\mathfrak{a})} \geq n - \sup_{m\geq 0} \frac{\dim(Y_m)}{m+1}. \tag{5.7}$$

For the opposite inequality, note that for every $m \geq 0$ and every irreducible component W of Y_m, the cylinder $\psi_m^{-1}(W)$ is closed and irreducible and does not dominate X since its image in X lies in Y. We thus have

$$(m+1)n - \dim(W) = \operatorname{codim}\big(\psi_m^{-1}(W)\big) \geq c \cdot \operatorname{ord}_{\psi_m^{-1}(W)}(\mathfrak{a}) \geq c(m+1).$$

This gives $\frac{\dim(W)}{m+1} \leq n - c$ and when we let W run over the irreducible components of Y_m and let m run over the nonnegative integers, we get

$$\sup_{m \geq 0} \frac{\dim(Y_m)}{m+1} \leq n - c,$$

which implies that the inequality in (5.7) is an equality. This completes the proof of the corollary. □

Remark 5.54. If $\operatorname{char}(k) = 0$, the we can use the description of the log canonical threshold in Corollary 5.53 to show that if X is a smooth variety and $Y \subset X$ is a smooth subvariety of codimension 1, and if $\mathfrak{a}$ is an ideal on X such that $\mathfrak{a} \cdot \mathcal{O}_Y \neq 0$, then there is an open neighborhood U of Y in X such that

$$\operatorname{lct}(\mathfrak{a}|_U) \geq \operatorname{lct}(\mathfrak{a} \cdot \mathcal{O}_Y)$$

(cf. Corollary 2.86, where this property is proved via results relying on vanishing theorems). Indeed, if this is not the case, then there is a divisorial valuation ord_E with center intersecting Y and such that $A_X(\operatorname{ord}_E) < c \cdot \operatorname{ord}_E(\mathfrak{a})$, where $c = \operatorname{lct}(\mathfrak{a} \cdot \mathcal{O}_Y)$ (this is where we need characteristic 0: this assertion is clear if we use the fact that we can compute $\operatorname{lct}(\mathfrak{a})$ via a log resolution). Note that $m = \operatorname{ord}_E(\mathfrak{a}) \geq 1$ and

$$C = \operatorname{Cyl}(\operatorname{ord}_E) \subseteq \psi_{m-1}^{-1}(Z_{m-1}),$$

where Z is the closed subscheme of X defined by $\mathfrak{a}$. Therefore there is an irreducible component W of Z_{m-1} with $\dim(W) > m(n-c)$ and such that $\varphi_{m,0}(W) \cap Y \neq \emptyset$ (note that $\varphi_{m,0}(W)$ is closed by Remark 5.20).

Recall that since Y is locally cut out in X by one equation, we know that $(Z \cap Y)_{m-1}$ is locally cut out in Z_{m-1} by m equations. Furthermore, we have $W \cap (Z \cap Y)_{m-1} \neq \emptyset$: indeed, it follows from Remark 5.20 that if $x \in \varphi_{m,0}(W) \cap Y$, then $\sigma_{m-1}(x) \in W$, hence $\sigma_{m-1}(x) \in W \cap (Y \cap Z)_{m-1}$. We thus conclude that

$$\dim(Z \cap Y)_{m-1} > m(n-c) - m = m\big((n-1) - c\big),$$

and Corollary 5.53 implies $\operatorname{lct}(\mathfrak{a}|_Y) < c$, a contradiction.

5.4. The Denef-Loeser motivic zeta function

We begin this chapter with a brief review of the Grothendieck ring of algebraic varieties. While the definitions can be given in a more general setup, we keep our usual framework, working over a fixed field[5] k (of arbitrary characteristic), all schemes being separated and of finite type over k. Fix such a scheme S.

Definition 5.55. The *Grothendieck group* $K_0(\mathrm{Var}/S)$ is the quotient of the free abelian group on the set of isomorphism classes of schemes of finite type X/S (denoted $[X/S]$) modulo the following relations:

i) $[X/S] = [X_{\mathrm{red}}/S]$ for every X over S, and

[5] To begin with, we do not assume that the field is algebraically closed.

ii) $[X/S] = [Y/S] + [U/S]$ if Y is a closed subscheme of X and $U = X \smallsetminus Y$ (both with the induced structures of schemes over S).

When S is understood from the context, we simply write $[X]$ instead of $[X/S]$. Also, when $S = \mathrm{Spec}(R)$, we write $K_0(\mathrm{Var}/R)$ for this Grothendieck group.

REMARK 5.56. By definition, $K_0(\mathrm{Var}/S)$ is an abelian group, but it becomes a commutative ring with multiplication induced by

$$[X] \cdot [Y] = [X \times_S Y].$$

Note that the unit element is $[S]$.

For every S, we denote by $\mathbf{L}$ the element $[\mathbf{A}^1 \times S/S] \in K_0(\mathrm{Var}/S)$. We will also consider the localization $K_0(\mathrm{Var}/S)[\mathbf{L}^{-1}]$.

EXAMPLE 5.57. In $K_0(\mathrm{Var}/k)$, we have

$$[\mathbf{P}^n] = 1 + \mathbf{L} + \ldots + \mathbf{L}^n. \tag{5.8}$$

Indeed, we have a closed immersion $\mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^n$, whose complement is $\mathbf{A}^n$. We thus get $[\mathbf{P}^n] = [\mathbf{P}^{n-1}] + \mathbf{L}^n$, and we get (5.8) by induction on n.

Given a morphism $f\colon S \to T$, we have two associated maps between Grothendieck groups. First, we have a group homomorphism

$$f_*\colon K_0(\mathrm{Var}/S) \to K_0(\mathrm{Var}/T), \quad [X/S] \rightsquigarrow [X/T].$$

Second, we have a ring homomorphism

$$f^*\colon K_0(\mathrm{Var}/T) \to K_0(\mathrm{Var}/S), \quad [X/T] \rightsquigarrow [X \times_T S/S].$$

They are related by the *projection formula*

$$f_*\big(f^*(\alpha) \cdot \beta\big) = \alpha \cdot f_*(\beta) \quad \text{for all} \quad \alpha \in K_0(\mathrm{Var}/T), \beta \in K_0(\mathrm{Var}/S).$$

In particular, for every $x \in S$, we have a ring homomorphism $i_x^*\colon K_0(\mathrm{Var}/S) \to K_0(\mathrm{Var}/k)$, where $i_x\colon \mathrm{Spec}\, k \to S$ corresponds to x.

By taking $T = \mathrm{Spec}\, k$, we see that $K_0(\mathrm{Var}/S)$ has a structure of $K_0(\mathrm{Var}/k)$-algebra. Given a morphism $f\colon S \to T$, we see that by tensoring f^* and f_* with $K_0(\mathrm{Var}/k)[\mathbf{L}^{-1}]$, we get an induced ring homomorphism

$$f^*\colon K_0(\mathrm{Var}/S)[\mathbf{L}^{-1}] \to K_0(\mathrm{Var}/T)[\mathbf{L}^{-1}]$$

and a group homomorphism

$$f_*\colon K_0(\mathrm{Var}/T)[\mathbf{L}^{-1}] \to K_0(\mathrm{Var}/S)[\mathbf{L}^{-1}]$$

related by an analogous projection formula.

We will need the following more general variant of the relation ii) in the definition of $K_0(\mathrm{Var}/S)$. Note that if X is a scheme over S and W is a locally closed subset of X, then we may consider $[W/S] \in K_0(\mathrm{Var}/S)$ since W carries a structure of locally closed subscheme in X (hence of scheme over S) and which one we choose does not matter because of property i) in the definition of the Grothendieck group. We will need the following extension of the cut-and-paste relation in the definition of the Grothendieck group to the case of several subsets.

LEMMA 5.58. *If X is a scheme over X and $W_1, \ldots, W_r$ are disjoint locally closed subsets of X with $X = \bigcup_{i=1}^r W_i$, then*

$$[X/S] = \sum_{i=1}^r [W_i/S].$$

PROOF. Arguing by Noetherian induction, we may assume that the result holds for every closed subscheme of X different from X. Let Z be an irreducible component of X. Note that there is an i_0 such that W_{i_0} contains a nonempty open subset U of Z (if i_0 is such that W_{i_0} contains the generic point of Z, then $Z \subseteq \overline{W_{i_0}}$ and $U = W_{i_0} \cap Z$ is an open subset of Z contained in W_{i_0}). After possibly replacing U by a smaller subset, we may assume that U is open in X, not just in W_{i_0} (simply replace U by $U \smallsetminus \bigcup_{Z'} Z'$, where Z' runs over the irreducible components of X different from Z. The defining property of $K_0(\mathrm{Var}/S)$ gives

$$[X/S] = [U/S] + \big[(X \smallsetminus U)/S\big] \quad \text{and} \quad [W_{i_0}/S] = [U/S] + \big[(W_{i_0} \smallsetminus U)/S\big]. \tag{5.9}$$

Applying the induction hypothesis for the locally closed decomposition

$$X \smallsetminus U = (W_{i_0} \smallsetminus U) \sqcup \bigsqcup_{i \neq i_0} W_i$$

gives

$$\big[(X \smallsetminus U)/S\big] = \big[(W_{i_0} \smallsetminus U)/S\big] + \sum_{i \neq i_0} [W_i/S]. \tag{5.10}$$

By combining (5.9) and (5.10), we obtain the equality in the statement of the lemma. □

COROLLARY 5.59. If $f\colon X \to Y$ is a morphism of reduced schemes over S which is piecewise trivial, with fiber F, then

$$[X/S] = [Y/S] \cdot [F_S/S],$$

where $F_S = F \times S$.

PROOF. By hypothesis, we have a decomposition $Y = Y_1 \sqcup \ldots \sqcup Y_r$, with all Y_i locally closed in Y, such that $f^{-1}(Y_i) \simeq Y_i \times F$ for all i (where both sides are viewed as reduced schemes over S). In this case, it follows from the lemma that

$$[X/S] = \sum_{i=1}^{r} [f^{-1}(Y_i)/S] = \sum_{i=1}^{r} (Y_i \times F/S) = \left(\sum_{i=1}^{r} [Y_i/S]\right) \cdot [F_S/S] = [Y/S] \cdot [F_S/S].$$

□

REMARK 5.60. Very little is known about general properties of $K_0(\mathrm{Var}/S)$, but (somewhat surprisingly) there is a good understanding of generators and relations (as an abelian group) in characteristic 0. First: it is elementary to see that if S is a scheme (separated and of finite type) over a field of characteristic 0, then $K_0(\mathrm{Var}/S)$ is generated as an abelian group by elements of the form $[X/S]$, where $X \to S$ is a projective morphism and X is smooth. Indeed, one shows by induction on n that if Y is a scheme over S with $\dim(Y) \leq n$, then $[Y/S]$ lies in the subgroup of $K_0(\mathrm{Var}/S)$ generated by the $[X/S]$ as above. This follows from the following observations:

i) Every Y has a cover by locally closed subsets that are affine and irreducible.
ii) If Y an irreducible closed subset of $\mathbf{A}^n_S$ and $\overline{Y}$ is its closure in $\mathbf{P}^n_S$, then $\dim(\overline{Y} \smallsetminus Y) < \dim(Y)$ and

$$[Y/S] = [\overline{Y}/S] + \big[(\overline{Y} \smallsetminus Y)/S\big].$$

iii) Given a projective morphism $Y \to S$, with Y an integral scheme, by Hironaka's theorem we have a projective morphism $f\colon \widetilde{Y} \to Y$ that gives a resolution of singularities of Y. If U is a nonempty open subset of Y such that $f^{-1}(U) \to U$ is an isomorphism, then

$$[Y/S] = [\widetilde{Y}/S] - \left[f^{-1}(Y \smallsetminus U)/S\right] + [(Y \smallsetminus U)/S]$$

and $\dim(Y \smallsetminus U)$, $\dim\left(f^{-1}(Y \smallsetminus U)\right) < \dim(Y)$.

A deeper result due to Bittner [**Bit04**] says that the relations between the above generators are generated by relations of the form

$$[X/S] - [Z/S] = [\widetilde{X}/S] - [E/S],$$

where $X \to S$ is a projective morphism, with X a smooth variety, Z is a smooth subvariety of X, and $Y \to X$ is the blow-up along Z, with exceptional divisor E. The main ingredient in the proof of this result is the Weak Factorization Theorem [**AKMW02**].

The Grothendieck group of algebraic varieties is a nice abstract construct, but it is quite hard to extract information from it. One way to do this is via *Euler-Poincaré characteristics*: these are ring homomorphisms $K_0(\mathrm{Var}/S) \to A$, where A is a ring (typically easier to understand). We only discuss briefly a few examples.

Example 5.61. If k is a finite field, then we have a ring homomorphism $K_0(\mathrm{Var}/k) \to \mathbf{Z}$ that maps $[X]$ to $|X(k)|$.

Example 5.62. If $k = \mathbf{C}$, then we have the usual topological characteristic

$$K_0(\mathrm{Var}/\mathbf{C}) \to \mathbf{Z}, \quad [X] \rightsquigarrow \chi^{\mathrm{top}}(X^{\mathrm{an}}) = \sum_{i=0}^{2\dim(X)} (-1)^i \dim_{\mathbf{C}} H^i(X^{\mathrm{an}}, \mathbf{C}). \tag{5.11}$$

Recall first that the usual Euler-Poincaré characteristic agrees for complex algebraic varieties with the Euler-Poincaré characteristic with compact supports (see [**Ful93**, p. 141-142]):

$$\chi^{\mathrm{top}}(X) = \chi_c^{\mathrm{top}}(X) := \sum_{i=0}^{2\dim(X)} (-1)^i \dim_{\mathbf{C}} H_c^i(X^{\mathrm{an}}, \mathbf{C}).$$

If Y is a closed subscheme of X and $U = X \smallsetminus Y$, then we have a long exact sequence for the cohomology with compact supports:

$$\ldots \to H_c^i(U^{\mathrm{an}}, \mathbf{C}) \to H_c^i(X^{\mathrm{an}}, \mathbf{C}) \to H_c^i(Y^{\mathrm{an}}, \mathbf{C}) \to H_c^{i+1}(U^{\mathrm{an}}, \mathbf{C}) \to \ldots.$$

This implies that indeed, we get a group homomorphism as in in (5.11). The fact that it is a ring homomorphism follows from the definition and the Künneth theorem.

We next turn to the main topic of this chapter, the motivic zeta function of Denef and Loeser [**DL98**] From now on we assume that the ground field k is algebraically closed, of characteristic 0. Let$X$ be a smooth variety over k, of dimension n. If $C \subseteq X_\infty(k)$ is a locally closed cylinder, we define $\mu_X(C) \in K_0(\mathrm{Var}/X)[\mathbf{L}^{-1}]$ as follows. By assumption, there is $m \in \mathbf{Z}_{\geq 0}$ and a locally closed $T \subseteq X_m$ such that $C = \psi_m^{-1}(T)$. In this case we put

$$\mu_X(C) := [T/X] \cdot \mathbf{L}^{-(m+1)n} \in K_0(\mathrm{Var}/X)[\mathbf{L}^{-1}]$$

Note that by Proposition 5.23, for every $p > 0$, the morphism $\varphi^{-1}_{m+p,m}(T) \to T$ is locally trivial, with fiber $\mathbf{A}^{pn}$, hence Corollary 5.59 implies $[\varphi^{-1}_{m+p,m}(T)/X] = [T/X] \cdot \mathbf{L}^{pn}$ and we see that the definition of $\mu_X(C)$ is independent of the choice of m.

REMARK 5.63. It follows from the definition of μ_X and Lemma 5.58 that if $C = C_1 \sqcup \ldots \sqcup C_r$ and $C, C_1, \ldots, C_r$ are locally closed cylinders in $X_\infty(k)$, then

$$\mu_X(C) = \sum_{i=1}^{r} \mu_X(C_i).$$

DEFINITION 5.64. Let X be a smooth variety and Y a proper closed subscheme of X. The *motivic zeta function* of Y is

$$Z_Y = \int_{X_\infty} \mathbf{L}^{-s\cdot \mathrm{ord}_Y} := \sum_{m\geq 0} \mu_X\big(\mathrm{Cont}^m(Y)\big)\mathbf{L}^{-sm}.$$

We note that this is an element in $R[\![T]\!]$, where $R = K_0(\mathrm{Var}/X)[\mathbf{L}^{-1}]$ and $T = \mathbf{L}^{-s}$ is a variable (the symbol $\mathbf{L}^{-s}$ is due to the analogy and connection with Igusa's p-adic zeta function, but it also makes certain formulas look better). More generally, suppose that W is another proper closed subscheme of X, with $\mathrm{Supp}(W) \subseteq \mathrm{Supp}(Y)$. We then define the motivic zeta function of (Y, W) to be

$$Z_{Y,W} = \int_{X_\infty} \mathbf{L}^{-s\cdot \mathrm{ord}_Y - \mathrm{ord}_W} := \sum_{m_1,m_2\geq 0} \mu_X\big(\mathrm{Cont}^{m_1}(Y) \cap \mathrm{Cont}^{m_2}(W)\big)\mathbf{L}^{-m_1 s - m_2}.$$

Note that since $\mathrm{Supp}(W) \subseteq \mathrm{Supp}(Y)$, if I_W and I_Y are the ideals defining the two subschemes, then there is q such that $I_Y^q \subseteq I_W$. This implies that $\mathrm{ord}_W \leq q \cdot \mathrm{ord}_Y$, hence for every $m_1 \geq 0$, there are only finitely many $m_2 \geq 0$ such that $\mathrm{Cont}^{m_1}(Y) \cap \mathrm{Cont}^{m_2}(W) \neq \emptyset$. Therefore $Z_{Y,W}$ is a well-defined element of $R[\![T]\!]$.

This definition satisfies the following transformation rule with respect to smooth blow-ups (this is, in fact, the reason for introducing the more general $Z_{Y,W}$).

PROPOSITION 5.65. *Let X be a smooth variety over k, and let Y and W be proper closed subschemes of X, defined by the ideals $\mathcal{I}_Y$ and $\mathcal{I}_W$, respectively, with $\mathrm{Supp}(W) \subseteq \mathrm{Supp}(Y)$. If Z is a smooth subvariety of X, with $Z \subseteq \mathrm{Supp}(Y)$, and if $f\colon X' \to X$ is the blow-up along Z, then*

$$Z_{Y,W} = Z_{Y',W'},$$

where Y' is defined by $\mathcal{I}_Y \cdot \mathcal{O}_{X'}$ and W' is defined by $\mathcal{I}_W \cdot \mathcal{O}_{X'}(-K_{X'/X})$.

PROOF. We may and will assume that $\mathrm{codim}_X(Z) = r \geq 2$, since otherwise the assertion is trivial. Note that if E is the exceptional divisor of f, then $K_{X'/X} = (r-1)E$ (see Example 2.19). By hypothesis, we have $Z \subseteq \mathrm{Supp}(Y)$, hence

$$\mathrm{Supp}(W') = f^{-1}\big(\mathrm{Supp}(W)\big) \cup E \subseteq \mathrm{Supp}\big(f^{-1}(Y)\big) = \mathrm{Supp}(Y').$$

Therefore $Z_{Y',W'}$ is well-defined.

Note that since $Z \subseteq \mathrm{Supp}(Y)$, for every $m \geq 0$, we have

$$\mathrm{Cont}^m(Y) \subseteq \bigcup_{0\leq i\leq m} \mathrm{Cont}^i(Z).$$

It follows that for every $m_1, m_2 \geq 0$, we get a disjoint decomposition with only finitely many nonempty terms into locally closed cylinders

$$\mathrm{Cont}^{m_1}(Y) \cap \mathrm{Cont}^{m_2}(W) = \bigsqcup_{a \geq 0} C_{m_1,m_2,a}, \quad \text{where}$$

$$C_{m_1,m_2,a} = \mathrm{Cont}^{m_1}(Y) \cap \mathrm{Cont}^{m_2}(Z) \cap \mathrm{Cont}^{a}(Z).$$

Note that for every m_1, m_2, and a, we have

$$f_\infty^{-1}(C_{m_1,m_2,a}) = \mathrm{Cont}^{m_1}(Y') \cap \mathrm{Cont}^{m_2+(r-1)a}(W') \cap \mathrm{Cont}^{a}(E).$$

Furthermore, it follows from Proposition 5.37 that if $m \gg 0$, then the induced morphism

$$\psi_m^{X'}\big(f_\infty^{-1}(C_{m_1,m_2,a})\big) \to \psi_m^{X}(C_{m_1,m_2,a})$$

is locally trivial, with fiber $\mathbf{A}^{(r-1)a}$, hence Corollary 5.59 implies

$$\mu_{X'}\big(f_\infty^{-1}(C_{m_1,m_2,a})\big) = \mu_X(C_{m_1,m_2,a}) \cdot \mathbf{L}^{(r-1)a}.$$

We thus conclude that

$$Z_{Y,W} = \sum_{m_1,m_2 \geq 0} \mu_X\big(\mathrm{Cont}^{m_1}(Y) \cap \mathrm{Cont}^{m_2}(W)\big)\mathbf{L}^{-m_1 s - m_2}$$

$$= \sum_{m_1,m_2,a \geq 0} \mu_X\big(C_{m_1,m_2,a}\big)\mathbf{L}^{-m_1 s - m_2}$$

$$= \sum_{m_1,m_2,a \geq 0} \mu_{X'}\big(\mathrm{Cont}^{m_1}(Y') \cap \mathrm{Cont}^{m_2+(r-1)a}(W') \cap \mathrm{Cont}^{a}(E)\big)\mathbf{L}^{-m_1 s - m_2 - (r-1)a}$$

$$= \sum_{m_1,q} \mu_{X'}\big(\mathrm{Cont}^{m_1}(Y') \cap \mathrm{Cont}^{q}(W')\big)\mathbf{L}^{-m_1 s - q} = Z_{Y',W'}.$$

$\square$

We can now prove the rationality of the motivic zeta function Z_Y (and, more generally, of the motivic zeta function $Z_{Y,W}$). Note that by (a strong form of) Hironaka's theorem on resolution of singularities, if Y and W are proper closed subschemes of X, defined by the ideals $\mathcal{I}_Y$ and $\mathcal{I}_W$, respectively, with $\mathrm{Supp}(W) \subseteq \mathrm{Supp}(Y)$, there is a log resolution of the pair $(X, \mathcal{I}_Y \cdot \mathcal{I}_W)$ given by a composition f of morphisms

$$X' = X_N \xrightarrow{f_N} X_{N-1} \xrightarrow{f_{N-1}} \ldots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = X,$$

where for each i, with $1 \leq i \leq N$, the morphism f_i is the blow-up along a smooth subvariety of X_{i-1} that lies inside the inverse image of Y. Given such a log resolution, we write

$$f^{-1}(Y) = \sum_{i=1}^{d} a_i E_i,\ f^{-1}(W) = \sum_{i=1}^{d} b_i E_i, \text{ and } K_{X'/X} = \sum_{i=1}^{d} k_i E_i. \tag{5.12}$$

For every subset $J \subseteq \{1, \ldots, d\}$, we put $E_J^\circ = \bigcap_{i \in J} E_i \smallsetminus \bigcup_{i \notin J} E_i$.

THEOREM 5.66. *If X is a smooth n-dimensional variety and Y and W are proper closed subschemes of X, with* $\mathrm{Supp}(W) \subseteq \mathrm{Supp}(Y)$*, then the motivic zeta function $Z_{Y,W}$ is rational, that is, it lies in $R(T)$. In fact, with the above notation for a log resolution, we have*

$$Z_{Y,W} = \mathbf{L}^{-n} \cdot \sum_{J \subseteq \{1,\ldots,N\}} [E_J^\circ/X] \cdot \prod_{i \in J} \frac{\mathbf{L}^{-a_i s-(k_i+b_i+1)}(\mathbf{L}-1)}{1-\mathbf{L}^{-a_i s-(k_i+b_i+1)}}. \tag{5.13}$$

PROOF. Arguing by induction on N, using Proposition 5.65, we easily see that

$$Z_{Y,W} = Z_{Y',W'},$$

where Y' is the subscheme of X' defined by $\mathcal{I}_Y \cdot \mathcal{O}_{X'}$ and W' is the subscheme of X' defined by $\mathcal{I}_Z \cdot \mathcal{O}_{X'}(-K_{X'/X})$. It is then clear that in order to prove formula (5.13), we may assume that $X' = X$ and that

$$Y = \sum_{i=1}^d a_i E_i \quad \text{and} \quad W = \sum_{i=1}^d b_i E_i,$$

are simple normal crossing divisors, with all $a_i > 0$. Note that for every $m_1 \geq 0$, we have a finite decomposition

$$\mathrm{Cont}^{m_1}(Y) = \bigsqcup_{\sum_i a_i\nu_i = m} \mathrm{Cont}^{\nu}(E),$$

where the union is over those $\nu = (\nu_1, \ldots, \nu_d) \in \mathbf{Z}^d_{\geq 0}$ such that $\sum_i a_i\nu_i = m_1$, and where $\mathrm{Cont}^\nu(E) = \bigcap_i \mathrm{Cont}^{\nu_i}(E_i)$. Of course, we have $\mathrm{Cont}^\nu(E) \subseteq \mathrm{Cont}^{b(\nu)}(W)$, where $b(\nu) = \sum_i b_i\nu_i$. Therefore it follows from Remark 5.63 that for every $m_1, m_2 \geq 0$, we have

$$\mu_X\big(\mathrm{Cont}^{m_1} \cap \mathrm{Cont}^{m_2}(W)\big) = \sum_\nu \mu_X(\mathrm{Cont}^\nu(E), \tag{5.14}$$

where the (finite) sum is over those $\nu = (\nu_i)$ with $\sum_i a_i\nu_i = m_1$ and $\sum_i b_i\nu_i = m_2$.

We next compute $\mu_X\big(\mathrm{Cont}^\nu(E)\big)$. Let $\mathrm{supp}(\nu) := \{i \mid \nu_i \geq 1\} \subseteq \{1, \ldots, d\}$ and $|\nu| := \sum_i \nu_i$. Note that if $\mathrm{supp}(\nu) = J$, then $\psi_0\big(\mathrm{Cont}^\nu(E)\big) \subseteq E_J^\circ$. It is clear that if $m \geq \max_i\{\nu_i\}$, then $\mathrm{Cont}^\nu(E) = \psi_m^{-1}\big(\psi_m(\mathrm{Cont}^\nu(E))\big)$. We claim that the induced map $\psi_m\big(\mathrm{Cont}^\nu(E)\big) \to E_J^\circ$ is locally trivial, with fiber $\mathbf{A}^{mn-|\nu|} \times \big(\mathbf{A}^1 \smallsetminus \{0\}\big)^{|J|}$. The assertion is local on X, hence we may assume that we have an algebraic system of coordinates $x_1, \ldots, x_n$ on X such that the divisors E_i with $i \in J$ are the ones defined by (x_i), with $1 \leq i \leq |J|$. Note that by Remark 5.25 in this case we have an isomorphism $X_m \simeq X \times \big(tk[t]/(t^{m+1})\big)^{\oplus n}$ of schemes over X. Via this isomorphism, $\mathrm{Cont}^\nu(E)$ corresponds to

$$\begin{aligned}
&E_J^\circ \times \big\{u = (u_1, \ldots, u_n) \in tk[t]/(t^{m+1}) \mid \mathrm{ord}(u_i) = \nu_i \text{ for } 1 \leq i \leq |J|\big\} \\
&\simeq E_J^\circ \times \big(tk[t]/(t^{m+1})\big)^{n-|J|} \times \prod_{1 \leq i \leq |J|} \big((\mathbf{A}^1 \smallsetminus \{0\}) \times \mathbf{A}^{m-\nu_i}\big).
\end{aligned}$$

This easily gives our claim.

We thus get using (5.14) that

$$Z_{Y,W} = \sum_{J \subseteq \{1,\ldots,N\}} \sum_{\mathrm{supp}(\nu)=J} \mu_X\big(\mathrm{Cont}^\nu(E)\big)\mathbf{L}^{-s\sum_i a_i\nu_i - \sum_i b_i\nu_i}$$

$$= \sum_{J\subseteq\{1,\ldots,N\}} \sum_{\mathrm{supp}(\nu)=J} [E_J^\circ/X] \cdot (\mathbf{L}-1)^{|J|} \mathbf{L}^{-n-s\sum_i a_i\nu_i - \sum_i (b_i+1)\nu_i}$$

$$= \mathbf{L}^{-n} \cdot \sum_{J\subseteq\{1,\ldots,N\}} [E_J^\circ/X] \cdot \prod_{i\in J} (\mathbf{L}-1) \cdot \sum_{\nu_i\geq 1} \mathbf{L}^{-sa_i\nu_i - \nu_i(b_i+1)}$$

$$= \mathbf{L}^{-n} \cdot \sum_{J\subseteq\{1,\ldots,N\}} [E_J^\circ/X] \cdot \prod_{i\in J} \frac{\mathbf{L}^{-a_is-(b_i+1)}(\mathbf{L}-1)}{1-\mathbf{L}^{-a_is-(b_i+1)}}.$$

This completes the proof of the theorem. □

Remark 5.67. A very interesting invariant that comes out of the motivic zeta function Z_Y is the *motivic Milnor fiber*[6] of Denef and Loeser [**DL98**]. This is obtained by expanding the rational function Z_Y in terms of T^{-1} (instead of $T = \mathbf{L}^{-s}$) and taking the constant term in R. With the notation in Theorem 5.66, this becomes

$$\mathbf{L}^{-n} \cdot \sum_{J\subseteq\{1,\ldots,d\}} (1-\mathbf{L})^{|J|} \cdot [E_J^\circ/X].$$

[6]Actually, the motivic Milnor fiber of [**DL98**] is associated to a hypersurface Y defined by a regular function $f \in \mathcal{O}_X(X)$. In that case, the motivic zeta function that one considers is a refined version of the one that we discussed, which takes values in a Grothendieck ring of varieties endowed with the action of a group of roots of unity.

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