Homework Set 8

Solutions are due Thursday, March 22.

Problem 1. Let $X = \mathrm{MaxProj}(S)$. For every positive integer d, let

$$S^{(d)} = \bigoplus_{j\geq 0} S_{jd}.$$

i) Show that the map taking $\mathfrak{q}$ to $\mathfrak{q} \cap S^{(d)}$ gives an isomorphism $f\colon X \to Y = \mathrm{MaxProj}(S^{(d)})$.
ii) Show that $f^*\big(\mathcal{O}_Y(1)\big) \simeq \mathcal{O}_X(d)$.

Problem 2. Let X be a closed subvariety of $\mathbb{P}^n$ and S_X the homogeneous coordinate ring of X.

i) Show that for $d \gg 0$, we have the following property: for every $j \geq 1$, the canonical multiplication map

$$\mathrm{Sym}_k^j \Gamma\big(X, \mathcal{O}_X(d)\big) \to \Gamma\big(X, \mathcal{O}_X(jd)\big)$$

is surjective.
ii) Show that for d as in i), we have

$$X \simeq \mathrm{MaxProj}\left(\bigoplus_{j\geq 0} \Gamma(X, \mathcal{O}_X(jd))\right).$$

Problem 3. Let $X = \mathbb{P}^n$ and $S = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$, with irrelevant ideal $\mathfrak{m} = (x_0, \ldots, x_n)$. A homogeneous ideal J in S is *saturated* if for every element $u \in S$ such that $u \cdot \mathfrak{m} \subseteq J$, we have $u \in J$.

i) Show that for every homogeneous ideal J in S, there is a unique saturated ideal J^{sat} in S such that $J \subseteq J^{\mathrm{sat}}$ and for some non-negative integer r, we have $\mathfrak{m}^r \cdot J^{\mathrm{sat}} \subseteq J$.
ii) Show that if J_1 and J_2 are homogeneous ideals in S, then $\widetilde{J_1} = \widetilde{J_2}$ if and only if $J_1^{\mathrm{sat}} = J_2^{\mathrm{sat}}$.
iii) In particular, given any coherent ideal sheaf $\mathcal{J}$ on $\mathbb{P}^n$, there is a unique saturated ideal J in S such that $\widetilde{J} = \mathcal{J}$. Show that this is the unique largest homogeneous ideal I such that $\widetilde{I} = \mathcal{J}$, and it is equal to

$$\bigoplus_{m\geq 0} \Gamma\big(\mathbb{P}^n, \mathcal{J}(m)\big) \subseteq S = \bigoplus_{m\geq 0} \Gamma\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)\big).$$

iv) Show that if J is a homogeneous, radical ideal different from $\mathfrak{m}$, then J is saturated. Show also that if $\mathcal{J}$ is a radical coherent ideal on $\mathbb{P}^n$, then the saturated ideal J such that $\widetilde{J} = \mathcal{J}$ is radical.

The following problem deals with two related invariants of algebraic varieties, the Grothendieck groups of vector bundles and of coherent sheaves.

Given an algebraic variety X, the *Grothendieck group* $K^0(X)$ *of vector bundles* on X is the quotient of the free Abelian group on the set of isomorphism classes of locally free sheaves on X, by the subgroup generated by relations of the form $[\mathcal{E}] - [\mathcal{E}'] - [\mathcal{E}'']$, where

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is an exact sequence of locally free sheaves on X. We denote by $[\mathcal{E}]$ the image in $K^0(X)$ of the isomorphism class of $\mathcal{E}$.

Similarly, the *Grothendieck group* $K_0(X)$ *of coherent sheaves* on a scheme X is the quotient of the free Abelian group on isomorphism classes of coherent sheaves on X by the subgroup generated by relations of the form $[\mathcal{F}] - [\mathcal{F}'] - [\mathcal{F}'']$, where

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence of coherent sheaves on X. We denote by $[\mathcal{F}]$ the image in $K^0(X)$ of the isomorphism class of $\mathcal{F}$.

Problem 4.

i) Show that the tensor product of locally free sheaves makes $K^0(X)$ a commutative ring, with identity.
ii) Show that the pull-back of vector bundles via a morphism makes the map taking X to $K^0(X)$ a contravariant functor from the category of algebraic varieties over k to the category of commutative rings.
iii) Show that $K_0(X)$ is a module over $K^0(X)$ via the tensor product with locally free sheaves.
iv) Show that we have a morphism of $K^0(X)$-modules $K^0(X) \to K_0(X)$ that maps $[\mathcal{E}]$ to $[\mathcal{E}]$.
v) Show that we have isomorphisms $K^0(X) = \mathbb{Z} = K_0(X)$ if X is a point or if $X = \mathbf{A}^1$.