Homework Set 9

Solutions are due Wednesday, November 29.

Problem 1. Show that if $f\colon X \to Y$ is a morphism of algebraic varieties, then for every non-negative integer m, the set X_m consisting of those $x \in X$ such that the fiber $f^{-1}\big(f(x)\big)$ has an irreducible component passing through x, of dimension $\geq m$, is closed.

Problem 2. Deduce that if f is a proper morphism of algebraic varieties, then we have the following semicontinuity result about the dimension of fibers: for every non-negative integer m, the set
$$\{y \in Y \mid \dim\big(f^{-1}(y)\big) \geq m\}$$
is closed in Y.

Problem 3. Let $f\colon X \to Y$ be a morphism of algebraic varieties. Suppose that Y is irreducible and that all fibers of f are irreducible, of the same dimension d (in particular, f is surjective). Show that if either one of the following two conditions holds:

a) X is pure-dimensional, or
b) f is closed,

then X is irreducible.

Remark. The above problem gives a very useful irreducibility criterion for projective varieties.