Homework Set 8

Solutions are due Thursday, March 15.

The first problem we could have discussed a while ago. It is included here because it is useful for the second problem, but the result is interesting in itself. It says that if $f\colon X \to Y$ is a finite morphism, then the functor f_* between the corresponding categories of coherent sheaves has a right adjoint (recall that it also has a left adjoint, namely f^*).

Problem 1. Let $f\colon X \to Y$ be a finite morphism.

i) Show that for every coherent (quasi-coherent) sheaf $\mathcal{G}$ on Y, there is a coherent (respectively, quasi-coherent) sheaf $f^!(\mathcal{G})$ on X such that we have an isomorphism

$$f_*\big(f^!(\mathcal{G})\big) \simeq \mathrm{Hom}_{\mathcal{O}_Y}\big(f_*(\mathcal{O}_X), \mathcal{G}\big).$$

Moreover, show that in this way we get a functor $f^!$ from $\mathcal{C}oh(Y)$ to $\mathcal{C}oh(X)$ (and similarly, from $\mathcal{Q}coh(Y)$ to $\mathcal{Q}coh(X)$).

ii) Show that for every quasi-coherent sheaf $\mathcal{F}$ on X and every quasi-coherent sheaf $\mathcal{G}$ on Y, we have a functorial isomorphism

$$f_*\mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{F}, f^!(\mathcal{G})\big) \simeq \mathcal{H}om_{\mathcal{O}_Y}\big(f_*(\mathcal{F}), \mathcal{G}\big).$$

iii) Deduce that the functor $f^!\colon \mathcal{C}oh(Y) \to \mathcal{C}oh(X)$ is the right adjoint of the functor $f_*\colon \mathcal{C}oh(X) \to \mathcal{C}oh(Y)$ (and similarly for the functors between the categories of quasi-coherent sheaves).

Problem 2. Prove the following result of Chevalley: if $f\colon X \to Y$ is a finite surjective morphism of algebraic varieties, and X is affine, then Y is affine. Hint: use the following steps:

i) Reduce to the case when both X and Y are irreducible.
ii) Show that if X and Y are irreducible, then there is a coherent sheaf $\mathcal{F}$ on X, and a morphism of sheaves $\phi\colon \mathcal{O}_Y^{\oplus r} \to f_*(\mathcal{F})$ for some $r \geq 1$, such that ϕ is an isomorphism over an open subset of Y .
iii) Use ii) and the functor $f^!$ to show that if X and Y are irreducible, then given a coherent sheaf $\mathcal{N}$ on Y, there is a coherent sheaf $\mathcal{M}$ on X and a morphism $f_*(\mathcal{M}) \to \mathcal{N}^{\oplus r}$ that is an isomorphism over an open subset of Y.
iv) Deduce Chevalley's result by Noetherian induction.

Problem 3. Suppose that we have a pointwise finite filtration $F_\bullet K^\bullet$ on a complex $K^\bullet$, leading to the spectral sequence

$$E_2^{p,q} \Rightarrow_p \mathcal{H}^{p+q}(K^\bullet).$$

Show that if this is a first-quadrant spectral sequence (that is, $E_2^{p,q} = 0$ unless $p \geq 0$ and $q \geq 0$), then we have an associated *five-term exact sequence*:

$$0 \to E_2^{1,0} \to \mathcal{H}^1(K^\bullet) \to E_2^{0,1} \to E_2^{2,0} \to \mathcal{H}^2(K^\bullet).$$