# Homework Set 8

Solutions are due Monday, November 20.

**Problem 1**. (The Segre embedding). Consider two projective spaces $\mathbf{P}^m$ and $\mathbf{P}^n$. Let $N = (m+1)(n+1)-1$, and let us denote the coordinates on $\mathbf{A}^{N+1}$ by $z_{i,j}$, with $0 \leq i \leq m$ and $0 \leq j \leq n$.

1) Show that the map $\mathbf{A}^{m+1} \times \mathbf{A}^{n+1} \to \mathbf{A}^{N+1}$ given by
$$((x_i)_i, (y_j)_j) \to (x_i y_j)_{i,j}$$
induces a morphism
$$\phi_{m,n} \colon \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N.$$
2) Consider the ring homomorphism
$$f_{m,n} \colon k[z_{i,j} \mid 0 \leq i \leq m, 0 \leq j \leq n] \to k[x_1, \ldots, x_m, y_1, \ldots, y_n], \quad f_{m,n}(z_{i,j}) = x_i y_j.$$
Show that $\ker(f_{m,n})$ is a homogeneous prime ideal that defines in $\mathbf{P}^N$ the image of $\phi_{m,n}$ (in particular, this image is closed).
3) Show that $\phi_{m,n}$ is a closed immersion.
4) Deduce that if $X$ and $Y$ are (quasi)projective varieties, then $X \times Y$ is a (quasi)projective variety.

**Problem 2**. Let $n$ and $d$ be positive integers, and let $M_0, \ldots, M_N$ be all monomials in $k[x_0, \ldots, x_n]$ of degree $d$ (hence $N = \binom{n+d}{d} - 1$).

1) Show that there is a morphism $\nu_{n,d} \colon \mathbf{P}^n \to \mathbf{P}^N$ that takes the point $[a_0, \ldots, a_n]$ to the point $\big[M_0(a), \ldots, M_N(a)\big]$.
2) Consider the ring homomorphism $f_d \colon k[z_0, \ldots, z_N] \to k[x_0, \ldots, x_n]$ defined by $f_d(z_i) = M_i$. Show that $\ker(f_d)$ is a homogeneous prime ideal that defines in $\mathbf{P}^N$ the image of $\nu_{n,d}$ (in particular, this image is closed).
3) Show that $\nu_{n,d}$ is a closed immersion.
4) Show that if $Z$ is a hypersurface of degree $d$ in $\mathbb{P}^n$ (this means that $I(Z) = (F)$, where $F$ is a homogeneous polyomial of degree $d$), then there is a hyperplane $H$ in $\mathbf{P}^N$ such that for every projective variety $X \subseteq \mathbf{P}^n$, the morphism $\nu_{n,d}$ induces an isomorphism between $X \cap Z$ and $\nu_{n,d}(X) \cap H$. This shows that the Veronese embedding allows to reduce the intersection with a hypersurface to the intersection with a hyperplane.
5) The *rational normal curve* in $\mathbf{P}^n$ is the image of the Veronese embedding $\nu_{1,d} \colon \mathbb{P}^1 \to \mathbf{P}^d$, mapping $[a, b]$ to $[a^d, a^{d-1}b, \ldots, b^d]$. Show that the rational normal curve is the zero-locus of the $2 \times 2$-minors of the matrix
$$\begin{pmatrix} z_0 & z_1 & \ldots & z_{d-1} \\ z_1 & z_2 & \ldots & z_d \end{pmatrix}.$$