Homework Set 7

Solutions are due Monday, November 13.

Problem 1.

i) Show that if $X_1, \ldots, X_n$ are algebraic varieties, then on the disjoint union $X = \bigsqcup_{i=1}^n X_i$ there is a unique structure of algebraic variety such that each inclusion map $X_i \hookrightarrow X$ is an open immersion .
ii) Show that every variety X is a disjoint union of connected open subvarieties; each of these is a union of irreducible components of X.
iii) Show that if X is an affine variety and $R = \mathcal{O}(X)$, then X is disconnected if and only if there is an isomorphism $R \simeq R_1 \times R_2$ for suitable nonzero k-algebras R_1 and R_2.

Problem 2. A *hypersurface* in $\mathbf{P}^n$ is a closed subset defined by

$$\{[x_0, \ldots, x_n] \in \mathbf{P}^n \mid F(x_0, \ldots, x_n) = 0\},$$

for some homogeneous polynomial F, of positive degree. Given a closed subset $X \subseteq \mathbf{P}^n$, show that the following are equivalent:

i) X is a hypersurface.
ii) The ideal $I(X)$ is a principal ideal.
iii) All irreducible component of X have codimension 1 in $\mathbf{P}^n$.

Note that if X is any irreducible variety and U is a nonempty open subset of X, then the map taking $Z \subseteq U$ to $\overline{Z}$ and the map taking $W \subseteq X$ to $W \cap U$ give inverse bijections (preserving the irreducible decompositions) between the nonempty closed subsets of U and the nonempty closed subsets of X that have no irreducible component contained in the $X \smallsetminus U$. This applies, in particular, to the open immersion

$$\mathbf{A}^n \hookrightarrow \mathbf{P}^n, \quad (x_1, \ldots, x_n) \to [1, x_1, \ldots, x_n].$$

The next exercise describes this correspondence at the level of ideals.

Problem 3. Let $S = k[x_0, \ldots, x_n]$ and $R = k[x_1, \ldots, x_n]$. For an ideal J in R, we put

$$J^{\mathrm{hom}} := \left(f^{\mathrm{hom}} \mid 0 \neq f \in J\right),$$

where $f^{\mathrm{hom}} = x_0^{\deg(f)} \cdot f(x_1/x_0, \ldots, x_n/x_0) \in S$. On the other hand, if $\mathfrak{a}$ is a homogeneous ideal in S, then we put $\overline{\mathfrak{a}} := \{h(1, x_1, \ldots, x_n) \mid h \in \mathfrak{a}\} \subseteq R$.

An ideal $\mathfrak{a}$ in S is called x_0-saturated if $(\mathfrak{a}\colon x_0) = \mathfrak{a}$ (recall that $(\mathfrak{a}\colon x_0) := \{u \in S \mid x_0 u \in \mathfrak{a}\}$).

i) Show that the above maps give inverse bijections between the ideals in R and the x_0-saturated homogeneous ideals in S.

ii) Show that we get induced bijections between the radical ideals in R and the homogeneous x_0-saturated radical ideals in S. Moreover, a homogeneous radical ideal $\mathfrak{a}$ is x_0-saturated if and only if either no irreducible component of $V(\mathfrak{a})$ is contained in the hyperplane $(x_0 = 0)$, or if $\mathfrak{a} = S$.

iii) The above correspondence induces a bijection between the prime ideals in R and the prime ideals in S that do not contain x_0.

iv) Consider the open immersion

$$\mathbf{A}^n \hookrightarrow \mathbf{P}^n, \ \ (u_1, \dots, u_n) \to (1\colon u_1 \colon \ \dots \colon u_n),$$

which allows us to identify $\mathbb{A}^n$ with the complement of the hyperplane $(x_0 = 0)$ in $\mathbf{P}^n$. Show that for every ideal J in R we have $\overline{V_{\mathbf{A}^n}(J)} = V_{\mathbb{P}^n}(J^{\text{hom}})$.

v) Show that for every homogeneous ideal $\mathfrak{a}$ in S, we have $V_{\mathbf{P}^n}(\mathfrak{a}) \cap \mathbf{A}^n = V_{\mathbf{A}^n}(\overline{\mathfrak{a}})$.

Problem 4. Recall that $GL_{n+1}(k)$ denotes the set of invertible $(n+1)\times(n+1)$ matrices with entries in k. Let $PGL_{n+1}(k)$ denote the quotient $GL_{n+1}(k)/k^*$, where k^* acts on $GL_{n+1}(k)$ by

$$\lambda \cdot (a_{i,j})_{i,j} = (\lambda a_{i,j})_{i,j}.$$

i) Show that $PGL_{n+1}(k)$ has a natural structure of linear algebraic group, and that it is irreducible.

ii) Prove that $PGL_{n+1}(k)$ acts algebraically on $\mathbf{P}^n$.

Definition 0.1. Two subsets of $\mathbf{P}^n$ are *projectively equivalent* if they differ by an automorphism in $PGL_{n+1}(k)$ (we will see later that these are, indeed, all automorphisms of $\mathbf{P}^n$).

Definition 0.2. A *linear subspace* of $\mathbf{P}^n$ is a closed subvariety of $\mathbf{P}^n$ defined by an ideal generated by homogeneous polynomials of degree one. A *hyperplane* is a linear subspace of codimension one.

Problem 5. Consider the projective space $\mathbf{P}^n$.

i) Show that a closed subset Y of $\mathbf{P}^n$ is a linear subspace if and only if the affine cone $C(Y) \subseteq \mathbf{A}^{n+1}$ is a linear subspace.

ii) Show that if L is a linear subspace in $\mathbf{P}^n$ of dimension r, then there is an isomorphism $L \simeq \mathbf{P}^r$.

iii) Show that the hyperplanes in $\mathbf{P}^n$ are in bijection with the points of "another" projective space $\mathbf{P}^n$, usually denoted by $(\mathbf{P}^n)^*$.

iv) Show that given two sets of points in $\mathbf{P}^n$

$$\Gamma = \{P_0, \dots, P_{n+1}\} \text{ and } \Gamma' = \{Q_0, \dots, Q_{n+1}\},$$

such that no $(n+1)$ points in the same set lie in a hyperplane, there is a unique $A \in PGL_n(k)$ such that $A \cdot P_i = Q_i$ for every i.

Problem 6. Let $X \subseteq \mathbf{P}^n$ be an irreducible closed subset of codimension r. Show that if $H \subseteq \mathbf{P}^n$ is a hypersurface such that X is not contained in H, then every irreducible component of $X \cap H$ has codimension $r+1$ in $\mathbf{P}^n$.

Problem 7. Let $X \subseteq \mathbf{P}^n$ be a closed subset of dimension r. Show that there is a linear space $L \subseteq \mathbf{P}^n$ of dimension $(n - r - 1)$ such that $L \cap X = \emptyset$.