Homework Set 4

Solutions are due Thursday, February 8.

Problem 1. Show that if X is a normal irreducible variety and U is an affine open subset of X, then every irreducible component of $X \smallsetminus U$ has codimension 1 in X.

Problem 2. Show that if L/K is a finite field extension, then L/K is separable if and only if $\Omega_{L/K} = 0$.

Problem 3. Let $f\colon X \to Y$ be a morphism of algebraic varieties.

i) Show that for every coherent sheaf $\mathcal{F}$ on Y and for every point $x \in X$, we have a canonical isomorphism of k-vector spaces
$$f^*(\mathcal{F})_{(x)} \simeq \mathcal{F}_{(f(x))}.$$
ii) Deduce that the canonical morphism $f^*(\Omega_Y) \to \Omega_X$ induces for every point $x \in X$ a morphism of k-vector spaces $(\Omega_Y)_{f(x)} \to (\Omega_X)_{(x)}$; show that this gets identified to the dual of the canonical morphism $df_x\colon T_xX \to T_{f(x)}Y$.

Problem 4. Compute the normalization morphism for each of the following varieties:

i) X is the curve in $\mathbf{A}^2$ given by $x^2 - y^3 = 0$.
ii) Y is the surface in $\mathbf{A}^3$ given by $x^2 - y^2z = 0$.

Problem 5. Let X be an irreducible variety and $\pi\colon X^{\rm norm} \to X$ the normalization morphism. Show that for every dominant morphism $f\colon Z \to X$, with Z irreducible and normal, there is a unique morphism $g\colon Z \to X^{\rm norm}$ such that $\pi \circ g = f$.