Homework Set 4

Solutions are due Wednesday, October 18.

Problem 1. Suppose that S is the image of a morphism of semigroups $\phi\colon \mathbb{N}^r \to \mathbb{Z}^m$ (this is how semigroups are usually described). Show that the kernel of the induced surjective k-algebra homomorphism

$$k[x_1, \ldots, x_r] \simeq k[\mathbb{N}^r] \to k[S]$$

is the ideal generated by

$$\{x^a - x^b \mid a, b \in \mathbb{N}^r, \phi(a) = \phi(b)\}.$$

The next two problems describe the torus-invariant subvarieties of $\mathrm{TV}(S)$ and the orbits of the torus action. We begin by defining the corresponding concept at the level of the semigroup.

Definition A face F of a semigroup S is a subsemigroup such that whenever u_1, $u_2 \in S$ satisfy $u_1 + u_2 \in F$, we have $u_1 \in F$ and $u_2 \in F$.

Note that if F is a face of S, then $S \smallsetminus F$ is a subsemigroup of S. Moreover, if S is generated by $u_1, \ldots, u_n$. then a face F of S is generated by those u_i that lie in F. In particular, if S is an integral, finitely generated semigroup, then S has only finitely many faces, and each of these is an integral, finitely generated semigroup.

Problem 2. Let $X = \mathrm{TV}(S)$ be an affine toric variety, with torus $T \subset X$. A subset Y of X is *torus-invariant* if $t \cdot Y \subseteq Y$ for every $t \in T$.

- i) Show that a closed subset Y of X is torus-invariant if and only if each irreducible component of Y is torus-invariant.
- ii) Show that the torus-invariant irreducible closed subsets of X are precisely the closed subsets defined by ideals of the form

  $$\bigoplus_{u \in S \smallsetminus F} k\chi^u,$$

  where F is a face of S.
- iii) Show that if Y is the closed subset defined by the ideal in ii), then we have $\mathcal{O}(Y) \simeq k[F]$, hence Y has a natural structure of affine toric variety.

Problem 3. Let $X = \mathrm{TV}(S)$ be an affine toric variety, with torus $T_X \subseteq X$.

- i) Show that if $M \hookrightarrow M'$ is an injective homomorphism of finitely generated, free Abelian groups, then the induced morphism of tori $\mathrm{TV}(M') \to \mathrm{TV}(M)$ is surjective.

ii) Show that if F is a face of S with corresponding closed invariant subset Y, then the inclusion of semigroups $F \subseteq S$ induces a morphism of toric varieties $f_Y \colon X \to Y$, which is a retract of the inclusion $Y \hookrightarrow X$. Show that if we denote by O_F the torus in Y, then O_F is an orbit for the action of T_X on X.

iii) Show that the map $F \to O_F$ gives a bijection between the faces of S and the orbits for the T_X-action on X.

Problem 4. We have seen in Problem 2 that if X is an affine toric variety and Y is a torus-invariant closed, irreducible subset, then Y has a natural structure of toric variety. Show that if X is normal, then every such Y is normal.