Homework Set 3

Solutions are due Monday, October 9.

definition A *linear algebraic group* over k is an affine variety G over k that is also a group, and such that the multiplication $\mu\colon G \times G \to G$, $\mu(g,h) = gh$, and the inverse map $\iota\colon G \to G$, $\iota(g) = g^{-1}$ are morphisms of algebraic varieties. If G_1 and G_2 are linear algebraic groups, a *morphism of algebraic groups* is a morphism of affine varieties $f\colon G_1 \to G_2$ that is also a group homomorphism.

Linear algebraic groups over k form a category. In particular, we have a notion of isomorphism between linear algebraic groups: this is an isomorphism of affine algebraic varieties that is also a group isomorphism.

Problem 1.

i) Show that $(k,+)$ and $(k^*,\cdot)$ are linear algebraic groups.
ii) Show that the set $\mathrm{GL}_n(k)$ of $n \times n$ invertible matrices with coefficients in k has a structure of linear algebraic group.
iii) Show that the set $\mathrm{SL}_n(k)$ of $n \times n$ matrices with coefficients in k and with determinant 1 has a structure of linear algebraic group.
iv) Show that if G and H are linear algebraic groups, then the product $G \times H$ has an induced structure of linear algebraic group. In particular, the (algebraic) *torus* $(k^*)^n$ is a linear algebraic group with respect to component-wise multiplication.

Definition. Let G be a linear algebraic group and X a quasi-affine variety. An *algebraic group action* of G on X is a (say, left) action of G on X such that the map $G \times X \to X$ giving the action is a morphism of algebraic varieties.

Problem 2. Show that $\mathrm{GL}_n(k)$ has an algebraic action on $\mathbf{A}^n$.

Problem 3. Let G be a linear algebraic group acting algebraically on an affine variety X. Show that in this case G has an induced linear action on $\mathcal{O}(X)$ given by

$$(g \cdot \phi)(u) = \phi\big(g^{-1}(u)\big).$$

While $\mathcal{O}(X)$ has in general infinite dimension over k, show that the action of G on $\mathcal{O}(X)$ has the following finiteness property: every element $f \in \mathcal{O}(X)$ lies in some finite-dimensional vector subspace V of $\mathcal{O}(X)$ that is preserved by the G-action (Hint: consider the image of f by the corresponding k-algebra homomorphism $\mathcal{O}(X) \to \mathcal{O}(G) \otimes_k \mathcal{O}(X)$).

Problem 4. Let G and X be as in the previous problem. Consider a system of k-algebra generators $f_1, \ldots, f_m$ of $\mathcal{O}(X)$, and apply the previous problem to each of these elements to show that there is a morphism of algebraic groups $G \to \mathrm{GL}_N(k)$, and an isomorphism of X with a closed subset of $\mathbf{A}^N$, such that the action of G on X is induced by the standard action of $\mathrm{GL}_N(k)$ on $\mathbf{A}^N$. Use a similar argument to show that every linear algebraic group is isomorphic to a closed subgroup of some $\mathrm{GL}_N(k)$.

Problem 5. Show that the linear algebraic group $\mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$ has an algebraic action on the space $M_{m,n}(k)$ (identified to $\mathbf{A}^{mn}$), induced by left and right matrix multiplication. What are the orbits of this action ? Note that the orbits are locally closed subsets of $M_{m,n}(k)$ (as we will see later, this is a general fact about orbits of algebraic group actions).