Homework Set 2

Solutions are due Tuesday, January 23.

Given two vector bundles $\mathcal{E}$ and $\mathcal{F}$ on a variety X, a *morphism of vector bundles* $\mathcal{E} \to \mathcal{F}$ is a morphism of sheaves such that the map

$$x \to \operatorname{rank}\big(E_{(x)} \to F_{(x)}\big)$$

is constant on each connected component of X.

Problem 1. Show that if $\phi\colon \mathcal{E} \to \mathcal{F}$ is a morphism of vector bundles, then $\operatorname{coker}(\phi)$, $\operatorname{Im}(\phi)$, and $\ker(\phi)$ are vector bundles.

Problem 2. Show that the composition of two morphisms of vector bundles might not be a morphism of vector bundles.

Problem 3. Show that for every coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ on X, there is a canonical morphism of $\mathcal{O}_X$-modules

$$\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}).$$

Show that this is an isomorphism if either $\mathcal{E}$ of $\mathcal{F}$ is locally free.

Problem 4. Let $\mathcal{E}$ be a coherent sheaf on X.

i) Show that there is a canonical morphism of $\mathcal{O}_X$-modules

$$\mathcal{E} \to (\mathcal{E}^\vee)^\vee,$$

which is an isomorphism if $\mathcal{E}$ is locally free.

ii) Show that there is a canonical morphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$$

which is an isomorphism if $\mathcal{E}$ is locally free, of rank 1.

Given a vector bundle $\mathcal{E}$, a *subbundle* of $\mathcal{E}$ is a subsheaf $\mathcal{F}$ of $\mathcal{E}$, which is a vector bundle, and such that the inclusion map $\mathcal{F} \hookrightarrow \mathcal{E}$ is a morphism of vector bundles.

Problem 5. Let $\mathcal{E}$ be a vector bundle on the algebraic variety X. Show that a subsheaf $\mathcal{F}$ of $\mathcal{E}$ is a subbundle if and only if it is a vector bundle and for all $x \in X$, the induced map $\mathcal{F}_{(x)} \to \mathcal{E}_{(x)}$ is injective.