Homework Set 2

Solutions are due Monday, October 2.

Problem 1. Show that the image of a morphism of affine algebraic varieties $f\colon X \to Y$ might not be locally closed in Y (you can use, for example, the morphism $f\colon \mathbf{A}^2 \to \mathbf{A}^2$ given by $f(x, y) = (x, xy)$).

Problem 2. Suppose that $\mathrm{char}(k) = p > 0$, and consider the map $f\colon \mathbf{A}^n \to \mathbf{A}^n$ given by $f(a_1, \ldots, a_n) = (a_1^p, \ldots, a_n^p)$. Show that f is a morphism of affine algebraic varieties, and that it is a homeomorphism, but it is not an isomorphism. This morphism is the *k-linear Frobenius morphism* on the affine space.

Problem 3. Let $f\colon X \to Y$ be a morphism of quasi-affine varieties, and let $Z \subseteq X$ be a closed irreducible subset. Recall that by a problem on the first homework set, we know that $W := \overline{f(Z)}$ is irreducible. Show that we have an induced morphism of k-algebras

$$g\colon \mathcal{O}_{Y,W} \to \mathcal{O}_{X,Z}$$

and that g is a local homomorphism of local rings (that is, it maps the maximal ideal of $\mathcal{O}_{Y,W}$ inside the maximal ideal of $\mathcal{O}_{X,Z}$). If X and Y are affine varieties, and

$$\mathfrak{p} = I_X(Z) \quad \text{and} \quad \mathfrak{q} = I_Y(W) = (f^\#)^{-1}(\mathfrak{p}),$$

then via the isomorphisms

$$\mathcal{O}_{Y,W} \simeq \mathcal{O}(Y)_\mathfrak{q} \quad \text{and} \quad \mathcal{O}_{X,Z} \simeq \mathcal{O}(X)_\mathfrak{p},$$

g gets identified to the homomorphism

$$\mathcal{O}(Y)_\mathfrak{q} \to \mathcal{O}(X)_\mathfrak{p}$$

induced by $f^\#$.

Problem 4.

i) Show that $\mathbf{A}^1 \smallsetminus \{0\}$ is an affine variety (recall: this means that it is isomorphic to a closed subset of an affine space).
ii) Let $U = \mathbf{A}^2 \smallsetminus \{(0,0)\}$. What is $\mathcal{O}(U)$?
iii) Deduce that U is not an affine variety.

Problem 5. Show that $\mathbf{A}^1$ is *not* isomorphic to any proper open subset of itself.