Homework Set 1

Solutions are due Tuesday, January 16.

Work out all of the following problems. Write solutions to four of them, of your choice.

Problem 1.

i) Show that if A is a Noetherian ring, then for every A-modules M and N, with M finitely generated, and for every multiplicative system S in A, we have a canonical isomorphism
$$S^{-1}\mathrm{Hom}_A(M,N) \simeq \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$
ii) Deduce that if X is an affine variety and $A = \mathcal{O}(X)$, then for every A-modules M and N, with M finitely generated, we have
$$\widetilde{\mathrm{Hom}_A(M,N)} \simeq \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}).$$
iii) Show that if A is a Noetherian ring, then for every finitely generated A-modules M and N, the A-module $\mathrm{Hom}_A(M,N)$ is finitely generated.
iv) Deduce that if X is an algebraic variety and $\mathcal{M}$, $\mathcal{N}$ are $\mathcal{O}_X$-modules, with $\mathcal{M}$ coherent and $\mathcal{N}$ quasi-coherent (coherent), then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is quasi-coherent (respectively, coherent).

For every sheaf of Abelian groups $\mathcal{F}$ on X, the *support* of $\mathcal{F}$ is the subset
$$\mathrm{Supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}.$$
If $Y = \mathrm{Supp}(\mathcal{F})$, we also say that $\mathcal{F}$ is *supported on* Y.

Problem 2. Let X be an algebraic variety. Show that if $\mathcal{M}$ is a coherent sheaf on X, then $\mathrm{Supp}(\mathcal{M})$ is a closed subset of X; in fact, if U is an affine open subset of X, then $\mathrm{Supp}(\mathcal{M}) \cap U$ is the zero-locus of $\mathrm{Ann}_{\mathcal{O}_X(U)}\mathcal{M}(U)$.

Given a coherent ideal $\mathcal{I} \hookrightarrow \mathcal{O}_X$, the *co-support* or *zero-locus* $V(\mathcal{I})$ of $\mathcal{I}$ is the support of the coherent sheaf $\mathcal{O}_X/\mathcal{I}$. It follows from the above problem that if U is an affine open subset of X, then $V(\mathcal{I}) \cap U$ is the zero-locus of $\mathcal{I}(U) \subseteq \mathcal{O}_X(U)$

Problem 3. Let X be an algebraic variety. Given an $\mathcal{O}_X$-module $\mathcal{M}$ on X, its *annihilator* $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{M})$ is the subsheaf of $\mathcal{O}_X$ given by
$$\Gamma\big(U, \mathrm{Ann}_{\mathcal{O}_X}(\mathcal{M})\big) := \mathrm{Ann}_{\mathcal{O}_X(U)}\mathcal{M}(U)$$
for every open subset U of X. Show that if $\mathcal{M}$ is a coherent sheaf on X, then $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{M})$ is a coherent ideal and for every irreducible closed subset $V \subseteq X$, we have
$$\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{M})_V \simeq \mathrm{Ann}_{\mathcal{O}_{X,V}}(\mathcal{M}_V).$$

Note that the zero-locus of $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{M})$ is equal to $\mathrm{Supp}(\mathcal{M})$.

Problem 4. Let Y be a closed subvariety of X and $i\colon Y \hookrightarrow X$ be the inclusion.

i) Show that the canonical morphism $\mathcal{O}_X \to i_*(\mathcal{O}_Y)$ is surjective.
ii) The *ideal sheaf defining* Y is defined as the kernel $\mathcal{I}_{Y/X}$ of the canonical morphism $\mathcal{O}_X \to i_*(\mathcal{O}_Y)$ (or equivalently, as $\mathrm{Ann}_{\mathcal{O}_X} i_*(\mathcal{O}_Y)$). Show that $\mathcal{I}_{Y/X}$ is a coherent ideal of $\mathcal{O}_X$.
iii) Show that for every affine open subset U of X, we have
$$\mathcal{I}_{Y/X}(U) = I_U(Y \cap U).$$

Problem 5. Let X be an algebraic variety.

i) If $\mathcal{I}$ is a coherent sheaf of ideals in $\mathcal{O}_X$, then the *radical* $\mathrm{rad}(\mathcal{I})$ of $\mathcal{I}$ is the sheaf given by
$$\Gamma\big(U, \mathrm{rad}(\mathcal{I})\big) = \mathrm{rad}\big(\Gamma(U, \mathcal{I})\big) \subseteq \Gamma(U, \mathcal{O}_X).$$
Show that $\mathrm{rad}(\mathcal{I})$ is a coherent sheaf.
ii) Show that if Y is a closed subvariety of X and $\mathcal{I} = \mathcal{I}_{Y/X}$, then $Y = V(\mathcal{I})$. Conversely, if $\mathcal{I}$ is any coherent ideal sheaf on X and $Y = V(\mathcal{I})$, then $\mathcal{I}_{Y/X} = \mathrm{rad}(\mathcal{I})$.

Problem 6. Let X be an algebraic variety, $\mathcal{M}$ a quasi-coherent sheaf on X, and $\mathcal{M}_1, \ldots, \mathcal{M}_r$ quasi-coherent subsheaves of $\mathcal{M}$.

i) The *sum* $\sum_{i=1}^r \mathcal{M}_i$ is the sheaf associated to the presheaf that maps an open subset $U \subseteq X$ to
$$\sum_{i=1}^r \mathcal{M}_i(U) \subseteq \mathcal{M}(U).$$
Show that this is quasi-coherent and it is coherent if all $\mathcal{M}_i$ are coherent.
ii) The *intersection* $\bigcap_{i=1}^r \mathcal{M}_i$ is the subsheaf of $\mathcal{M}$ that maps an open subset $U \subseteq X$ to
$$\bigcap_{i=1}^r \mathcal{M}_i(U) \subseteq \mathcal{M}(U).$$
Show that this is quasi-coherent and it is coherent if one of the $\mathcal{M}_i$ is coherent.

Problem 7. Let $f\colon X \to Y$ be a morphism of algebraic varieties and $\mathcal{I}$ a coherent sheaf of ideals on Y.

i) Show that the image of the induced morphism
$$f^*(\mathcal{I}) \to f^*(\mathcal{O}_Y) = \mathcal{O}_X$$
is a coherent sheaf of ideals on X, that we denote $\mathcal{I} \cdot \mathcal{O}_X$.
ii) Show that
$$V(\mathcal{I} \cdot \mathcal{O}_X) = f^{-1}\big(V(\mathcal{I})\big).$$

Problem 8. Show that if Y is a closed subvariety of X and $i\colon Y \hookrightarrow X$ is the inclusion, then the map $\mathcal{F} \to i_*(\mathcal{F})$ gives an equivalence of categories between $\mathcal{Q}coh(Y)$ (or $\mathcal{C}oh(Y)$) and the full subcategory of $\mathcal{Q}coh(X)$ (respectively, $\mathcal{C}oh(X)$) consisting of those $\mathcal{F}$ such that $\mathcal{I}_{Y/X} \subseteq \mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F})$.