Homework Set 12

Solutions are due Thursday, April 12.

Problem 1. Show that if $\mathfrak{a}$ is a proper ideal in a Noetherian ring R, then we have the inequality

$$\mathrm{depth}(\mathfrak{a}, R) \leq \mathrm{codim}(\mathfrak{a})$$

(recall that $\mathrm{codim}(\mathfrak{a}) = \min_{\mathfrak{p}\supseteq\mathfrak{a}} \mathrm{codim}(\mathfrak{p})$, where the minimum is over all prime ideals $\mathfrak{p}$ containing $\mathfrak{a}$, or equivalently, over the minimal prime ideals containing $\mathfrak{a}$).

Problem 2. Show that if X and Y are Cohen-Macaulay varieties, then $X \times Y$ is Cohen-Macaulay.

Problem 3. Show that the subring $k[x^4, x^3y, xy^3, y^4]$ of $k[x, y]$ is not Cohen-Macaulay.

Problem 4. Show that if X is a smooth, quasi-projective variety, then the canonical group homomorphism

$$K^0(X) \to K_0(X), \quad [\mathcal{E}] \to [\mathcal{E}]$$

is an isomorphism.

The following problem deals with the notion of *Castelnuovo-Mumford regularity*. By definition, a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is *m-regular* if

$$H^i\big(\mathbb{P}^n, \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(m-i)\big) = 0 \quad \text{for all} \quad i \geq 1.$$

Note that $\mathcal{F}$ is m-regular for all $m \gg 0$. The *Castelnuovo-Mumford regularity* of $\mathcal{F}$ is the smallest m such that $\mathcal{F}$ is m-regular.

Problem 5. Show that if $\mathcal{F}$ is an m-regular coherent sheaf on $\mathbb{P}^n$, the following hold:

i) $\mathcal{F}$ is also $(m+1)$-regular (this justifies the definition of *Castelnuovo-Mumford regularity*).

ii) The canonical map

$$\Gamma\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)\big) \otimes_k \Gamma\big(\mathbb{P}^n, \mathcal{F}(m)\big) \to \Gamma\big(\mathbb{P}^n, \mathcal{F}(m+1)\big),$$

given by multiplication of sections, is surjective.

iii) Deduce that $\mathcal{F}(m)$ is globally generated. (One of the reasons this notion is important is that it provides a way to deduce global generation from cohomology vanishing.)

Hint: use the Koszul complex on $\mathbb{P}^n$.

Extra credit problem. Fix $n \geq 3$ and let $\mathcal{S}$ be the set of subsets of $\mathbb{P}^2$ with n elements, not all of them lying on the same line.

i) Show that there is N such that for every set $\Lambda \in \mathcal{S}$, the subspace

$$\Gamma\big(\mathbb{P}^2, \mathcal{I}_\Lambda(n-1)\big) \subseteq \Gamma\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-1)\big)$$

has dimension N.

ii) Show that the map $\mathcal{S} \to G$, where G is the Grassmann variety of N-dimensional subspaces of $\Gamma\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-1)\big)$, is injective.