Homework Set 10

Solutions are due Friday, December 8.

Problem 1. Let m, n be positive integers and $r \leq \min\{m, n\}$. Consider the *generic determinantal variety*

$$M^r_{m,n}(k) = \{B \in M_{m,n}(k) \mid \operatorname{rank}(B) \leq r\}$$

(see Problem 3 on HW #1). Compute the dimension of $M^r_{m,n}(k)$, as follows (we identify in the obvious way $M_{m,n}(k)$ and $\operatorname{Hom}_k(k^n, k^m)$).

i) Show that the set
$$Y = \{(A, [W]) \in M_{m,n}(k) \times G(n-r, n) \mid W \subseteq \ker(A)\}$$
is a closed subset of $M_{m,n}(k) \times G(n-r, n)$.

ii) Show that the projection onto the second component induces a morphism $p\colon Y \to G(n-r, n)$ which is locally trivial[1], with fiber $\mathbf{A}^{rm}$. Deduce that Y is irreducible, of dimension $mr + nr - r^2$.

iii) Note that the image of the map $f\colon Y \to M_{m,n}(k)$ induced by the projection onto the first component is $M^r_{m,n}(k)$ (in particular, this is irreducible). Show that there is an open subset U of $M^r_{m,n}(k)$ such that f has finite fibers over U. Deduce that
$$\operatorname{codim}_{M_{m,n}(k)}\big(M^r_{m,n}(k)\big) = (m-r)(n-r).$$

iv) For extra credit, show that the morphism $Y \to M^r_{m,n}(k)$ is birational.

Let V be a vector space over k, with $\dim_k V = n$ and let $1 \leq \ell_1 < \ldots < \ell_r \leq n$. A *flag of type* $(\ell_1, \ldots, \ell_r)$ in V is a sequence of linear subspaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_r \subseteq V$, where $\dim_k(V_i) = \ell_i$. A *complete flag* is a flag of type $(1, 2, \ldots, n)$.

The *flag variety* $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ parametrizes flags in V of type $(\ell_1, \ldots, \ell_r)$. In other words, this is the set

$$\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V) := \{(V_1, \ldots, V_r) \in G(\ell_1, V) \times \cdots \times G(\ell_r, V) \mid V_1 \subseteq \cdots \subseteq V_r\}.$$

In particular, the *complete flag variety* $\mathrm{Fl}(V) = \mathrm{Fl}_{1,\ldots,n}(V)$ parametrizes complete flags in V.

Problem 2.

i) Show that the subset $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ of $G(\ell_1, V) \times \cdots \times G(\ell_r, V)$ is closed, hence $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ is a projective variety.

ii) Show that for every $(\ell_1, \ldots, \ell_r)$, the flag variety $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ is a smooth, irreducible variety, of dimension $\sum_{i=1}^r \ell_i(\ell_{i+1} - \ell_i)$, where $\ell_{r+1} = n$. In particular, the complete flag variety $\mathrm{Fl}(V)$ is a smooth, irreducible variety of dimension $\frac{n(n-1)}{2}$.

[1] A morphism of algebraic varieties $f\colon X \to Y$ is *locally trivial*, with fiber a variety F, if Y has an open cover $Y = \bigcup_i U_i$ such that for every i, we have an isomorphism $f^{-1}(U_i) \simeq U_i \times F$, compatible with the maps to U_i.

Problem 3. Given varieties X and Y, for every $x \in X$ and $y \in Y$, the projections $X \times Y \to X$ and $X \times Y \to Y$ induce a linear map

$$T_{(x,y)}(X \times Y) \to T_xX \times T_yY.$$

Show that this is an isomorphism. Deduce that x and y are smooth points of X and Y, respectively, if and only if (x, y) is a smooth point of $X \times Y$.