D-modules and singularities
Lecture notes for Math 732, Winter 2023

Mircea Mustaţă

Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109, USA

Email address: mmustata@umich.edu

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CHAPTER 1

Preface

The course will be devoted to an introduction to $\mathcal{D}$-module theory and some of its connections with invariants of singularities. After a discussion of the sheaf of differential operators and general facts about $\mathcal{D}$-modules, we give a presentation of the theory of holonomic $\mathcal{D}$-modules on the affine space. We next turn and discuss some general cohomological results concerning filtered modules over certain almost-commutative filtered rings. We will then define the main functors on $\mathcal{D}$-modules and prove the main results: the Kashiwara equivalence theorem, Bernstein's inequality, the existence of b-functions, and the preservation of holonomicity under push-forward and pull-back, as well as the 6-functor formalism for holonomic $\mathcal{D}$-modules. We then treat briefly the Riemann-Hilbert correspondence. Finally, we end the course with a detailed discussion of the V-filtration of Kashiwara and Malgrange and its connection with invariants of singularities, such as the Bernstein-Sato polynomial, multiplier ideals, and the minimal exponent. With the exception of the last chapter, almost everything that we discuss in this course can be found in [**HTT08**]. The presentation in several sections is also inspired from Bhargav Bhatt's Michigan course in Fall 2020.

We always work in the algebraic setting. In fact, with the exception of Chapter 7, we work over an arbitrary algebraically closed field k of characteristic 0. We only specialize to the case $k = \mathbf{C}$ when treating the Riemann-Hilbert correspondence, for which the classical topology plays an important role. It is important to keep in mind that the theory that we discuss can be also developed, with some care, in the setting of complex manifolds.

We assume basic familiarity with algebraic varieties and quasi-coherent sheaves on them. We also assume familiarity with derived functors. In $\mathcal{D}$-module theory, it is important to work at the level of derived categories: we do not assume familiarity with derived categories, so we include in the appendix a brief introduction to the basic features of this formalism.

CHAPTER 2

The sheaf of differential operators

Let k be a fixed algebraically closed field of characteristic 0 and let X be a smooth, irreducible, algebraic variety over k. By a variety we mean a reduced separated scheme of finite type over k, possibly reducible. Our goal in this chapter is to discuss the sheaf $\mathcal{D}_X$ of differential operators on X.

2.1. Grothendieck's definition of differential operators

We begin with Grothendieck's definition for the sheaf of differential operators. This is done by first defining inductively the sheaf $F_p\mathcal{D}_X$ of differential operators of order $\leq p$. All these are subsheaves of the sheaf $\mathcal{E}nd_k(\mathcal{O}_X)$ of k-linear endomorphisms of $\mathcal{O}_X$. Note that inside $\mathcal{E}nd_k(\mathcal{O}_X)$ we have two subsheaves: we have the sheaf $\mathcal{O}_X$ of $\mathcal{O}_X$-linear operators (where a regular function $f \in \mathcal{O}_X(U)$ corresponds to the endomorphism $\mathcal{O}_U \to \mathcal{O}_U$ given by multiplication with f) and the sheaf $\mathcal{D}er_k(\mathcal{O}_X)$ of k-linear derivations of $\mathcal{O}_X$. We also note that $\mathcal{E}nd_k(\mathcal{O}_X)$ is naturally a sheaf of (noncommutative) k-algebras with respect to composition. In particular, it carries the bracket $[-,-]$ given by $[P,Q] = PQ - QP$ whenever P and Q are two sections defined over the same open subset (by an abuse of notation, we often do not keep track of this open subset). The bracket satisfies the following compatibility formula with the product:

$$[PQ, R] = P[Q,R] + [P,R]Q \tag{2.1}$$

for all open subsets $U \subseteq X$ and all $P, Q, R \in \Gamma\big(U, \mathcal{E}nd_k(\mathcal{O}_X)\big)$.

The following fact will be often useful:

Lemma 2.1. *For every $D \in \mathrm{Der}_k(\mathcal{O}_X)$ and every $f \in \mathcal{O}_X$, we have $[D,f] = D(f)$ in $\mathcal{E}nd_k(\mathcal{O}_X)$.*

Proof. The assertion follows from the fact that a derivation satisfies the Leibniz formula. □

Definition 2.2. For $p \geq -1$, we define inductively the sheaf $F_p\mathcal{D}_X$ of *differential operators of order* $\leq p$ on X as follows:

i) We put $F_{-1}\mathcal{D}_X = 0$.
ii) Assuming that $p \geq 0$ and $F_{p-1}\mathcal{D}_X \subseteq \mathcal{E}nd_k(\mathcal{O}_X)$ is defined, for every open subset $U \subseteq X$, we define $\Gamma(U, F_p\mathcal{D}_X)$ to consist of those $P \in \Gamma(U, \mathcal{E}nd_k(\mathcal{O}_X))$ such that for every open subset $V \subseteq U$ and every $f \in \mathcal{O}_X(V)$, we have $[P,f] \in \Gamma(V, F_{p-1}\mathcal{D}_X)$.

The *sheaf of differential operators* on X is

$$\mathcal{D}_X = \bigcup_p F_p\mathcal{D}_X \subseteq \mathcal{E}nd_k(\mathcal{O}_X).$$

If X is affine and $R = \mathcal{O}_X(X)$, then we put $D_R := \Gamma(X, \mathcal{D}_X)$.

REMARK 2.3. It is straightforward to see that all $F_p\mathcal{D}_X$ and $\mathcal{D}_X$ are subsheaves of $\mathcal{E}nd_k(\mathcal{O}_X)$.

REMARK 2.4. It follows from the definition that if $U \subseteq X$ is an open subset, then $F_p\mathcal{D}_U = F_p\mathcal{D}_X|_U$ for all $p \geq 0$ and thus $\mathcal{D}_U = \mathcal{D}_X|_U$.

REMARK 2.5. It follows from the definition, by induction on p, that $F_p\mathcal{D}_X \subseteq F_{p+1}\mathcal{D}_X$ for all $p \geq -1$ (the case $p = -1$ being trivial).

EXAMPLE 2.6. We have $F_0\mathcal{D}_X = \mathcal{O}_X$. Indeed, it follows from the definition that $F_0\mathcal{D}_X(U)$ consists of those P such that $[P, f] = 0$ for every regular function $f \in \mathcal{O}_X(V)$, with $V \subseteq U$. This means that $P(fg) = fP(g)$ for every $f, g \in \mathcal{O}_X(V)$. This implies that P is given by multiplication with $P(1) \in \mathcal{O}_X(U)$.

EXAMPLE 2.7. It follows from the definition and Lemma 2.1 that we have $\mathcal{O}_X + \mathcal{D}er_k(\mathcal{O}_X) \subseteq F_1\mathcal{D}_X$. In fact, this is an equality. Indeed, if $P \in \Gamma(U, F_1\mathcal{D}_X)$, then it follows from the definition and Example 2.6 that for every $V \subseteq U$ and $f \in \mathcal{O}_X(V)$, we have $[P, f] \in \mathcal{O}_X(V)$. In other words, for every $f, g \in \mathcal{O}_X(V)$, we have

$$P(fg) - fP(g) = g\big(P(f) - P(1)\big).$$

If we put $Q = P - P(1)$, then the above relation becomes

$$Q(fg) - fQ(g) = gQ(f),$$

and since $Q(1) = 0$, it follows that $Q \in \Gamma\big(U, \mathrm{Der}_k(\mathcal{O}_X)\big)$.

We next recall the general definition of a filtration on a sheaf of rings:

DEFINITION 2.8. If $\mathcal{R}$ is a sheaf of rings on a topological space, then a *filtration* on $\mathcal{R}$ is given by a family of subsheaves $F_\bullet\mathcal{R} = (F_p\mathcal{R})_{p\in\mathbf{Z}}$, such that the following conditions hold:

i) $F_p\mathcal{R} \subseteq F_{p+1}\mathcal{R}$ for all $p \in \mathbf{Z}$.
ii) $F_{-1}\mathcal{R} = 0$ and $\mathcal{R} = \bigcup_{p\in\mathbf{Z}} F_p\mathcal{R}$.
iii) $1 \in F_0\mathcal{R}$.
iv) $F_p\mathcal{R} \cdot F_q\mathcal{R} \subseteq F_{p+q}\mathcal{R}$ for all $p, q \in \mathbf{Z}$.

In this case,

$$\mathrm{Gr}^F_\bullet(\mathcal{R}) := \bigoplus_{p\geq 0} \mathrm{Gr}^F_p(\mathcal{R}), \quad \text{where} \quad \mathrm{Gr}^F_p(\mathcal{R}) = F_p\mathcal{R}/F_{p-1}\mathcal{R}$$

is a sheaf of graded rings.

In our setting, $\mathcal{D}_X$ is a sheaf of rings and $F_\bullet\mathcal{D}_X$ is a filtration on $\mathcal{D}_X$ (called the *order filtration* on $\mathcal{D}_X$). Indeed, properties i)-iii) above are clear, while condition iv) and the fact that $\mathcal{D}_X$ is a sheaf of rings are the content of the next proposition.

PROPOSITION 2.9. For every $p, q \geq 0$, we have

$$F_p\mathcal{D}_X \cdot F_q\mathcal{D}_X \subseteq F_{p+q}\mathcal{D}_X.$$

In particular, $\mathcal{D}_X$ is a sheaf of subrings of $\mathcal{E}nd_k(\mathcal{O}_X)$.

PROOF. The first assertion is clearly true if $p = -1$ or $q = -1$ and the general case follows by induction on $p+q$, using the fact that by (2.1), for every $P \in F_p\mathcal{D}_X$, $Q \in F_q\mathcal{D}_X$, and $f \in \mathcal{O}_X$, using the induction hypothesis we have

$$[PQ, f] = P[Q, f] + [P, f]Q \in F_p\mathcal{D}_X \cdot F_{q-1}\mathcal{D}_X + F_{p-1}\mathcal{D}_X \cdot F_q\mathcal{D}_X \subseteq F_{p+q-1}\mathcal{D}_X.$$

The fact that $\mathcal{D}_X$ is a sheaf of subrings is an immediate consequence. $\square$

REMARK 2.10. The first assertion in Proposition 2.9 implies, by taking $p = 0$ or $q = 0$, that $\mathcal{D}_X$ and every $F_p\mathcal{D}_X$ are $\mathcal{O}_X$-modules using either left or right multiplication in $\mathcal{E}nd_k(\mathcal{O}_X)$ (but these structures are different, as follows for example from Lemma 2.1). From now on, unless explicitly mentioned otherwise, we always consider $\mathcal{D}_X$ as an $\mathcal{O}_X$-module via left multiplication.

Our next goal is to give a description of $\mathcal{D}_X$ in terms of local coordinates. We introduce some notation that we will always use in the presence of local coordinates. Suppose that $x_1, \ldots, x_n$ are algebraic coordinates in an open subset $U \subseteq X$: this means that $x_1, \ldots, x_n \in \mathcal{O}_X(U)$ are such that $dx_1, \ldots, dx_n$ trivialize the cotangent sheaf $\Omega_X = \Omega_{X/k}$ on U (equivalently, for every $Q \in U$, the functions $x_1 - x_1(Q), \ldots, x_n - x_n(Q)$ give a regular system of parameters of $\mathcal{O}_{X,Q}$). We will say that the coordinates are centered at some point $Q \in U$ if $x_i(Q) = 0$ for all i (note that $y_1 = x_1 - x_1(Q), \ldots, y_n = x_n - x_n(Q)$ are algebraic coordinates centered at Q and $dy_i = dx_i$ for all i). Given any algebraic coordinates $x_1, \ldots, x_n \in \mathcal{O}_X(U)$, we denote by $\partial_1 = \partial_{x_1}, \ldots, \partial_n = \partial_{x_n}$ the dual basis of $\mathcal{D}er_k(\mathcal{O}_X)|_U$. Note that $\partial_1, \ldots, \partial_n$ pairwise commute. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n_{\geq 0}$, we put $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \in \Gamma\big(U, \mathcal{E}nd_k(\mathcal{O}_X)\big)$. Using the fact that $[\partial_i, x_j] = \delta_{i,j}$ for all i and j, it is an easy exercise to show via (2.1) that for every $\alpha \in \mathbf{Z}^n_{\geq 0}$ and every $h \in \mathcal{O}_U$, we have

$$[h\partial^\alpha, x_i] = \alpha_i h \partial^{\alpha - e_i}, \tag{2.2}$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbf{Z}^n$.

THEOREM 2.11. *If $x_1, \ldots, x_n$ is an algebraic system of coordinates on the open subset $U \subseteq X$, then for every $p \geq 0$, the sheaf $F_p\mathcal{D}_U$ is a free $\mathcal{O}_U$-module, with basis $\{\partial^\alpha, |\alpha| \leq p\}$.*

Before giving the proof of the theorem, we make some preparations. Note first that the ∂^α are linearly independent over $\mathcal{O}_U$. Indeed, if $P = \sum_\alpha h_\alpha \partial^\alpha = 0$, then in order to see that h_α vanishes at every $Q \in U$, we may assume (after replacing each x_i by $x_i - x_i(Q)$) that $x_i(Q) = 0$ for all i. In this case the value of $P(x^\alpha)$ at Q is $\alpha! \cdot h_\alpha(Q)$, hence $h_\alpha(Q) = 0$.

Let us define $F'_p\mathcal{D}_U \subseteq \mathcal{D}_U$ to be the left $\mathcal{O}_U$-submodule of $\mathcal{E}nd_k(\mathcal{O}_U)$ generated by $\{\partial^\alpha, |\alpha| \leq p\}$. The inclusion $F_p\mathcal{D}_U \subseteq F'_p\mathcal{D}_U$ follows from Proposition 2.9 and Examples 2.6 and 2.7. The interesting assertion in the theorem is that this inclusion is an equality.

LEMMA 2.12. *If $\mathfrak{m}$ is the ideal defining a point $Q \in X$, then for every $q \geq p \geq 0$ and every $P \in \Gamma(X, F_p\mathcal{D}_X)$, we have $P(\mathfrak{m}^q) \subseteq \mathfrak{m}^{q-p}$.*

PROOF. We may and will assume that X is affine and let $y_1, \ldots, y_r \in \mathcal{O}_X(X)$ be generators of $\mathfrak{m}$. We argue by induction on $p + q$, with the cases $q = p$ and $p = 0$ being clearly true. For the induction step, note that if $q > p$, then every $h \in \mathfrak{m}^q$ can be written as $h = \sum_{i=1}^r y_i h_i$, with $h_i \in \mathfrak{m}^{q-1}$, hence

$$P(h) = \sum_{i=1}^r P y_i(h_i) = \sum_{i=1}^r y_i P(h_i) + \sum_{i=1}^r [P, y_i](h_i) \in \mathfrak{m}^{q-p}$$

since by the induction hypothesis we have $P(h_i) \in \mathfrak{m}^{q-1-p}$ (hence $y_i P(h_i) \in \mathfrak{m}^{q-p}$) and $[P, y_i](h_i) \in \mathfrak{m}^{q-p}$ (note that $[P, y_i] \in F_{p-1}\mathcal{D}_X$). $\square$

LEMMA 2.13. *Given coordinates $x_1, \ldots, x_n$ on $U \subseteq X$, if $P \in \Gamma(U, \mathcal{D}_X)$ is such that $[P, x_i] = 0$ for all i, then $P \in \mathcal{O}_X(U)$.*

PROOF. We may assume that U is affine. By Example 2.6, is enough to show that for every open subset $V \subseteq U$, every $f, h \in \mathcal{O}_X(V)$ and every point $Q \in V$, the function $[P, f](h)$ vanishes at Q. After replacing each x_i by $x_i - x_i(Q)$, we may and will assume that $x_1, \ldots, x_n$ generate the ideal $\mathfrak{m}$ defining Q.

Since P commutes with each x_i, it follows that P commutes with each x^α. Let p be such that $P \in \Gamma(U, F_p\mathcal{D}_X)$. After possibly replacing V by an open neighborhood of Q, we can write $f = f_1 + f_2$, with $f_1 \in k[x_1, \ldots, x_n]$ and $f_2 \in \mathfrak{m}^{p+1}$, so

$$[P, f](h) = [P, f_1](h) + [P, f_2](h) = [P, f_2](h) = P(f_2h) - f_2P(h) \in \mathfrak{m},$$

where the containment follows from Lemma 2.12. Therefore $[P, f](h)$ vanishes at Q. $\square$

LEMMA 2.14. *Given algebraic coordinates $x_1, \ldots, x_n$ on the affine open subset $U \subseteq X$, if $P_i \in \Gamma(U, F'_p\mathcal{D}_X)$ for $1 \leq i \leq n$ are such that $[P_i, x_j] = [P_j, x_i]$ for all i and j, then there is $P \in \Gamma(U, F'_{p+1}\mathcal{D}_X)$ such that $[P, x_i] = P_i$ for $1 \leq i \leq n$.*

PROOF. We argue by descending induction on m, with $1 \leq m \leq n+1$, that there is $P' \in \Gamma(U, F'_{p+1}\mathcal{D}_X)$ such that $[P', x_i] = P_i$ for $m \leq i \leq n$. The assertion trivially holds for $m = n+1$. Suppose now, by induction, that we have such P' for $m+1$. If $G = [P', x_m] - P_m$, then

(2.3)
$$[G, x_i] = \big[[P', x_i], x_m\big] - [P_m, x_i] = [P_i, x_m] - [P_m, x_i] = 0 \quad \text{for} \quad m+1 \leq i \leq n.$$

Since $P' \in \Gamma(U, F'_{p+1}\mathcal{D}_X)$, it follows from (2.2) that $[P', x_m] \in \Gamma(U, F'_p\mathcal{D}_X)$, and thus $G \in \Gamma(U, F'_p\mathcal{D}_X)$. If we write $G = \sum_{|\alpha| \leq p} h_\alpha \partial^\alpha$, then it follows from (2.3) using (2.2) that $h_\alpha = 0$ unless $\alpha_{m+1} = \ldots = \alpha_n = 0$. If we put

$$P'' = \sum_{|\alpha| \leq p} \tfrac{1}{\alpha_m + 1} h_\alpha \partial^{\alpha + e_m} \in \Gamma(U, F'_{p+1}\mathcal{D}_X),$$

then it is clear that $[P'', x_i] = 0$ for $m+1 \leq i \leq n$, while $[P'', x_m] = G$ by (2.2). We thus have $[P' - P'', x_i] = P_i$ for $m \leq i \leq n$. This completes the proof of the induction step. By taking $m = 1$, we obtain the conclusion of the lemma. $\square$

We can now prove the local description of the sheaf of differential operators.

PROOF OF THEOREM 2.11. We may and will assume that U is affine. We show that $F'_p\mathcal{D}_U \subseteq F_p\mathcal{D}_U$ by induction on p. Note that if $p = 0$, then the assertion follows from Example 2.6. Suppose now that $p \geq 1$ and we know the assertion for $p-1$. If $P \in \Gamma(U, F_p\mathcal{D}_X)$, then $P_i := [P, x_i] \in \Gamma(U, F_{p-1}\mathcal{D}_X) = \Gamma(U, F'_{p-1}\mathcal{D}_X)$. For all i and j, we have

$$[P_i, x_j] = \big[[P, x_i], x_j\big] = \big[[P, x_j], x_i\big] = [P_j, x_i],$$

hence it follows from Lemma 2.14 that there is $P' \in \Gamma(U, F'_p\mathcal{D}_X)$ such that $[P', x_i] = P_i$ for all i. Since $[P - P', x_i] = 0$ for all i, it follows from Lemma 2.13 that $P - P' \in \mathcal{O}_X(U)$, hence $P \in \Gamma(U, F'_p\mathcal{D}_X)$. This completes the proof of the theorem. $\square$

The following are immediate consequences of Theorem 2.11.

COROLLARY 2.15. The sheaf $\mathcal{D}_X$ is a quasi-coherent $\mathcal{O}_X$-module.

COROLLARY 2.16. The subsheaf $\mathcal{D}_X \subseteq \mathcal{E}nd_k(\mathcal{O}_X)$ is the sheaf of subrings generated by $\mathcal{O}_X, \mathcal{D}er_k(\mathcal{O}_X) \subseteq \mathcal{E}nd_k(\mathcal{O}_X)$.

REMARK 2.17. If X is affine and $P \in \Gamma\big(X, \mathcal{E}nd_k(\mathcal{O}_X)\big)$, then $P \in \Gamma(X, F_p\mathcal{D}_X)$ if and only if $[P, f] \in \Gamma(X, F_{p-1}\mathcal{D}_X)$ for all $f \in \mathcal{O}_X(X)$. Indeed, it is enough to show that in this case, for every nonzero $g \in \mathcal{O}_X(X)$, we have $\big[P, \frac{f}{g}\big] \in \Gamma(V, F_{p-1}\mathcal{D}_X)$, where $V = X \smallsetminus V(g)$. This follows from the fact that $\big[P, \frac{f}{g}\big] = \frac{1}{g}[g, P]\frac{f}{g} + \frac{1}{g}[P, f]$, which in turn follows by direct computation.

REMARK 2.18. While we will not pursue this, it is worth pointing out that there is a purely analytic side of the story. If X is a complex manifold, then the sheaf of differential operators $\mathcal{D}_X$ is the sheaf of subrings of $\mathcal{E}nd_{\mathbf{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\mathcal{D}er_{\mathbf{C}}(\mathcal{O}_X)$ (note that in this case $\mathcal{O}_X$ is the sheaf of holomorphic functions on X). Again, we have an order filtration $F_\bullet\mathcal{D}_X$ on $\mathcal{D}_X$ such that if $x_1, \ldots, x_n$ are *analytic coordinates* in an open subset $U \subseteq X$, then $F_p\mathcal{D}_X$ is a free left $\mathcal{O}_X$-module with basis $\{\partial^\alpha, |\alpha| \leq p\}$.

2.2. The sheaf of graded rings for the order filtration

The order filtration on $\mathcal{D}_X$ provides the main tool for reducing the study of $\mathcal{D}_X$ and of sheaves of $\mathcal{D}_X$-modules to the commutative setting. Note that in our case $\mathrm{Gr}^F_\bullet(\mathcal{D}_X)$ is a sheaf of k-algebras. It follows from Examples 2.6 and 2.7, we have $\mathrm{Gr}^F_0(\mathcal{D}_X) = \mathcal{O}_X$ and $\mathrm{Gr}^F_1(\mathcal{D}_X) = \mathcal{T}_X := \mathcal{D}er_k(\mathcal{O}_X)$.

The next lemma implies that $\mathrm{Gr}^F_\bullet(\mathcal{D}_X)$ is a sheaf of *commutative rings*.

LEMMA 2.19. *For every p and q, we have*

$$[F_p\mathcal{D}_X, F_q\mathcal{D}_X] \subseteq F_{p+q-1}\mathcal{D}_X.$$

PROOF. The assertion follows by induction on $p+q$, being trivial if $p = -1$ or $q = -1$. For the induction step, note that if $P \in \Gamma(U, F_p\mathcal{D}_X)$ and $Q \in \Gamma(U, F_q\mathcal{D}_X)$, then for every $f \in \mathcal{O}_X(V)$, where $V \subseteq U$, the Jacobi identity gives

$$\big[[P, Q], f\big] = \big[P, [Q, f]\big] + \big[[P, f], Q\big].$$

Since $[P, f] \in \Gamma(V, F_{p-1}\mathcal{D}_X)$ and $[Q, f] \in \Gamma(V, F_{q-1}\mathcal{D}_X)$, it follows from the inductive hypothesis that $\big[P, [Q, f]\big] \in \Gamma(V, F_{p+q-2}\mathcal{D}_X)$ and $\big[[P, f], Q\big] \in \Gamma(V, F_{p+q-2}\mathcal{D}_X)$. By definition, we thus have $[P, Q] \in \Gamma(U, F_{p+q-1}\mathcal{D}_X)$. □

REMARK 2.20. We stated Theorem 2.11 with respect to the action of $\mathcal{O}_X$ on $\mathcal{D}_X$ on the left, but the same holds with respect to the action on the right: with the notation in that theorem, we also have

$$F_p\mathcal{D}_X = \bigoplus_{|\alpha| \leq p} \partial^\alpha \mathcal{O}_X.$$

The assertion follows easily from that in Theorem 2.11 by induction on p, using the fact that for every α with $|\alpha| = p$, and every $h \in \mathcal{O}_X$, the difference $\partial^\alpha h - h\partial^\alpha$ lies in $F_{p-1}\mathcal{D}_X$ by Lemma 2.19.

THEOREM 2.21. *For every smooth complex algebraic variety X, we have a canonical isomorphism of sheaves of graded rings*

$$\mathcal{S}ym^\bullet_{\mathcal{O}_X}(\mathcal{T}_X) \simeq \mathrm{gr}^F_\bullet(\mathcal{D}_X).$$

Proof. It follows from Lemma 2.19 that $\mathcal{S} := \mathrm{gr}^F_\bullet(\mathcal{D}_X)$ is a sheaf of commutative graded $\mathcal{O}_X$-algebras. Since $\mathcal{S}_1 = \mathcal{T}_X$, it follows from the universal property of the symmetric algebra that we have a morphism of sheaves of graded commutative rings $\varphi\colon \mathcal{S}ym^\bullet_{\mathcal{O}_X}(\mathcal{T}_X) \to \mathcal{S}$ such that φ_1 is the identity. In order to check that φ is an isomorphism, we can argue locally, hence we may assume that X is affine and we have algebraic coordinates $x_1, \ldots, x_n \in \mathcal{O}_X(X)$. Since $\mathcal{T}_X$ is a free $\mathcal{O}_X$-module with basis $\partial_1, \ldots, \partial_n$, then the fact that each $\varphi_p\colon \mathcal{S}ym^p_{\mathcal{O}_X}(\mathcal{T}_X) \to F_p\mathcal{D}_X/F_{p-1}\mathcal{D}_X$ is an isomorphism follows from the fact that in light of Theorem 2.11, it maps an $\mathcal{O}_X$-module basis to a basis. □

Corollary 2.22. If X is a smooth complex algebraic variety, then for every affine open subset $U \subseteq X$, the ring $\mathcal{D}_X(U)$ is both left and right Noetherian.

Proof. Let $R = \mathcal{O}_X(U)$, so $D_R = \Gamma(U, \mathcal{D}_X)$. If I is a left (or right) ideal on D_R and we put $F_pI := I \cap F_pD_R$ for $p \geq 0$, then

$$\mathrm{gr}^F_\bullet(I) := \bigoplus_{p\geq 0}(F_pI/F_{p-1}I)$$

is an ideal in $\mathrm{gr}^F_\bullet(D_R)$, which is a commutative Noetherian ring by Theorem 2.21. Moreover, if $I \subseteq J$ are ideals in D_R, then $\mathrm{gr}^F_\bullet(I) \subseteq \mathrm{gr}^F_\bullet(J)$; furthermore, if this is an equality, then it follows by an easy induction that $J \cap F_pD_R \subseteq I \cap F_pD_R$ for all $p \geq 0$, hence $J = I$. Since $\mathrm{gr}^F_\bullet(D_R)$ contains no infinite strictly increasing sequences of ideals, it follows that D_R contains no infinite strictly increasing sequences of left (or right) ideals. □

2.3. Differential operators on the affine space

We next discuss a presentation of the ring of differential operators on the affine space $\mathbf{A}^n_k$.

Definition 2.23. For every field k, the *Weyl algebra* $A_n(k)$ is the quotient of the free associative k-algebra

$$k\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$$

by the 2-sided ideal generated by the following elements: $[x_i, x_j]$, $[\partial_i, \partial_j]$, and $[\partial_i, x_j] - \delta_{i,j}$ for $1 \leq i, j \leq n$.

From now on, we assume that k is algebraically closed and $R_n = k[x_1, \ldots, x_n]$, so $\mathrm{Spec}(R_n) = \mathbf{A}^n_k$. Note that $x_1, \ldots, x_n$ are algebraic coordinates on $\mathbf{A}^n_k$, so the description of $\mathcal{D}_{\mathbf{A}^n_k}$ in Theorem 2.11 applies in this case.

Proposition 2.24. We have an isomorphism of k-algebras $\varphi\colon A_n(k) \to D_{R_n}$ such that $\varphi(x_i) = x_i$ and $\varphi(\partial_i) = \partial_i$ for $1 \leq i \leq n$.

Proof. We clearly have a morphism of k-algebras as in the theorem, due to the fact that in D_{R_n} we have $[x_i, x_j] = 0$, $[\partial_i, \partial_j] = 0$, and $[\partial_i, x_j] = \delta_{i,j}$ for $1 \leq i, j \leq n$. We need to show that φ is bijective.

Note that we have a morphism of k-algebras $R_n \to A_n(k)$ that maps each x_i to x_i, hence we may consider $A_n(k)$ as a left R_n-module via this morphism. In this case, it is clear that φ is a morphism of left R_n-modules. Since D_{R_n} is a free R_n-module with basis $\{\partial^\alpha \mid \alpha \in \mathbf{Z}^n_{\geq 0}\}$ by Theorem 2.11, in order to show that φ is bijective, it is enough to show that $A_n(k)$ is generated as a left R_n-module by

$\{y^\alpha \mid \alpha \in \mathbf{Z}^n_{\geq 0}\}$. This is easy to see, using the fact that the relations in $A_n(k)$ imply that for every i and j, we have $\partial_i x_j = x_j\partial_i$ if $i \neq j$ and $\partial_i x_j = x_j\partial_i + 1$ if $i = j$. $\square$

2.4. A general presentation of $\mathcal{D}_X$

We next generalize the presentation of $\mathcal{D}_X$ in Proposition 2.24 to the case of an arbitrary smooth affine variety. This will be useful later for describing quasi-coherent sheaves of $\mathcal{D}_X$-modules. Let X be a smooth affine variety over k, and let $R = \mathcal{O}_X(X)$ and $T_R = \Gamma(X, \mathcal{T}_X)$.

PROPOSITION 2.25. With the above notation, the ring D_R is isomorphic to the free associative k-algebra A generated by

$$\{\widetilde{a} \mid a \in R\} \quad \text{and} \quad \{\widetilde{D} \mid D \in T_R\}$$

modulo the following relations:

i) $\widetilde{a_1a_2} = \widetilde{a_1} \cdot \widetilde{a_2}$ for every $a_1, a_2 \in R$;
ii) $\widetilde{aD} = \widetilde{a} \cdot \widetilde{D}$ for every $a \in R$ and $D \in T_R$;
iii) $[\widetilde{D}, \widetilde{a}] = \widetilde{D(a)}$ for every $a \in R$ and $D \in T_R$;
iv) $[\widetilde{D_1}, \widetilde{D_2}] = \widetilde{[D_1, D_2]}$ for every $D_1, D_2 \in T_R$.

PROOF. It is clear that we have a morphism of k-algebras $\varphi\colon A \to D_R$ that maps $\widetilde{a}$ to a and $\widetilde{D}$ to D for every $a \in R$ and $D \in T_R$. It follows from Corollary 2.16 that φ is surjective, hence we only need to show that it is also injective.

Let us consider A as a left R-module via $a \cdot u = \widetilde{a} \cdot u$. For every $p \geq 0$, we denote by F_pA the R-submodule of A generated by $\widetilde{D_1} \cdots \widetilde{D_k}$, where $k \leq p$ and $D_1, \ldots, D_k \in T_R$. It is easy to see, using the given relations that $F_pA \cdot F_qA \subseteq F_{p+q}A$ and $[F_pA, F_qA] \subseteq F_{p+q-1}A$ for every $p, q \geq 0$. Therefore $\mathrm{gr}^F_\bullet(A)$ is a commutative ring and it follows from the definition of A that we have a surjective k-algebra homomorphism $\psi\colon \mathrm{Sym}^\bullet_R(T_R) \to \mathrm{gr}^F_\bullet(A)$. Moreover, φ induces a ring homomorphism $\mathrm{gr}(\varphi)\colon \mathrm{gr}^F_\bullet(A) \to \mathrm{gr}^F_\bullet(D_R)$ such that the composition $\mathrm{gr}(\varphi) \circ \psi$ is an isomorphism by Theorem 2.21. Therefore ψ is an isomorphism, so $\mathrm{gr}(\varphi)$ is an isomorphism as well, which in turn immediately implies that φ is injective. This completes the proof of the proposition. $\square$

COROLLARY 2.26. With the notation in Proposition 2.25, if $x_1, \ldots, x_n \in R$ are algebraic coordinates on X, then D_R is isomorphic to the free associative k-algebra generated by $\{\widetilde{a} \mid a \in R\}$ and $\partial_1, \ldots, \partial_n$, modulo the following relations:

a) $\widetilde{a_1a_2} = \widetilde{a_1} \cdot \widetilde{a_2}$ for every $a_1, a_2 \in R$.
b) $[\partial_i, \widetilde{a}] = \widetilde{\frac{\partial a}{\partial x_i}}$ for every $a \in R$ and $1 \leq i \leq n$.
c) $[\partial_i, \partial_j] = 0$ for $1 \leq i, j \leq n$.

PROOF. Since $x_1, \ldots, x_n \in R$ are algebraic coordinates on X, the R-module T_R is free, with basis $\partial_1, \ldots, \partial_n$, and it is straightforward to check that the algebra A in Proposition 2.25 is isomorphic to the one in the statement of the corollary. $\square$

REMARK 2.27. For the sake of simplicity we made the assumption that the ground field k is algebraically closed, but all the results in this chapter hold over any field of characteristic 0.

CHAPTER 3

$\mathcal{D}_X$-modules: basic properties

As in the previous chapter, we assume that X is a smooth algebraic variety over an algebraically closed field k of characteristic 0. Let $n = \dim(X)$.

3.1. $\mathcal{D}_X$-modules

DEFINITION 3.1. A left (right) $\mathcal{D}$-*module* on X is a sheaf of left (respectively, right) $\mathcal{D}_X$-modules on X and a morphism of $\mathcal{D}$-modules is a morphism of sheaves of $\mathcal{D}_X$-modules. Unless explicitly mentioned otherwise, we assume that all our $\mathcal{D}_X$-modules are *left* $\mathcal{D}_X$-modules.

Note that a left or right $\mathcal{D}_X$-module $\mathcal{M}$ is automatically an $\mathcal{O}_X$-module via the injective homomorphism $\mathcal{O}_X \hookrightarrow \mathcal{D}_X$. We say that $\mathcal{M}$ is *quasi-coherent* if it is quasi-coherent as an $\mathcal{O}_X$-module. A *coherent* (left or right) $\mathcal{D}_X$-module $\mathcal{M}$ is a $\mathcal{D}_X$-module which is quasi-coherent and that is locally finitely generated over $\mathcal{D}_X$ (that is, for every[1] affine open subset $U \subseteq X$, the (left or right) $\mathcal{D}_X(U)$-module $\mathcal{M}(U)$ is finitely generated).

REMARK 3.2. It is clear that the category of quasi-coherent left (or right) $\mathcal{D}_X$-modules is Abelian. The same property holds for the category of coherent left (or right) $\mathcal{D}_X$-modules by Corollary 2.22.

EXAMPLE 3.3. It is clear that $\mathcal{D}_X$ has a natural structure of left $\mathcal{D}_X$-module and of right $\mathcal{D}_X$-module.

EXAMPLE 3.4. Since $\mathcal{D}_X \subseteq \mathcal{E}nd_k(\mathcal{O}_X)$, it follows that $\mathcal{D}_X$ naturally acts on $\mathcal{O}_X$ making $\mathcal{O}_X$ a left $\mathcal{D}_X$-module.

REMARK 3.5. Part of the motivation for the development of the theory of $\mathcal{D}$-modules is that it provides an algebraic approach to studying solutions of linear partial differential equations. For a concrete example, suppose that $k = \mathbf{C}$, $X = \mathbf{A}^n_{\mathbf{C}}$, and $A_n = A_n(\mathbf{C})$. If $M = \mathbf{C}[x_1, \ldots, x_n]$ and $N = A_n/A_n(P_1, \ldots, P_r)$, for some $P_1, \ldots, P_r \in A_n$, then the elements of $\mathrm{Hom}_{A_n}(N, M)$ are the polynomial solutions f of the system of equations

$$P_1 f = \ldots = P_r f = 0.$$

Of course, depending on the context, one can replace M by other spaces of functions (for example, by the rings of holomorphic or smooth functions on $\mathbf{C}^n$).

[1] It is standard to see, using the fact that $\mathcal{D}_X$ is a quasi-coherent $\mathcal{O}_X$-module by Corollary 2.15, that it is enough to know that X has an affine open covering $X = \bigcup_{i \in I} U_i$ such that $\mathcal{M}(U_i)$ is a finitely generated $\mathcal{D}_X(U_i)$-module.

3.2. Integrable connections

Our next goal is to describe the structure of left $\mathcal{D}_X$-modules in terms of integrable connections.

DEFINITION 3.6. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. A *connection* on $\mathcal{M}$ is a k-linear morphism

$$\nabla\colon \mathcal{M} \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

that satisfies the *Leibniz condition*

$$\nabla(fu) = f\nabla(u) + df \otimes u \quad \text{for all} \quad f \in \mathcal{O}_X, u \in \mathcal{M}.$$

REMARK 3.7. Since Ω_X is a locally free $\mathcal{O}_X$-module whose dual is $\mathcal{T}_X$, it follows that giving a connection ∇ on $\mathcal{M}$ is equivalent to giving, for every open subset $U \subseteq X$ and $\xi \in \mathcal{T}_X(U)$, a k-linear map $\nabla_\xi\colon \mathcal{M}|_U \to \mathcal{M}|_U$ (compatible with restriction), that satisfies

i) $\nabla_{f\xi} = f\nabla_\xi$ for all $f \in \mathcal{O}_U$;
ii) $\nabla_\xi(fu) = f\nabla_\xi(u) + \xi(f)u$ for all $f \in \mathcal{O}_U$ and $u \in \mathcal{M}|_U$.

Indeed, given ∇, we take ∇_ξ to be the composition

$$\mathcal{M}|_U \xrightarrow{\nabla|_U} \Omega_U \otimes_{\mathcal{O}_U} \mathcal{M}|_U \xrightarrow{\xi\otimes 1_{\mathcal{M}}} \mathcal{M}|_U.$$

We also note that if $x_1, \ldots, x_n$ are algebraic coordinates on $U \subseteq X$, then

$$\nabla(u) = \sum_{i=1}^n dx_i \otimes \nabla_{\partial_i}(u) \quad \text{for every} \quad u \in \mathcal{O}_U.$$

Given a connection ∇ on $\mathcal{M}$, we define

$$\nabla\colon \Omega_X^i \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega_X^{i+1} \otimes_{\mathcal{O}_X} \mathcal{M}$$

for $i \geq 1$ by

$$\nabla(\eta \otimes u) = d\eta \otimes u + (-1)^i \eta \wedge \nabla(u) \quad \text{for all} \quad \eta \in \Omega_X^i, u \in \mathcal{M},$$

where $\eta \wedge \nabla(u)$ denotes the image of $\eta \otimes \nabla(u) \in \Omega_X \otimes \Omega_X^i \otimes \mathcal{M}$ via

$$\Omega_X \otimes \Omega_X^i \otimes \mathcal{M} \to \Omega_X^{i+1} \otimes \mathcal{M}, \quad \eta_1 \otimes \eta_2 \otimes u \to (\eta_1 \wedge \eta_2) \otimes u.$$

EXERCISE 3.8. Show that for every $\eta \in \Omega_X^i$ and every $u \in \mathcal{M}$, we have

$$\nabla^2(\eta \otimes u) = \eta \wedge \nabla^2(u).$$

DEFINITION 3.9. A connection ∇ on $\mathcal{M}$ is *integrable* (or *flat*) if $\nabla^2(u) = 0$ for every $u \in \mathcal{M}$.

PROPOSITION 3.10. Giving a left $\mathcal{D}_X$-module $\mathcal{M}$ is equivalent to giving an $\mathcal{O}_X$-module $\mathcal{M}$, with an integrable connection ∇; the relation between the two structures is given by

$$\xi \cdot u = \nabla_\xi(u) \quad \text{for every} \quad \xi \in \mathcal{T}_X, u \in \mathcal{M}.$$

PROOF. Giving a left $\mathcal{D}_X$-module structure on the sheaf $\mathcal{M}$ is equivalent to giving a morphism of sheaves of rings $\mathcal{D}_X \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M})$. The assertion follows easily from the description of $\mathcal{D}_X$ on affine open subsets in Proposition 2.25: for example, the relation ii) in that proposition corresponds to the fact that $\nabla_{f\xi} = f\nabla_\xi$, the relation iii) corresponds to the Leibniz condition for ∇, while the relation iv) is equivalent with the integrability condition (this is easiest to see using local coordinates). We leave checking the details as an exercise. $\square$

EXAMPLE 3.11. The standard $\mathcal{D}_X$-module structure on $\mathcal{O}_X$ corresponds to the *de Rham connection* on $\mathcal{O}_X$ given by $d\colon \mathcal{O}_X \to \Omega_X$.

REMARK 3.12. If $(\mathcal{M}, \nabla)$ and $(\mathcal{N}, \nabla)$ are $\mathcal{O}_X$-modules with integrable connection, then a morphism of modules with integrable connection $f\colon (\mathcal{M}, \nabla) \to (\mathcal{N}, \nabla)$ is a morphism of $\mathcal{O}_X$-modules such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\nabla"] \arrow[d, "f"'] & \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \arrow[d, "1\otimes f"] \\ \mathcal{N} \arrow[r, "\nabla"] & \Omega_X \otimes_{\mathcal{O}_X} \mathcal{N}, \end{tikzcd}$$

is commutative. It is then clear that a morphism of left $\mathcal{D}_X$-modules corresponds to a morphism of $\mathcal{O}_X$-modules with integrable connection.

REMARK 3.13. If $\mathcal{M}$ is a left $\mathcal{D}_X$-module on X and ∇ is the corresponding integrable connection, then it follows from Exercise 3.8 that we have a complex

$$0 \to \mathcal{M} \xrightarrow{\nabla} \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\nabla} \ldots \xrightarrow{\nabla} \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{M} \to 0,$$

where $n = \dim(X)$. This is the *de Rham complex* $\mathrm{DR}_X(\mathcal{M})$ of $\mathcal{M}$. Note that the maps are *not* $\mathcal{O}_X$-linear, but only k-linear. Our convention will be to consider $\mathrm{DR}_X(\mathcal{M})$ as placed in cohomological degrees $-n, \ldots, 0$. Note that if $x_1, \ldots, x_n$ are algebraic coordinates on an open subset $U \subseteq X$, then the differential in the de Rham complex of $\mathcal{M}$ is given on U by

$$d\big(\eta \otimes u\big) = d\eta \otimes u + \sum_{j=1}^{n} dx_j \wedge \eta \otimes \partial_j u. \tag{3.1}$$

EXAMPLE 3.14. If $\mathcal{M} = \mathcal{O}_X$, then we recover the usual de Rham complex on X (shifted by n).

We now use the description of left $\mathcal{D}_X$-modules in terms of integrable connections to show that any localisation of a D_R-module is again a D_R-module.

EXAMPLE 3.15. Suppose that X is affine and $R = \mathcal{O}_X(X)$. If M is a left D_R-module and $S \subseteq R$ is any multiplicative system, then the localization $S^{-1}M$ carries a unique structure of D_R-module such that the canonical map $M \to S^{-1}M$ is a morphism of D_R-modules. Indeed, the fact that $M \to S^{-1}M$ is a morphism of modules with connection and the Leibniz rule imply that for every $\xi \in T_R = \mathcal{T}_X(X)$, we need to have

$$\nabla_\xi(u) = \nabla_\xi\left(s \cdot \frac{u}{s}\right) = s\nabla_\xi\left(\frac{u}{s}\right) + \xi(s) \cdot \frac{u}{s},$$

hence we need to have

$$\nabla_\xi\left(\frac{u}{s}\right) = \frac{\nabla_\xi(u)}{s} - \frac{\xi(s)u}{s^2}.$$

It is then straightforward to see that this formula defines a map $\nabla_\xi\colon S^{-1}M \to S^{-1}M$ for every ξ and that in this way we get an integrable connection on $S^{-1}M$.

In particular, by globalizing this construction, we see that for every smooth variety X and every nonzero $f \in \mathcal{O}_X(X)$, the sheaf $\mathcal{O}_X\left[\frac{1}{f}\right]$ is a quasi-coherent $\mathcal{D}_X$-module.

EXAMPLE 3.16. If $f \in \mathcal{O}_X(X)$ defines a smooth hypersurface in X, then $\mathcal{O}_X\left[\frac{1}{f}\right]$ is generated over $\mathcal{D}_X$ by $\frac{1}{f}$; in particular, it is a coherent $\mathcal{D}_X$-module (note that it is not coherent as an $\mathcal{O}_X$-module, unless f is invertible). Indeed, working locally we may assume that we have algebraic coordinates $x_1, \ldots, x_n$ on X such that $f = x_1$. The assertion then follows from the fact that for every $j \geq 1$, we have $\partial_1^j \frac{1}{x_1} = \frac{(-1)^j j!}{x_1^{j+1}}$.

It is an important result, that we will discuss later, saying for every nonzero f, the $\mathcal{D}_X$-module $\mathcal{O}_X\left[\frac{1}{f}\right]$ is coherent.

EXERCISE 3.17. It follows from the above example that there are left ideals I and J in $A_n(k)$ such that

$$A_n(k)/I \simeq k[x_1, \ldots, x_n]_{x_1} \quad \text{and} \quad A_n(k)/J \simeq k[x_1, \ldots, x_n]_{x_1}/k[x_1, \ldots, x_n].$$

Show that $I = A_n(k) \cdot (\partial_1 x_1, \partial_2, \ldots, \partial_n)$ and $J = A_n(k) \cdot (x_1, \partial_2, \ldots, \partial_n)$.

3.3. $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules

The "best" $\mathcal{D}_X$-modules are the ones that are coherent as $\mathcal{O}_X$-modules. We now show that these are, in fact, locally free $\mathcal{O}_X$-modules (that is, in light of Proposition 3.10, these are precisely locally free sheaves on X with integrable connection).

PROPOSITION 3.18. If $\mathcal{M}$ is a left $\mathcal{D}_X$-module which is a coherent $\mathcal{O}_X$-module, then $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module.

PROOF. We need to show that for every point $P \in X$, the stalk $\mathcal{M}_P$ is a free $\mathcal{O}_X$-module. After possibly replacing X by a suitable open neighborhood of P, we may and will assume that we have algebraic coordinates $x_1, \ldots, x_n \in \mathcal{O}_X(X)$ centered at P. Since $\mathcal{M}_P$ is a finitely generated $\mathcal{O}_{X,P}$-module, we may choose a minimal system of generators $u_1, \ldots, u_d \in \mathcal{M}_P$ over $\mathcal{O}_{X,P}$. We need to show that there is no relation $\sum_{i=1}^d f_i u_i = 0$, with $f_i \in \mathcal{O}_{X,P}$ not all 0. If such a relation exists, let $r = \min_i \mathrm{ord}_P(f_i)$. Recall that for $f \in \mathcal{O}_{X,P}$, we put $\mathrm{ord}_P(f) = \infty$ if $f = 0$ and $\mathrm{ord}_P(f) = k$ if $f \in \mathfrak{m}_P^k \smallsetminus \mathfrak{m}_P^{k+1}$, where $\mathfrak{m}_P$ is the maximal ideal in $\mathcal{O}_{X,P}$; equivalently, we have $\mathrm{ord}_P(f) = \min\left\{|\alpha|, \partial^\alpha f(P) \neq 0\right\}$.

We argue by induction on r. If $r = 0$, then there is i such that $f_i(P) \neq 0$, so u_i lies in the $\mathcal{O}_{X,P}$-submodule generated by $\{u_j \mid j \neq i\}$, a contradiction. On the other hand, if $r \geq 1$ and i_0 is such that $\mathrm{ord}_P(f_{i_0}) = r$, then there is j such that $\mathrm{ord}_P\left(\frac{\partial f_{i_0}}{\partial x_j}\right) = r - 1$. In this case we have

$$0 = \sum_{i=1}^n \partial_j f_i u_i = \sum_{i=1}^n f_i(\partial_j u_i) + \sum_{i=1}^n \frac{\partial f_i}{\partial x_j} u_i.$$

If we write

$$\partial_j u_i = \sum_{k=1}^n g_{ik} u_k \quad \text{for} \quad 1 \leq i \leq n,$$

with $g_{ik} \in \mathcal{O}_{X,P}$, then we obtain the relation $\sum_{k=1}^n h_k u_k = 0$, where

$$h_k = \frac{\partial f_k}{\partial x_j} + \sum_i f_i g_{ik}.$$

In particular, we have $\mathrm{ord}_P(h_{i_0}) = r - 1$, hence we are done by induction. $\square$

3.4. Good filtrations on coherent $\mathcal{D}_X$-modules

Since $\mathcal{D}_X$ is a sheaf of *noncommutative* rings, the way to study a $\mathcal{D}_X$-module $\mathcal{M}$ is by considering a suitable filtration $F_\bullet\mathcal{M}$ on $\mathcal{M}$ such that $\mathrm{gr}^F_\bullet(\mathcal{M})$ becomes a sheaf of modules over the sheaf of *commutative* rings $\mathrm{gr}^F_\bullet(\mathcal{D}_X)$.

It is convenient to give the relevant definitions and results in a more general setting. Let $\mathcal{R}$ be a sheaf of rings on X (here X can be any Noetherian scheme). We assume that we have a morphism of sheaves of rings $\mathcal{O}_X \to \mathcal{R}$, so every left $\mathcal{R}$-module is automatically a $\mathcal{O}_X$-module; we say that it is *quasi-coherent* if it is quasi-coherent as an $\mathcal{O}_X$-module. We assume that we have a fixed filtration $F_\bullet\mathcal{R}$ on $\mathcal{R}$ by quasi-coherent $\mathcal{O}_X$-submodules (so, in particular, $\mathcal{R}$ is quasi-coherent) that satisfies the following two extra conditions:

- C_1) $\mathrm{Gr}^F_\bullet(\mathcal{R})$ is a sheaf of *commutative* rings.
- C_2) For every affine open subset $U \subseteq X$, the $\mathcal{O}_X(U)$-algebra $\Gamma\big(U, \mathrm{Gr}^F_\bullet(\mathcal{R})\big)$ is generated by finitely many homogeneous elements of degree 1.

Note that these conditions are satisfied when $\mathcal{R} = \mathcal{D}_X$, with $F_\bullet\mathcal{D}_X$ being the order filtration. Arguing as in the proof of Corollary 2.22, we see that condition C_2) above implies that for every affine open subset $U \subseteq X$, the ring $\mathcal{R}(U)$ is both left and right Noetherian.

DEFINITION 3.19. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{R}$-module. A *filtration* $F_\bullet\mathcal{M}$ on $\mathcal{M}$ (with respect to the filtration $F_\bullet\mathcal{R}$) is given by a family $(F_q\mathcal{M})_{q\in\mathbf{Z}}$ of quasi-coherent $\mathcal{O}_X$-submodules of $\mathcal{M}$ that satisfies the following conditions:

- i) $F_q\mathcal{M} \subseteq F_{q+1}\mathcal{M}$ for every $q \in \mathbf{Z}$;
- ii) $F_q\mathcal{M} = 0$ for $q \ll 0$;
- iii) $\mathcal{M} = \bigcup_{q\in\mathbf{Z}} F_q\mathcal{M}$;
- iv) $F_p\mathcal{D}_X \cdot F_q\mathcal{M} \subseteq F_{p+q}\mathcal{M}$ for all $p, q \in \mathbf{Z}$.

If $F_\bullet\mathcal{M}$ is a filtration on the $\mathcal{R}$-module $\mathcal{M}$, it is clear that if

$$\mathrm{gr}^F_\bullet(\mathcal{M}) = \bigoplus_{q\in\mathbf{Z}} \mathrm{gr}^F_q(\mathcal{M}), \quad \text{where} \quad \mathrm{gr}^F_m(\mathcal{M}) = F_q\mathcal{M}/F_{q-1}\mathcal{M},$$

then $\mathrm{gr}^F_\bullet(\mathcal{M})$ is a sheaf of $\mathrm{gr}^F_\bullet(\mathcal{R})$-modules on X, which is quasi-coherent as an $\mathcal{O}_X$-module.

REMARK 3.20. In the case of interest for us, when $\mathcal{R} = \mathcal{D}_X$, with the order filtration, consider the canonical morphism $\pi\colon Y = \mathcal{S}pec(\mathrm{gr}^F_\bullet(\mathcal{D}_X) \to X$. In this case, we have a quasi-coherent sheaf $\mathcal{F}$ on Y such that $\mathrm{gr}^F_\bullet(\mathcal{M}) \simeq \pi_*(\mathcal{F})$. Recall that by Theorem 2.21, the morphism π is canonically isomorphic to the cotangent bundle $T^*X \to X$ of X.

Returning to the general setting, we will be interested in filtrations that satisfy an extra finiteness condition.

DEFINITION 3.21. The filtration $F_\bullet\mathcal{M}$ on the $\mathcal{D}_X$-module $\mathcal{M}$ is a *good filtration* if it satisfies the following extra conditions:

- i) $F_q\mathcal{M}$ is a coherent $\mathcal{O}_X$-module for every $q \in \mathbf{Z}$.
- ii) There is q such that

$$F_{p+q}\mathcal{M} = F_p\mathcal{D}_X \cdot F_q\mathcal{M} \quad \text{for all} \quad p \geq 0$$

(in this case we will say that the filtration is generated at level q).

PROPOSITION 3.22. The filtration $F_\bullet\mathcal{M}$ on the $\mathcal{R}$-module $\mathcal{M}$ is good if and only if $\mathrm{gr}_\bullet^F(\mathcal{M})$ is a locally finitely generated $\mathrm{gr}_\bullet^F(\mathcal{R})$-module.

PROOF. Both properties are local, hence we may and will assume that X is an affine variety and $A = \mathcal{O}_X(X)$, $R = \Gamma(X, \mathcal{R})$, and $M = \Gamma(X, \mathcal{M})$. Recall that by assumption, there are $y_1, \ldots, y_r \in F_1R$ such that $\mathrm{Gr}_\bullet^F(R)$ is generated as an A-algebra by $\overline{u_1}, \ldots, \overline{u_r} \in \mathrm{Gr}_1^F(R)$.

It is clear that if $\bigoplus_{q\in\mathbf{Z}}(F_qM/F_{q-1}M)$ is generated by the homogeneous elements $u_1, \ldots, u_N$, with $\deg(u_i) = d_i$, then

$$F_qM = F_{q-1}M + \sum_{i=1}^{N} F_{q-d_i}R \cdot u_i \quad \text{for all} \quad q \in \mathbf{Z}. \tag{3.2}$$

Since $F_qM = 0$ for $q \ll 0$, it follows by induction on q that F_qM is a finitely generated A-module if we know that F_qR is a finitely generated A-module for every q. This in turn follows by induction on q, using the fact that

$$F_qR = F_{q-1}R + \sum_{|\alpha|=q} Ay^\alpha. \tag{3.3}$$

Furthermore, we deduce from (3.2) and (3.3) that $F_qM = F_1R \cdot F_{q-1}M$ for $q \geq \max_i d_i$.

Conversely, suppose that $F_\bullet M$ is a good filtration on M. Condition ii) in the definition implies that $\mathrm{Gr}_\bullet^F(M)$ is generated over $\mathrm{Gr}_\bullet^F(R)$ by $\bigoplus_{i\leq q}\mathrm{Gr}_i^F(M)$. Since we have only finitely many nonzero such summands and each of them is finitely generated over A, it follows that $\mathrm{Gr}_\bullet^F(M)$ is a finitely generated $\mathrm{Gr}_\bullet^F(R)$-module. $\square$

REMARK 3.23. Since $\mathrm{Gr}_\bullet^F(\mathcal{R})$ is clearly finitely generated over itself, it follows that $F_\bullet\mathcal{R}$ is a good filtration on $\mathcal{R}$.

The following result will allow us to compare two good filtrations.

PROPOSITION 3.24. If $F_\bullet\mathcal{M}$ and $F'_\bullet\mathcal{M}$ are filtrations on $\mathcal{M}$, and $F_\bullet\mathcal{M}$ is good, then there is $\ell \geq 0$ such that

$$F_k\mathcal{M} \subseteq F'_{k+\ell}\mathcal{M} \quad \text{for all} \quad k \in \mathbf{Z}. \tag{3.4}$$

If $F'_\bullet\mathcal{M}$ is good too, then we may choose ℓ such that we also have

$$F'_{k-\ell}\mathcal{M} \subseteq F_k\mathcal{M} \quad \text{for all} \quad k \in \mathbf{Z}.$$

PROOF. Let q be such that

$$F_{p+q}\mathcal{M} = F_p\mathcal{R} \cdot F_q\mathcal{M} \quad \text{for all} \quad p \geq 0. \tag{3.5}$$

Since $F_q\mathcal{M}$ is a coherent $\mathcal{O}_X$-module and $\mathcal{M} = \bigcup_i F'_i\mathcal{M}$, it follows that there is $\ell \geq 0$ such that $F_q\mathcal{M} \subseteq F'_{q+\ell}\mathcal{M}$. In this case, it follows from (3.5) that

$$F_{p+q}\mathcal{M} \subseteq F'_{p+q+\ell}\mathcal{M} \quad \text{for all} \quad p \geq 0.$$

Therefore (3.4) holds for all $k \geq q$. On the other hand, since there are only finitely many nonzero $F_k\mathcal{M}$ with $k < q$ and they are all coherent $\mathcal{O}_X$-modules, we can ensure that (3.4) also holds for $k < q$ after possibly enlarging ℓ. The last assertion in the proposition follows by switching the roles of $F_\bullet\mathcal{M}$ and $F'_\bullet\mathcal{M}$. $\square$

We next give a criterion for the existence of a good filtration.

PROPOSITION 3.25. A quasi-coherent $\mathcal{R}$-module $\mathcal{M}$ has a good filtration if and only if it is locally finitely generated over $\mathcal{R}$.

PROOF. If $\mathcal{M}$ has a good filtration $F_\bullet\mathcal{M}$ which is generated at level q, it follows that $\mathcal{M}$ is generated by $F_q\mathcal{M}$ as an $\mathcal{R}$-module. Since $F_q\mathcal{M}$ is a coherent $\mathcal{O}_X$-module, it follows that $\mathcal{M}$ is locally finitely generated over $\mathcal{R}$.

Conversely, suppose that $\mathcal{M}$ is a coherent R-module. The first observation is that there is a coherent $\mathcal{O}_X$-submodule $\mathcal{M}_0$ of $\mathcal{M}$ that generates $\mathcal{M}$ over $\mathcal{R}$. Indeed, we can cover X by finitely many affine open subsets U_i, and for every i, we can find a coherent $\mathcal{O}_{U_i}$-submodule $\mathcal{M}_i \subseteq \mathcal{M}|_{U_i}$ that generates $\mathcal{M}|_{U_i}$ over $\mathcal{R}|_{U_i}$. We can fine a coherent submodule $\mathcal{M}'_i$ of $\mathcal{M}$ such that $\mathcal{M}'_i|_{U_i} = \mathcal{M}_i$ (see [**Har77**, Exercise II.5.15]). It is then clear that $\mathcal{M}_0 = \sum_i \mathcal{M}'_i$ has the desired property.

Given $\mathcal{M}_0$ as above, let us define $F_p\mathcal{M} = F_p\mathcal{R} \cdot \mathcal{M}_0$. It is straightforward to check that this is a filtration on $\mathcal{M}$ and the fact that it is a good filtration follows from the fact that this is the case for $F_\bullet\mathcal{D}_X$ (see Remark 3.23). $\square$

We now go back to the setting of interest of us, when $\mathcal{R} = \mathcal{D}_X$ and $F_\bullet\mathcal{R}$ is the order filtration. A *filtered* $\mathcal{D}_X$-module is a coherent $\mathcal{D}_X$-module, endowed with a good filtration.

REMARK 3.26. Given a filtered $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$, the de Rham complex of $\mathcal{M}$ has a filtration such that

$$F_p\mathrm{DR}_X(\mathcal{M}): \ \ 0 \to F_p\mathcal{M} \to \Omega^1_X \otimes_{\mathcal{O}_X} F_{p+1}\mathcal{M} \to \ldots \to \Omega^n_X \otimes_{\mathcal{O}_X} F_{p+n}\mathcal{M} \to 0$$

(the fact that this is a subcomplex of $\mathrm{DR}_X(\mathcal{M})$ follows from the formula (3.1)). Moreover, that formula shows that each graded piece $\mathrm{Gr}^F_p\mathrm{DR}_X(\mathcal{M})$

$$0 \to \mathrm{Gr}^F_p(\mathcal{M}) \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathrm{Gr}^F_{p+1}(\mathcal{M}) \to \ldots \to \Omega^n_X \otimes_{\mathcal{O}_X} \mathrm{Gr}^F_{p+n}(\mathcal{M}) \to 0$$

is a complex of $\mathcal{O}_X$-modules: indeed, if $x_1, \ldots, x_n$ are algebraic coordinates on $U \subseteq X$, then the differential is given on U by

$$d(\eta \otimes \overline{u}) = \sum_{j=1}^n dx_i \wedge \eta \otimes \overline{\partial_i u}$$

and this is clearly $\mathcal{O}_X$-linear.

REMARK 3.27. Whenever we have a good filtration $F_\bullet\mathcal{M}$ on a $\mathcal{D}_X$-module $\mathcal{M}$, the coherent $\mathcal{O}_X$-modules $F_p\mathcal{M}$ and $\mathrm{Gr}^F_p(\mathcal{M})$ provide invariants of $(\mathcal{M}, F_\bullet\mathcal{M})$ in the familiar realm of coherent sheaves on X. In general, these depend very much on the choice of the filtration. However, when discussing Hodge modules in the next chapter, we will see that many interesting $\mathcal{D}_X$-modules carry canonical good filtrations, in which case, by looking at the (graded) pieces of the filtration, we get interesting invariants of these $\mathcal{D}_X$-modules.

We end this section by discussing the most important geometric invariant associated to a coherent $\mathcal{D}_X$-module. This is defined by choosing a good filtration, but in such a way that it is independent of the choice of filtration.

DEFINITION 3.28. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module on X. Let $F_\bullet\mathcal{M}$ be a good filtration on X, $\pi\colon T^*X \to X$ the cotangent bundle of X, and $\mathcal{F}$ the coherent sheaf on T^*X such that $\pi_*(\mathcal{F}) \simeq \mathrm{Gr}^F_\bullet(\mathcal{M})$. The *characteristic variety* $\mathrm{Char}(\mathcal{M})$ is the support of $\mathcal{F}$ (that is, it is the closed subset of T^*X defined by $\mathrm{Ann}_{\mathcal{O}_{T^*X}}(\mathcal{F})$).

REMARK 3.29. Note that since $\mathrm{Gr}^F_\bullet(\mathcal{M})$ is a sheaf of *graded* $\mathrm{Gr}^F_\bullet(\mathcal{D}_X)$-modules, its annihilator is a sheaf of homogeneous ideals, hence $\mathrm{Char}(\mathcal{M}) \subseteq T^*X$ is a *conical* subvariety (that is, it is preserved by the standard action of k^* on T^*X).

PROPOSITION 3.30. The characteristic variety of $\mathcal{M}$ is independent of the choice of good filtration $F_\bullet\mathcal{M}$.

PROOF. We need to show that if $\mathcal{F}$ and $\mathcal{F}'$ are the coherent sheaves on T^*X corresponding, respectively, to the good filtrations $F_\bullet\mathcal{M}$ and $F'_\bullet\mathcal{M}$, then $\mathrm{rad}\big(\mathrm{Ann}(\mathcal{F})\big) = \mathrm{rad}\big(\mathrm{Ann}(\mathcal{F}')\big)$. By Proposition 3.24, there is $k \geq 0$ such that

$$F'_q\mathcal{M} \subseteq F_{q+k}\mathcal{M} \subseteq F'_{q+2k}\mathcal{M} \quad \text{for all} \quad q \in \mathbf{Z}. \tag{3.6}$$

If $f \in \Gamma\big(U, \mathrm{Gr}^F_d(\mathcal{D}_X)\big)$, then f lies in $\mathrm{Ann}(\mathcal{F})$ if and only if $f \cdot F_q\mathcal{M} \subseteq F_{q+d-1}\mathcal{M}$ on U for all q. In this case, it follows from (3.6) that for every $r \geq 1$, we have on U

$$f^r \cdot F'_q\mathcal{M} \subseteq f^r \cdot F_{q+k}\mathcal{M} \subseteq F_{q+k+r(d-1)}\mathcal{M} \subseteq F'_{q+2k+r(d-1)}\mathcal{M}$$

for all $q \in \mathbf{Z}$. It follows that if $r \geq 2k+1$, so that $q + 2k + r(d-1) \leq q + rd - 1$ for all q, then $f^r \in \mathrm{Ann}(\mathcal{F}')$. This shows that $\mathrm{rad}\big(\mathrm{Ann}(\mathcal{F})\big) \subseteq \mathrm{rad}\big(\mathrm{Ann}(\mathcal{F}')\big)$ and the opposite inclusion follows by symmetry. □

REMARK 3.31. For a coherent $\mathcal{D}_X$-module $\mathcal{M}$, we have $\mathrm{Char}(\mathcal{M}) = \emptyset$ if and only if $\mathrm{Gr}^F_\bullet(\mathcal{M}) = 0$, which is the case if and only if $\mathcal{M} = 0$.

DEFINITION 3.32. The *dimension* $\dim(\mathcal{M})$ of a nonzero coherent $\mathcal{D}_X$-module $\mathcal{M}$ is the dimension of its characteristic variety. It is clear that we have $\dim(\mathcal{M}) \leq 2n$. We make the convention that the dimension of the zero module is -1.

The following proposition describes the behavior of the characteristic variety for the $\mathcal{D}_X$-modules in a short exact sequence.

PROPOSITION 3.33. Given a short exact sequence of coherent $\mathcal{D}_X$-modules

$$0 \to \mathcal{M}' \xrightarrow{i} \mathcal{M} \xrightarrow{p} \mathcal{M}'' \to 0$$

and a good filtration $F_\bullet\mathcal{M}$ on $\mathcal{M}$, if we put $F_q\mathcal{M}' = i^{-1}(F_q\mathcal{M})$ and $F_q\mathcal{M}'' = p(F_q\mathcal{M})$, then

i) $F_\bullet\mathcal{M}'$ and $F_\bullet\mathcal{M}''$ are good filtrations on $\mathcal{M}'$ and $\mathcal{M}''$, respectively, and we have a short exact sequence of $\mathrm{gr}^F_\bullet(\mathcal{D}_X)$-modules
$$0 \to \mathrm{Gr}^F_\bullet(\mathcal{M}') \to \mathrm{Gr}^F_\bullet(\mathcal{M}) \to \mathrm{Gr}^F_\bullet(\mathcal{M}'') \to 0.$$
ii) In particular, we have $\mathrm{Char}(\mathcal{M}) = \mathrm{Char}(\mathcal{M}') \cup \mathrm{Char}(\mathcal{M}'')$ and $\dim(\mathcal{M}) = \max\big\{\dim(\mathcal{M}'), \dim(\mathcal{M}'')\big\}$.

PROOF. The fact that $F_\bullet\mathcal{M}'$ and $F_\bullet\mathcal{M}''$ are filtrations and the exactness of the sequence in i) are straightforward to check. Since $\mathrm{Gr}^F_\bullet(\mathcal{M})$ is a locally finitely generated $\mathrm{Gr}^F_\bullet(\mathcal{D}_X)$-module, we deduce from the exact sequence that $\mathrm{Gr}^F_\bullet(\mathcal{M}')$ and $\mathrm{Gr}^F_\bullet(\mathcal{M}'')$ are locally finitely generated as well, hence the filtrations on $\mathcal{M}'$ and $\mathcal{M}''$ are good by Proposition 3.22. The assertions in ii) follow from the behavior of the support for modules in a short exact sequence. □

EXAMPLE 3.34. If $\mathcal{E}$ is a nonzero $\mathcal{D}_X$-module on X which is coherent as an $\mathcal{O}_X$-module, then we have a good filtration on $\mathcal{E}$ given by $F_p\mathcal{E} = \mathcal{E}$ for $p \geq 0$ and $F_p\mathcal{E} = 0$ for $p < 0$. In this case we see that $\mathrm{Gr}^F_\bullet(\mathcal{E})$ is the structure sheaf of the 0-section of T^*X, hence $\mathrm{Char}(\mathcal{E})$ is the 0-section. In particular, we have $\dim(\mathcal{E}) = n$.

EXAMPLE 3.35. We have $\mathrm{Char}(\mathcal{D}_X) = T^*X$ (we can use the order filtration on $\mathcal{D}_X$ as a good filtration), so $\dim(\mathcal{D}_X) = 2n$.

EXAMPLE 3.36. Let X be affine with $A = \mathcal{O}_X(X)$, and let $P \in D_R$ be nonzero $\mathcal{M} = D_R/D_R \cdot P$. Show that $\dim(M) = 2n - 1$.

EXERCISE 3.37. Show that if $X = \mathbf{A}^1_k$ and $M = k[x, x^{-1}]/k[x]$, then $\mathrm{Char}(M)$ is the fiber of T^*X over $0 \in X$. In particular, we have $\dim(M) = 1$.

A fundamental result about $\mathcal{D}_X$-modules, which underpins much of the theory, is the following

THEOREM 3.38. *If X is a smooth n-dimensional variety and $\mathcal{M}$ is a nonzero coherent $\mathcal{D}_X$-module, then* $\dim(\mathcal{M}) \geq n$. *Moreover, every irreducible component of* $\mathrm{Char}(X)$ *has dimension* $\geq n$.

A version of this theorem was proved by Bernstein for $\mathcal{D}$-modules on the affine space in [**Ber71**] (we will discuss this proof, following [**Cou95**], in Chapter 4). We will give a proof of the theorem in the general setting in Chapter 6, after discussing the Sato-Kashiwara filtration and the Kashiwara equivalence theorem. In the next section we discuss, without proof, a stronger version of the theorem that makes reference to the symplectic variety structure on the cotangent bundle.

The above theorem leads to the following central concept:

DEFINITION 3.39. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ on the smooth n-dimensional variety X is *holonomic* if $\mathcal{M} = 0$ or $\dim(\mathcal{M}) = n$.

We do not begin to discuss now the main properties of this notion, leaving this for Chapter 6.

3.5. Involutivity of the characteristic variety

We will not make use of the material in this section, but it is hard to discuss $\mathcal{D}$-modules without mentioning it. Recall that a symplectic variety over k is a smooth irreducible variety Y over k, together with a 2-form ω which is nondegenerate[2] and closed[3].

If X is a smooth, irreducible variety over k, then its cotangent bundle has a canonical structure of symplectic variety over k. This is given as follows. Let $\pi\colon Y = T^*X \to X$ be the canonical projection. We first define a 1-form λ on Y by mapping $u \in T_\alpha Y$, with $\alpha \in T^*_x X$ to $\lambda(\alpha) := \langle \alpha, \pi_*(\alpha)\rangle \in k$, where $\pi_*\colon T_\alpha Y \to T_x X$ is the tangent map of π at α and $\langle -, -\rangle\colon T^*_x X \times T_x X \to k$ is the standard pairing.

In order to see that this is an algebraic 1-form, consider algebraic coordinates $x_1, \ldots, x_n$ on an open subset $U \subseteq X$. We get an isomorphism $\varphi\colon \pi^{-1}(U) \simeq U \times k^n$, and we consider on $\pi^{-1}(U)$ the algebraic coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$, where $p_i = x_i \circ \pi$ and $\varphi(v) = \big(\pi(v), q_1(v), \ldots, q_n(v)\big)$.

[2] This means that for every $P \in Y$, the corresponding alternating bilinear form $T_P Y \times T_P Y \to k$ is non-degenerate.

[3] This means that $d\omega = 0$.

EXERCISE 3.40. Show that with the above notation, we have

$$\lambda|_{\pi^{-1}(U)} = \sum_{i=1}^{n} q_i dp_i.$$

It follows from the above exercise that λ is an algebraic 1-form. We take $\omega = d\lambda$. This is clearly closed. It is also nondegenerate, since with the notation in the exercise, we have $\omega|_{\pi^{-1}(U)} = \sum_{i=1}^{n} dq_i \wedge dp_i$.

Recall now that if V is a finite-dimensional k-vector space and $\eta \colon V \times V \to k$ is a non-degenerate alternating bilinear form, then for a linear subspace $W \subseteq V$, its *orthogonal subspace*

$$W^{\perp} := \{v \in V \mid \eta(v, w) = 0 \text{ for all } w \in W\}$$

has dimension $2n - \dim(W)$, where $\dim(V) = 2n$ (note that the dimension of V has to be even). The linear subspace W is *isotropic* if $W \subseteq W^{\perp}$, it is *coisotropic* if $W^{\perp} \subseteq W$, and it is Lagrangian if $W^{\perp} = W$. The formula for $\dim(W^{\perp})$ implies that if W is isotropic, then $\dim(W) \leq n$ and if W is coisotropic, then $\dim(W) \geq n$. Moreover, W is Lagrangian if and only if it is coisotropic and $\dim(W) = n$.

DEFINITION 3.41. Let (Y, ω) be an arbitrary symplective variety over k. A subvariety Z of Y (not necessarily irreducible) is *isotropic* (*involutive*, *Lagrangian*) if for every smooth point $z \in Z$, the linear subspace $T_z Z \subseteq T_z Y$ is isotropic (respectively coisotropic, Lagrangian).

REMARK 3.42. We will see in Proposition 3.50 below that a subvariety Z of a sympletic variety (Y, ω) is involutive if and only if $T_z Z^{\perp} \subseteq T_z Z$ for *every* $z \in Z$; moreover, it is enough to check this property on some dense open subset of X. This immediately implies that Z is involutive (Lagrangian) if and only if every irreducible component of Z satisfies this property.

REMARK 3.43. Note that if Z is an involutive (Lagrangian) subvariety of (Y, ω), then every irreducible component of Z has dimension $\geq \frac{1}{2}\dim(Y)$ (respectively, $= \frac{1}{2}\dim(Y)$). Moreover, Z is Lagrangian if and only if it is involutive and $\dim(Z) = \frac{1}{2}\dim(Y)$ if and only if $\omega|_{Z_{\mathrm{sm}}} = 0$ and every irreducible component of Z has dimension $\frac{1}{2}\dim(Y)$.

In light of the above remark, Bernstein's inequality for the dimensions of the irreducible components of the characteristic variety (see Theorem 3.38) is a consequence of the following deep result.

THEOREM 3.44. *If X is a smooth irreducible variety over k and $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module on X, then* $\mathrm{Char}(\mathcal{M}) \subseteq T^*X$ *is an involutive subvariety with respect to the standard symplectic structure on the cotangent bundle.*

The first proof of this theorem (in the analytic setting) was due to Sato, Kashiwara, and Kawai [**SKK73**] and used hard analytic tools. An algebraic proof of a more general result was obtained by Gabber in [**Gab81**]. While quite intricate, this is elementary, but we do not discuss it in this course (though we give the statement of Gabber's result below).

REMARK 3.45. In light of Remark 3.43, it follows from Theorem 3.44 that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ on a smooth variety X is holonomic if and only if every irreducible component of $\mathrm{Char}(\mathcal{M})$ is a Lagrangian subvariety of T^*X. Recall that $\mathrm{Char}(\mathcal{M})$ is a conical subvariety of T^*), hence all its irreducible components have the same property. We note that the irreducible conical Lagrangian subvarieties of $T^*(X)$ are easy to describe. Note first that if Z is a smooth subvariety of X with ideal sheaf $\mathcal{I}_Z$, then the *conormal subvariety* of Z in X is

$$T^*_Z X = \mathcal{S}pec\big(\mathrm{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{N}_{Z/X})\big) \subseteq \pi^{-1}(Z) \subseteq T^*X,$$

where $\mathcal{N}_{Z/X} = (\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee$ is the normal sheaf of Z in X. The conormal subvariety has dimension $n = \dim(X)$ and it is clearly conical. Moreover, if $x_1, \ldots, x_n$ are local coordinates on $U \subseteq X$ such that $Z \cap U$ is defined in U by $(x_1, \ldots, x_r)$, and if $p_1, \ldots, p_n, q_1, \ldots, q_n$ are the local coordinates on $\pi^{-1}(U) \subseteq T^*X$ in Exercise 3.40, then $T^*_Z X \cap \pi^{-1}(U)$ is defined in $\pi^{-1}(U)$ by $(p_1, \ldots, p_r, q_{r+1}, \ldots, q_n)$. Since $\omega|_{\pi^{-1}(U)} = \sum_i dq_i \wedge dp_i$, we conclude that $\omega|_{T^*_Z X} = 0$. Therefore $T^*_Z X$ is a conical Lagrangian subvariety of T^*X. More generally, if Z is a closed irreducible subvariety of X, with smooth locus $Z_{\mathrm{sm}} = V \cap Z$, for some open subset V in X, then $T^*_Z X := \overline{T^*_{Z_{\mathrm{sm}}} V} \subseteq T^*X$ is a conical Lagrangian subvariety of T^*X. It is not hard to see that every irreducible conical Lagrangian subvariety Λ of T^*X is of this form: more precisely if $Z = \pi(\Lambda)$ (note that this is closed, since Λ is a conical subvariety), then $\Lambda = T^*_Z X$ (see [**CG10**, Lemma 1.3.27]).

In order to state Gabber's result, we first introduce the notion of Poisson algebra.

Definition 3.46. Let A be a commutative k-algebra. A *Poisson bracket* on A is a map $\{-,-\}\colon A \times A \to A$ such that the following conditions are satisfied:

i) The pair $\big(A, \{-,-\}\big)$ is a Lie algebra over k.
ii) For every $a \in T$, then map $\{a,-\}\colon T \to T$ satisfies the Leibniz rule (hence it is a k-derivation).

A commutative algebra endowed with a Poisson bracket is a *Poisson algebra*.

Example 3.47. Suppose that B is an associative k-algebra, endowed with a filtration $F_\bullet B$. If the associated graded ring $S = \mathrm{Gr}^F_\bullet(B)$ is commutative, then S is endowed with a Poisson bracket, defined as follows.

Given two elements $a \in \mathrm{Gr}^F_p(B)$ and $b \in \mathrm{Gr}^F_q(B)$, let us choose lifts $a' \in F_pB$ and $b' \in F_qB$. Since a and b commute in S, it follows that $a'b' - b'a' \in F_{p+q-1}B$. We define $\{a,b\} \in \mathrm{Gr}^F_{p+q-1}(B)$ to be the class of $a'b' - b'a'$. It is straightforward to see that this is independent of the choice of lifts for a and b and that it extends by additivity to a map $S \times S \to S$ that is a Poisson bracket.

Example 3.48. Suppose that (Y, ω) is a symplectic affine algebraic variety over k and $A = \mathcal{O}_X(X)$. Since ω is nondegenerate, for every $f \in A$, we have a unique derivation $\xi_f \in \Gamma(X, \mathcal{T}_X)$ given by $\omega(-, \xi_f) = df$. If for $f, g \in A$ we put

$$\{f, g\} := \omega(\xi_f, \xi_g) = \xi_f(g),$$

then one can check that this gives a Poisson bracket on A.

Remark 3.49. Suppose that X is a smooth irreducible affine algebraic variety over k and $R = \mathcal{O}_X(X)$ and $T_X = \Gamma(X, \mathcal{T}_X)$. In this case the order filtration on D_R induces by Example 3.47 a Poisson algebra structure on $A = \mathrm{Gr}^F_\bullet(D_R) \simeq \mathcal{O}_Y(Y)$, where $Y = T^*X$. This coincides with the Poisson structure induced by the standard symplectic form on Y via Example 3.48. In order to check this, we may assume that we have $x_1, \ldots, x_n \in R$ giving algebraic coordinates on X, and let $p_1, \ldots, p_n, q_1, \ldots, q_n \in \operatorname{Sym}^\bullet_R(T_X)$ be the corresponding algebraic coordinates on Y, so $p_i = x_i$ and $q_i = \overline{\partial_i} \in T_X$. Recall that in this case we have $\omega = \sum_{i=1}^n dq_i \wedge dp_i$ and an easy computation gives

$$\xi_{p_i} = -\partial_{q_i} \quad \text{and} \quad \xi_{q_i} = \partial_{p_i} \quad \text{for} \quad 1 \leq i \leq n,$$

hence the bracket associated to the symplectic form satisfies

$$\{p_i, g\} = -\frac{\partial g}{\partial q_i} \quad \text{and} \quad \{q_i, g\} = \frac{\partial g}{\partial p_i}.$$

It is easy to deduce that for every $f_1, f_2 \in R$, we have $\{f_1, f_2\} = 0$, $\{q_i, f_1\} = \frac{\partial f_1}{\partial x_i}$, and $\{q_i, q_j\} = 0$ for all i and j. These formulas are also satisfied by the Poisson bracket given by Example 3.47 and they uniquely determine a Poisson bracket.

For a symplectic affine algebraic variety, we can describe the involutive property of a subvariety in terms of the corresponding Poisson bracket.

PROPOSITION 3.50. Let Y be an affine symplectic algebraic variety over k and consider on $S = \mathcal{O}_Y(Y)$ the Poisson bracket defined in Example 3.48. If Z is a closed subvariety of Y, with corresponding radical ideal $I \subseteq S$, and if $Z_0 \subseteq Z$ is an open dense subset, then the following are equivalent:

i) The ideal I satisfies $\{I, I\} \subseteq I$.
ii) $T_z Z^\perp \subseteq T_z Z$ for every $z \in Z_0$.

PROOF. Note first that for every $z \in Z$, we have

$$T_z Z = \big\{u \in T_z Y \mid df_z(u) = 0 \quad \text{for all} \quad f \in I\big\}.$$

It follows that $v \in T_z X$ lies in $T_z Z^\perp$ if and only if

$$\bigcap_{f \in I} \ker(df_z) \subseteq \operatorname{Ker}\big(\omega_z(-, v)\big),$$

which is the case if and only if $\omega_z(-, v)$ lies in the linear span of $df_z = \omega_z\big(-, \xi_f(z)\big)$, for $f \in I$. We thus conclude that

$$T_z Z^\perp = \big\{\xi_f(z) \mid f \in I\big\}.$$

Note now that since Z_0 is dense in Z, the condition in i) is equivalent to the fact that for every $f, g \in I$, we have $\{f, g\}(z) = 0$ for all $z \in Z_0$. By definition of the Poisson bracket, we have

$$\{f, g\}(z) = \omega\big(\xi_f(z), \xi_g(z)\big) = dg_z\big(\xi_f(z)\big).$$

By what we have seen, for a given $g \in I$ we have $\{f, g\}(z) = 0$ for all $f \in I$ if and only if dg_z vanishes on $T_z Z^\perp$. This is turn holds for all $g \in I$ if and only if $T_z Z^\perp \subseteq T_z Z$, completing the proof of the equivalence. $\square$

The following is Gabber's theorem [**Gab81**].

THEOREM 3.51. *Let $(B, F_\bullet B)$ be an associative ring with a filtration such that $\operatorname{Gr}^F_\bullet(B)$ is commutative and Noetherian. If M is a B-module with a filtration $F_\bullet M$ compatible with $F_\bullet B$ such that $\operatorname{Gr}^F_\bullet(M)$ is a finitely generated $\operatorname{Gr}^F_\bullet(B)$-module, with annihilator J, and $I = \operatorname{rad}(J)$, then*

$$\big\{I, I\big\} \subseteq I.$$

REMARK 3.52. If X is a smooth variety over k and $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, in order to check that $\mathrm{Char}(\mathcal{M})$ is an involutive variety, it is enough to do it after restricting to a suitable affine open cover of X. We may thus assume that X is affine, with $R = \mathcal{O}_X(X)$, and applying Gabber's theorem for the order filtration on D_R, we obtain the fact that $\mathrm{Char}(\mathcal{M})$ is involutive using the interpretation in Proposition 3.50.

REMARK 3.53. In the setting of Theorem 3.51, it is easy to see that $\{J, J\} \subseteq J$. Indeed, if $a, b \in J$ are homogeneous, of degree p and q, respectively, and if we pick lifts $a' \in F_pB$ and $b' \in F_qB$, then

$$a' \cdot F_iM \subseteq F_{i+p-1}M \quad \text{and} \quad b' \cdot F_iM \subseteq F_{i+q-1}M \quad \text{for all} \quad i \in \mathbf{Z}.$$

Let $c = a'b' - b'a' \in F_{p+q-1}A$, so that c lies over $\{a, b\}$. For every $u \in F_iM$, we have

$$cu = a'(b'u) - b'(a'u) \in a' \cdot F_{i+q-1}M + b' \cdot F_{i+p-1}M \subseteq F_{i+p+q-2}M,$$

hence $\{a, b\} \in I$.

The subtlety regarding the statement of Theorem 3.51 is that in general it is not true that if J is an ideal such that $\{J, J\} \subseteq J$, then $\{\sqrt{J}, \sqrt{J}\} \subseteq \sqrt{J}$.

3.6. Left versus right $\mathcal{D}_X$-modules

The discussion in Section 3.4 applies as well for right $\mathcal{D}_X$-modules if instead of taking $\mathcal{R} = \mathcal{D}_X$, we take $\mathcal{R} = \mathcal{D}_X^{\mathrm{op}}$. In particular, we may consider the characteristic variety and the dimension of a right $\mathcal{D}_X$-module.

In what follows, we will be mainly interested in left $\mathcal{D}_X$-modules, but right $\mathcal{D}_X$ modules appear naturally when defining duality and push-forward for $\mathcal{D}_X$-modules. In this section we discuss an equivalence between the categories of left and right $\mathcal{D}_X$-modules that will allow us to switch between the two categories whenever convenient.

We will define this equivalence globally, but in order to check the desired properties, we will argue locally, in the presence of coordinates. The main ingredient is the following

LEMMA 3.54. *If $x_1, \ldots, x_n$ are algebraic coordinates on the smooth variety X, then there is an isomorphism $\tau\colon \mathcal{D}_X \to \mathcal{D}_X^{\mathrm{op}}$ of sheaves of rings that is the identity on $\mathcal{O}_X$ and such that $\tau(\partial_i) = -\partial_i$ for $1 \leq i \leq n$. Moreover, we have $\tau^2 = \mathrm{Id}$.*

PROOF. It is enough to prove the assertion when X is affine. In this case it is an immediate consequence of the presentation of $\mathcal{D}_X$ in Corollary 2.26. ☐

As usual, we work on a smooth irreducible n-dimensional algebraic variety X over k. We review two definitions that are algebraic versions of familiar definitions in the context of smooth manifolds.

DEFINITION 3.55. If P is a vector field on X, then for every i, the *contraction by P* is the morphism of sheaves

$$i_P\colon \Omega_X^{i+1} \to \Omega_X^i \quad \text{given by} \quad i_P\eta(Q_1, \ldots, Q_i) = \omega(P, Q_1, \ldots, Q_i)$$

(here we think of forms as alternating multilinear functions on vector fields). The *Lie derivative* with respect to P is the morphism of sheaves

$$\mathcal{L}_P\colon \Omega_X^i \to \Omega_X^i \quad \text{given by} \quad \mathcal{L}_P = i_P \circ d + d \circ i_P.$$

REMARK 3.56. We will only be interested in the Lie derivative on $\omega_X = \Omega_X^n$. Note that $\mathcal{L}_P(\eta) = d(i_P\eta)$ for every n-form η. Let us describe this in terms of algebraic coordinates $x_1, \ldots, x_n$ on $U \subseteq X$: if $P = \sum_{i=1}^n P_i\partial_i$ is a vector field on U and $\eta = f\,dx$ is an n-form on U, where $dx = dx_1 \wedge \ldots \wedge dx_n$, then

$$i_P(\eta)(\partial_1, \ldots, \widehat{\partial_i}, \ldots, \partial_n) = \eta(P, \partial_1, \ldots, \widehat{\partial_i}, \ldots, \partial_n) = (-1)^{i-1} f P_i,$$

hence

$$i_P(\eta) = \sum_{i=1}^n (-1)^{i-1} f P_i\, dx_1 \wedge \ldots \widehat{dx_i} \wedge \ldots \wedge dx_n$$

and thus

$$\mathcal{L}_P(\eta) = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} P_i + \frac{\partial P_i}{\partial x_i} f \right) dx = \left(P(f) + f \sum_{i=1}^n \frac{\partial P_i}{\partial x_i} \right) dx.$$

PROPOSITION 3.57. The standard structure of $\mathcal{O}_X$-module on ω_X can be (uniquely) extended to a right $\mathcal{D}_X$-module structure such that for every $\eta \in \Gamma(U, \omega_X)$ and $P \in \Gamma(U, \mathcal{T}_X)$, we have

$$\eta \cdot P = -\mathcal{L}_P(\eta).$$

PROOF. It is enough to check this locally, hence we may assume that we have coordinates $x_1, \ldots, x_n$ on X, in which case we have an isomorphism $\mathcal{O}_X \simeq \omega_X$ that maps 1 to $dx = dx_1 \wedge \ldots \wedge dx_n$. Via this isomorphism, and using Lemma 3.54 and the fact that a right $\mathcal{D}_X$-module structure is the same as a left $\mathcal{D}_X^{\mathrm{op}}$-module structure, we see that the assertion in the proposition is equivalent to the fact that the standard $\mathcal{O}_X$-module structure on $\mathcal{O}_X$ can be (uniquely) extended to a left $\mathcal{D}_X$-module structure such that for every vector field P and every regular function f, if we write $\tau(P) = Q + g$, for a vector field Q and a regular function g, and if $L_Q(f dx) = h\,dx$, then $P \cdot f = fg - h$. If $P = \sum_i P_i\partial_i$, then

$$\tau(P) = -\sum_{i=1}^n \partial_i P_i = -P + \sum_{i=1}^n \tfrac{\partial P_i}{\partial x_i},$$

hence $Q = -P$ and $g = \sum_i \frac{\partial P_i}{\partial x_i}$. It follows from the description of the Lie derivative in Remark 3.56 that

$$h = -P(f) + f \cdot \sum_{i=1}^n \tfrac{\partial P_i}{\partial x_i},$$

hence $fg - h = P(f)$. Therefore this is precisely the standard $\mathcal{D}_X$-module structure on $\mathcal{O}_X$. $\square$

Recall that $\mathcal{E}nd_k(\mathcal{O}_X)$ has compatible structures of left and right $\mathcal{O}_X$-modules (in other words, it is an $\mathcal{O}_X$-$\mathcal{O}_X$-bimodule) such that for local sections $\varphi \in \mathcal{E}nd_k(\mathcal{O}_X)$ and $a, b \in \mathcal{O}_X$, $a \cdot \varphi \cdot b$ maps u to $a\varphi(ub)$. If $\mathcal{L}$ is a line bundle on X, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E}nd_k(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}$ is a sheaf of rings on X, with multiplication given by

$$(a \otimes \varphi \otimes b) \cdot (c \otimes \psi \otimes d) = a \otimes \varphi\alpha(bc)\psi \otimes d,$$

where $\alpha\colon \mathcal{L}^{-1} \otimes \mathcal{L} \to \mathcal{O}_X$ is the canonical isomorphism. In fact, we have a ring homomorphism

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E}nd_k(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \to \mathcal{E}nd_k(\mathcal{L}) \tag{3.7}$$

that maps $a\otimes\varphi\otimes b$ to the map $u\mapsto\varphi\big(\alpha(b\otimes u)\big)a$. By looking at an open cover on which $\mathcal{L}$ is trivial, we see that (3.7) is an isomorphism.

We apply this with $\mathcal{L}=\omega_X$. The right $\mathcal{D}_X$-module structure on ω_X corresponds to a left $\mathcal{D}_X^{\rm op}$-module structure, which gives a morphism of sheaves of rings

$$\mathcal{D}_X^{\rm op}\to\mathcal{E}nd_k(\omega_X)\simeq\omega_X\otimes_{\mathcal{O}_X}\mathcal{E}nd_k(\mathcal{O}_X)\otimes_{\mathcal{O}_X}\omega_X^{-1}. \tag{3.8}$$

We can easily describe this locally: if $x_1,\ldots,x_n$ are algebraic coordinates on $U\subseteq X$, then (3.8) maps P to $dx\otimes\tau(P)\otimes dx^{-1}$. As a consequence, we conclude that (3.8) gives an isomorphism of sheaves of rings

$$\mathcal{D}_X^{\rm op}\simeq\omega_X\otimes_{\mathcal{O}_X}\mathcal{D}_X\otimes_{\mathcal{O}_X}\omega_X^{-1}. \tag{3.9}$$

We can now give the equivalence between the categories of left and right $\mathcal{D}_X$-modules. Let us denote by $\mathcal{D}_X-\text{mod}$ and $\text{mod}-\mathcal{D}_X$ the categories of left, respectively right, $\mathcal{D}_X$-modules. Note that if $\mathcal{M}$ is a left $\mathcal{D}_X$-module, then $\omega_X\otimes_{\mathcal{O}_X}\mathcal{M}$ is a left module over $\omega_X\otimes_{\mathcal{O}_X}\mathcal{D}_X\otimes_{\mathcal{O}_X}\omega_X^{-1}$, with scalar multiplication given by

$$(a\otimes P\otimes b)\cdot(c\otimes u)=a\otimes P\alpha(b\otimes c)u.$$

Via the isomorphism (3.9), this makes $\omega_X\otimes_{\mathcal{O}_X}\mathcal{M}$ a left $\mathcal{D}_X^{\rm op}$-module, hence a right $\mathcal{D}_X$-module. This structure is easy to make explicit locally: if we have algebraic coordinates $x_1,\ldots,x_n$ on $U\subseteq X$, then on U we have $(dx\otimes u)P=dx\otimes\tau(P)u$. Let $F\colon\mathcal{D}_X-\text{mod}\to\text{mod}-\mathcal{D}_X$ be this functor.

Going in the opposite direction, if $\mathcal{N}$ is a right $\mathcal{D}_X$-module, then $\mathcal{N}\otimes_{\mathcal{O}_X}\omega_X^{-1}$ is a right module over $\omega_X\otimes_{\mathcal{O}_X}\mathcal{D}_X\otimes_{\mathcal{O}_X}\omega_X^{-1}$, with scalar multiplication given by

$$(u\otimes c)\cdot(a\otimes P\otimes b)=u\alpha(c\otimes a)P\otimes b.$$

Via the isomorphism (3.9), this makes $\mathcal{N}\otimes_{\mathcal{O}_X}\omega_X^{-1}$ a right $\mathcal{D}_X^{\rm op}$-module, hence a left $\mathcal{D}_X$-module. If $x_1,\ldots,x_n$ are algebraic coordinates on U, then on U we have $P\cdot(u\otimes dx^{-1})=u\tau(P)\otimes dx^{-1}$. We denote by G this functor $\text{mod}-\mathcal{D}_X\to\mathcal{D}_X-\text{mod}$.

Note that if we start with a left $\mathcal{D}_X$-module $\mathcal{M}$, then we have an isomorphism of $\mathcal{O}_X$-modules

$$G\big(F(\mathcal{M})\big)=\omega_X\otimes_{\mathcal{O}_X}\mathcal{M}\otimes_{\mathcal{O}_X}\omega_X^{-1}\simeq\mathcal{M}.$$

By looking in local coordinates, we see that this is in fact an isomorphism of left $\mathcal{D}_X$-modules. Similarly, we see that for every right $\mathcal{D}_X$-module $\mathcal{N}$, we have a functorial isomorphism of right $\mathcal{D}_X$-modules $G\big(F(\mathcal{N})\big)\simeq\mathcal{N}$. We thus have proved

PROPOSITION 3.58. The above functors F and G give inverse equivalences between $\mathcal{D}_X-\text{mod}$ and $\text{mod}-\mathcal{D}_X$.

REMARK 3.59. It is clear that via the equivalence of categories in the proposition the left $\mathcal{D}_X$-module $\mathcal{O}_X$ corresponds to the right $\mathcal{D}_X$-module ω_X. Another interesting example is that of a left $\mathcal{D}_X$-module $\mathcal{M}=\mathcal{D}_X\otimes_{\mathcal{O}_X}\mathcal{F}$, for some $\mathcal{O}_X$-module $\mathcal{F}$, that uses the left $\mathcal{D}_X$-module structure on $\mathcal{D}_X$. In this case we have

$$F(\mathcal{M})=\omega_X\otimes_{\mathcal{O}_X}\mathcal{D}_X\otimes_{\mathcal{O}_X}\mathcal{F}\simeq(\omega_X\otimes_{\mathcal{O}_X}\mathcal{D}_X\otimes_{\mathcal{O}_X}\omega_X^{-1})\otimes_{\mathcal{O}_X}(\omega_X\otimes_{\mathcal{O}_X}\mathcal{F}),$$

hence $F(\mathcal{M})\simeq(\omega_X\otimes_{\mathcal{O}_X}\mathcal{F})\otimes_{\mathcal{O}_X}\mathcal{D}_X$, where the right-hand side uses the right $\mathcal{D}_X$-module structure on $\mathcal{D}_X$.

EXERCISE 3.60. Show that if $\mathcal{M}$ and $\mathcal{N}$ are left $\mathcal{D}_X$-modules and $\mathcal{M}^r$ and $\mathcal{N}^r$ are the corresponding right $\mathcal{D}_X$-modules, then there is a canonical isomorphism

$$\mathcal{M}^r\otimes_{\mathcal{D}_X}\mathcal{N}\simeq\mathcal{N}^r\otimes_{\mathcal{D}_X}\mathcal{M}.$$

DEFINITION 3.61. The *Spencer complex* of a right $\mathcal{D}_X$-module $\mathcal{N}$ is the de Rham complex of the corresponding left $\mathcal{D}_X$-module.

REMARK 3.62. Since the left $\mathcal{D}_X$-module $\mathcal{M}$ associated to a right $\mathcal{D}_X$-module $\mathcal{N}$ is isomorphic as an $\mathcal{O}_X$-module to $\mathcal{N} \otimes_{\mathcal{O}_X} \omega_X^{-1}$ and for every p we have

$$\Omega_X^p \otimes_{\mathcal{O}_X} \omega_X^{-1} \simeq (\Omega_X^{n-p})^\vee \simeq \wedge^{n-p}\mathcal{T}_X,$$

it follows that the Spencer complex of $\mathcal{N}$ is given by the complex

$$0 \to \mathcal{N} \otimes_{\mathcal{O}_X} \wedge^n \mathcal{T}_X \to \ldots \to \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{T}_X \to \mathcal{N} \to 0,$$

placed in cohomological degrees $-n, \ldots, 0$. Moreover, it is easy to describe the differential $d\colon \mathcal{N} \otimes_{\mathcal{O}_X} \wedge^p \mathcal{T}_X \to \mathcal{N} \otimes_{\mathcal{O}_X} \wedge^{p-1}\mathcal{T}_X$ in the presence of local coordinates $x_1, \ldots, x_n$: it is given by

$$d(u \otimes \partial_{i_1} \wedge \ldots \wedge \partial_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} u\partial_{i_j} \otimes \partial_{i_1} \wedge \ldots \wedge \widehat{\partial_{i_j}} \wedge \ldots \wedge \partial_{i_p}.$$

REMARK 3.63. As we have already mentioned at the beginning of this section, we may consider filtrations on a right $\mathcal{D}_X$-module $\mathcal{N}$ in the same way that we considered filtrations on a left $\mathcal{D}_X$-module. Given such a filtration $F_\bullet\mathcal{N}$, then the Spencer complex of $\mathcal{N}$ is a filtered complex, where the pth filtered piece of the Spencer complex is given by

$$0 \to F_{p-n}\mathcal{N} \otimes_{\mathcal{O}_X} \wedge^n \mathcal{T}_X \to \ldots \to F_{p-1}\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{T}_X \to F_p\mathcal{N} \to 0.$$

An important convention is that when dealing with filtered $\mathcal{D}_X$-modules, we shift the filtration when passing from left to right $\mathcal{D}_X$-modules as follows: if $F_\bullet\mathcal{M}$ is a filtration on the left $\mathcal{D}_X$-module $\mathcal{M}$, then the corresponding filtration on $\mathcal{M}^r = \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is given by $F_{p-n}\mathcal{M}^r = \omega_X \otimes_{\mathcal{O}_X} F_p\mathcal{M}$. Note that with this convention, the pth filtered piece of the Spencer complex of $\mathcal{M}^r$ is equal to $F_p\mathrm{DR}_X(\mathcal{M})$.

REMARK 3.64. Suppose that $\mathcal{M}$ is a coherent left $\mathcal{D}_X$-module and $\mathcal{N}$ is the corresponding right $\mathcal{D}_X$-module. Note that if $F_\bullet\mathcal{M}$ is a good filtration on $\mathcal{M}$ and $F_\bullet\mathcal{N}$ is the corresponding filtration on $\mathcal{N}$ (with the convention in the previous remark), then we have an isomorphism of graded $\mathrm{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{T}_X)$-modules

$$\mathrm{Gr}^F_\bullet(\mathcal{N}) \simeq \omega_X \otimes_{\mathcal{O}_X} \mathrm{Gr}^F_\bullet(\mathcal{M})(n).$$

In particular, we see that $\mathrm{Char}(\mathcal{M}) = \mathrm{Char}(\mathcal{N})$ and thus $\dim(\mathcal{M}) = \dim(\mathcal{N})$.

EXAMPLE 3.65. Let's consider the de Rham complex of $\mathcal{D}_X$, with the standard left $\mathcal{D}_X$-module structure:

$$0 \to \mathcal{D}_X \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \ldots \to \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to 0. \tag{3.10}$$

Note that this is a complex of right $\mathcal{D}_X$-modules: the fact that the maps are $\mathcal{D}_X$-linear follows from formula (3.1). This gives a resolution of ω_X by free right $\mathcal{D}_X$-modules, where the map $\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \omega_X$ is given by $\eta \otimes P \mapsto \eta P$. In order to show that the resulting complex $C^\bullet$ is exact, it is enough to show that $F_pC^\bullet$ is exact for all p, which in turn follows if we show that $\mathrm{Gr}^F_p(C^\bullet)$ is exact for all p. Recall that the filtration on $\mathrm{DR}_X(\mathcal{D}_X)$ is induced by the order filtration on $\mathcal{D}_X$, while we take the filtration on ω_X such that $\mathrm{Gr}^F_p(\omega_X) = 0$ unless $p = -n$. We thus conclude that $\mathrm{Gr}^F_p(C^\bullet)$ is the complex

$$0 \to \mathrm{Sym}^p_{\mathcal{O}_X}(\mathcal{T}_X) \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathrm{Sym}^{p+1}(\mathcal{T}_X) \to \ldots \to \mathrm{Sym}^{p+n}_{\mathcal{O}_X}(\mathcal{T}_X) \to \mathcal{A}_p \to 0,$$

where $\mathcal{A}_p = \omega_X$ if $p = -n$ and $\mathcal{A}_p = 0$ otherwise. If $\mathcal{S} = \mathrm{Sym}^{\bullet}_{\mathcal{O}_X}(\mathcal{T}_X)$, then the direct sum of these complexes is

$$0 \to \mathcal{S} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{S} \to \ldots \to \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{S} \to \Omega^n_X \to 0.$$

To check that this is an exact complex, we can argue locally: the assertion then follows from the fact that it is obtained by tensoring with Ω^n_X the Koszul complex corresponding to the regular sequence $y_1, \ldots, y_n \in \mathcal{O}_X[y_1, \ldots, y_n]$.

Finally, applying our equivalence of categories to pass from right $\mathcal{D}_X$-modules to left $\mathcal{D}_X$-modules and using Remark 3.59, we see that the complex (3.10) induces a resolution of $\mathcal{O}_X$ by left $\mathcal{D}_X$-modules

$$0 \to \mathcal{D}_X \otimes \wedge^n \mathcal{T}_X \to \ldots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{T}_X \to \mathcal{D}_X \to \mathcal{O}_X \to 0.$$

CHAPTER 4

Holonomic modules over the Weyl algebra

The material in this chapter is not logically necessary for what follows: we treat here the case of modules over the Weyl algebra, as in the original paper of Bernstein [**Ber71**]. Several of the results that we will later prove more generally, via more sophisticated methods, can be seen in a rather elementary fashion in this setting, giving a quick access point to holonomic $\mathcal{D}$-modules on the affine space. The presentation we give follows [**Cou95**].

4.1. The Bernstein filtration

In this chapter it is convenient to work over an arbitrary field k of characteristic 0. We will study modules over the Weyl algebra $A_n(k)$.

PROPOSITION 4.1. The k-algebra $A_n(k)$ has a basis given by $x^\alpha\partial^\beta$, for $\alpha,\beta\in\mathbf{Z}^n_{\geq 0}$.

PROOF. If k is algebraically closed, then the assertion follows from Proposition 2.24 and Theorem 2.11. The general case follows immediately from this one after passing to the algebraic closure of k: note that it follows from the definition of the Weyl algebra that for every field extension K/k, we have an isomorphism $A_n(k)\otimes_k K\simeq A_n(K)$. □

From now on we simply write A_n for $A_n(k)$. What distinguishes this ring of differential operators is that we have another natural filtration on A_n, whose terms are finite-dimensional vector spaces over k. More precisely, we have the following

DEFINITION 4.2. The *Bernstein filtration* on A_n is given by

$$B_pA_n=\bigoplus_{|\alpha|+|\beta|\leq p}kx^\alpha\partial^\beta.$$

REMARK 4.3. Note that in this case B_pA_n is not a module over $k[x_1,\ldots,x_n]$ anymore, but just a k-vector space. All properties of a filtration in Definition 2.8 are clearly satisfied, with the exception of iv), which will be proved in the next lemma. The new feature is that $\dim_k B_pA_n<\infty$ for all p.

LEMMA 4.4. *For every $p,q\in\mathbf{Z}_{\geq 0}$, we have*

i) $B_pA_n\cdot B_qA_n\subseteq B_{p+q}A_n$.
ii) $[B_pA_n,B_qA_n]\subseteq B_{p+q-2}A_n$.

PROOF. The assertion in i) follows if we show that for every $\alpha,\beta\in\mathbf{Z}^n_{\geq 0}$, with $|\alpha|=r$ and $|\beta|=s$, $\partial^\beta x^\alpha\in B_{r+s}A_n$. We argue by induction on s, the case $s=0$ being trivial. The case $s=1$ is also easy: by Lemma 2.1, we have $\partial_ix^\alpha=x^\alpha\partial_i+\alpha_ix^{\alpha-e_i}\in B_{r+1}A_n$. For the induction step, note that if $\beta'=\beta-e_i\in\mathbf{Z}^n_{\geq 0}$, then $\partial^{\beta'} x^{\alpha} \in B_{p+q-1}A_n$ by induction. We thus conclude that $\partial^{\beta} x^{\alpha} \in B_{p+q}A_n$ by the case $s = 1$.

For the assertion in ii), we need to show that for every $\alpha, \beta, \alpha', \beta' \in \mathbf{Z}^n_{\geq 0}$, we have $[x^{\alpha}\partial^{\beta}, x^{\alpha'}\partial^{\beta'}] \in B_{N-2}A_n$, where $N = |\alpha|+|\beta|+|\alpha'|+|\beta'|$. Using the behavior of the bracket with respect to products (see equation (2.1)) and the assertion in i), we see that it is enough to prove this when $|\alpha| + |\beta| = 1 = |\alpha'| + |\beta'|$. This case is clear: the only nontrivial commutators are $\pm[\partial_i, x_i] = \pm 1 \in B_0 A_n$. $\square$

Assertion i) in the above lemma implies that $B_{\bullet}A_n$ is a filtration on the ring A_n and assertion ii) implies that $\mathrm{Gr}^B_{\bullet}(A_n)$ is commutative. Since $\mathrm{Gr}^B_m(A_n)$ is free, with a basis given by the monomials in $x_1, \ldots, x_n, y_1, \ldots, y_n$ of degree m, where $y_i = \overline{\partial_i}$ for $1 \leq i \leq n$, it follows that the unique morphism of graded k-algebras

$$k[x_1, \ldots, x_n, y_1, \ldots, y_n] \to \mathrm{Gr}^B_{\bullet}(A_n)$$

that maps the x_i to x_i and the y_i to y_i is an isomorphism. In particular, $(A_n, B_{\bullet}A_n)$ satisfies conditions C_1) and C_2) in Section 3.4 (note that we are using the formalism about filtrations in Section 3.4 with $X = \mathrm{Spec}(k)$, not $X = \mathbf{A}^n_k$).

Remark 4.5. As in Corollary 2.22, from the fact that $\mathrm{Gr}^B_{\bullet}(A_n)$ is Noetherian, we deduce that A_n is both left and right Noetherian (though, of course, at least when k is algebraically closed, we have already shown this via the order filtration).

In order to fix ideas, we only consider left A_n-modules. For every such module M, we may consider filtrations $B_{\bullet}M$ with respect to $B_{\bullet}A_n$, as introduced in Section 3.4. We note each B_qM is a k-vector subspace of M, but it is not true anymore that B_qM is a module over $k[x_1, \ldots, x_n]$. In fact, if $B_{\bullet}M$ is a good filtration, then $\dim_k B_qM < \infty$ for all q.

Given a finitely generated A_n-module M and a good filtration $B_{\bullet}M$ on M, we again consider the radical I of the annihilator of $\mathrm{Gr}^B_{\bullet}(M)$ over $\mathrm{Gr}^B_{\bullet}(A_n)$. The subvariety of $\mathbf{A}^{2n}_k = \mathrm{Spec}\big(\mathrm{Gr}^B_{\bullet}(A_n)\big)$ defined by I is the *characteristic variety* $\mathrm{Char}_B(M)$. The argument in the proof of Proposition 3.30 applies verbatim to show that the characteristic variety is independent on the choice of good filtration. As before, we have $\mathrm{Char}_B(M) \neq \emptyset$ if and only if $M \neq 0$, and if this is the case, then the *dimension* $\dim(M)$ is $\dim\big(\mathrm{Char}_B(M)\big)$ (we make the convention that $\dim(M) = -1$ if $M = 0$).

Remark 4.6. Note that if $X = \mathbf{A}^n_k$, with k algebraically closed, and M is a finitely generated $A_n(k)$-module, we have two notions of characteristic variety of M, depending whether we work with the order filtration or the Bernstein filtration. The resulting varieties are really different: consider for example the module $M = A_1(k)/A_1(k) \cdot (x^2 + \partial^2)$. In this case $\mathrm{Char}(M)$ is defined by the ideal in $(y) \subseteq k[x, y]$, while $\mathrm{Char}_B(M)$ is defined by $(x^2 + y^2)$ (in particular, one variety is irreducible, while the other one is not). However, we will see in the next chapter that the two notions of dimension coincide, since they admit the same cohomological interpretation.

The advantage in the setting of the Bernstein filtration is that $\mathrm{Gr}^B_{\bullet}(A_n)$ is a polynomial ring in $2n$ variables with the *standard grading* (note that with respect to the order filtration, half of the variables have degree 0). In this case, it is a basic result of commutative algebra that if $B_{\bullet}M$ is a good filtration on the nonzero

A_n-module M and if P is the Hilbert polynomial of the graded $\mathrm{Gr}^B_\bullet(A_n)$-module $\mathrm{Gr}^B_\bullet(M)$, we have

$$\dim\bigl(\mathrm{Gr}^B_\bullet(M)\bigr) = \deg(P) + 1$$

(with the convention that the degree of the zero polynomial is -1). Recall that by definition $P \in \mathbf{Q}[y]$ is such that

$$\dim_k \mathrm{Gr}^F_i(M) = P(i) \quad \text{for all} \quad i \gg 0.$$

For us, it will be more convenient to consider the function $f\colon \mathbf{Z} \to \mathbf{Z}$ given by $f(i) = \dim_k B_iM$. Since $f(i) - f(i-1) = P(i)$ for $i \gg 0$, it follows that there is a polynomial $Q \in \mathbf{Q}[y]$ of degree $d = \dim(M)$ such that $f(i) = Q(i)$ for $i \gg 0$. It is well-known (and easy to see) that since $Q(i) \in \mathbf{Z}_{>0}$ for $i \gg 0$, the top degree term of Q is of the form $\frac{e(M)}{d!}y^d$, for some $e(M) \in \mathbf{Z}_{>0}$. By definition, $e(M)$ is the *multiplicity* of M

REMARK 4.7. If $B_\bullet M$ and Q are as above, then $d = \dim(M)$ and $e = e(M)$ are characterized by the fact that there is $C > 0$ such that

$$|\dim_k(B_iM) - \frac{e}{d!}i^d| \leq Ci^{d-1} \quad \text{for all} \quad i \gg 0.$$

In particular, it follows from the comparison of the terms in two good filtrations (see Proposition 3.24) that the multiplicity of M does not depend on the choice of good filtration.

REMARK 4.8. Given a short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

of finitely generated $A_n(k)$-modules, arguing as in the proof of Proposition 3.33, we see that we can choose filtrations $B_\bullet M'$, $B_\bullet M$, and $B_\bullet M''$ such that for every i we have a short exact sequence of k-vector spaces

$$0 \to B_iM' \to B_iM \to B_iM'' \to 0,$$

hence $\dim_k B_iM = \dim_k B_iM' + \dim_k B_iM''$. This implies that

$$\dim(M) = \max\bigl\{\dim(M'), \dim(M'')\bigr\}.$$

Moreover, if all three modules have the same dimension, then $e(M) = e(M') + e(M'')$.

4.2. The Bernstein inequality for modules over the Weyl algebra

The following is the special case of Theorem 3.38 in the setting of modules over the Weyl algebra.

THEOREM 4.9. *If M is a nonzero finitely generated A_n-module, then* $\dim(M) \geq n$.

REMARK 4.10. Note that unlike in Theorem 3.38, we here do not bound the dimension of every irreducible component of $\mathrm{Char}_B(M)$.

PROOF OF THEOREM 4.9. Let $B_\bullet M$ be a good filtration on M, compatible with $B_\bullet A_n$. After possibly replacing all B_pM by $B_{p+p_0}M$ for some p_0, we may and will assume that $B_0M \neq 0$. The key point is the following
Claim. For every $p \geq 0$, the map

$$B_pA_n \to \mathrm{Hom}_k\bigl(B_pM, B_{2p}M\bigr), \quad Q \mapsto (u \mapsto Qu)$$

is injective.

We prove the claim by induction on p, the case $p = 0$ being clear, since $B_0A_n = k$ and $B_0M \neq 0$. Suppose now that $p \geq 1$ and we know the assertion for $p-1$. If $0 \neq Q \in B_pA_n$, we need to show that there is $u \in B_pM$ such that $Qu \neq 0$. This is clear if $Q \in k$. Otherwise, if we write $Q = \sum_{\alpha,\beta} c_{\alpha,\beta}c_{\alpha,\beta}x^\alpha\partial^\beta$, there is $(\alpha', \beta') \neq (0,0)$ such that $c_{\alpha',\beta'} \neq 0$.

If $\alpha' \neq 0$, let i be such that $\alpha'_i \neq 0$. In this case, using Lemma 2.1, we see that

$$[Q, \partial_i] = -\sum_{\alpha,\beta} c_{\alpha,\beta}\alpha_i c_{\alpha,\beta}x^{\alpha-e_i}\partial^\beta$$

is a nonzero element of $B_{p-1}A_n$. By induction, it follows that we have $v \in B_{p-1}A_n$ such that $[Q, \partial_i]v \neq 0$. Since $Q\partial_i v \neq \partial_i Q_v$, it follows that either $Q\partial_i v \neq 0$ (in which case we may take $u = \partial_i v \in B_pM$) or $\partial_i Qv \neq 0$, hence $Qv \neq 0$, so we may take $u = v \in B_{p-1}M \subseteq B_pM$.

The case when $\beta' \neq 0$ is similar. This completes the proof of the claim. Note now that $\dim_k B_pA_n$ grows like p^{2n}, while if $d = \dim(M)$, then it follows that $\dim_k \mathrm{Hom}_k(B_pM, B_{2p}M)$ grows like $p^d \cdot (2p)^d = 2^d p^{2d}$, and therefore $2d \geq 2n$. This gives the inequality in the theorem. □

As in Section 3.4, we say that a finitely generated A_n-module is *holonomic* if either $M = 0$ or $\dim(M) = n$.

THEOREM 4.11. *The following hold:*

i) *Given an exact sequence*
$$0 \to M' \to M \to M'' \to 0$$
of finitely generated A_n-modules, then M is holonomic if and only if both M' and M'' are holonomic. Moreover, in this case we have $e(M) = e(M') + e(M'')$.

ii) *The category of all holonomic A_n-modules is an Abelian subcategory of the category of all A_n-modules. Moreover, all objects in this category have finite length; in fact, for every M holonomic, we have $\ell(M) \leq e(M)$.*

PROOF. The assertion in i) follows from Remark 4.8. This implies that the category of holonomic A_n-modules has kernels and cokernels. Then the fact that it is an Abelian category is immediate.

We also see that if $M_1 \subsetneq M_2 \subseteq M$ are A_n-modules, with M holonomic, then $e(M_1) < e(M_2) \leq e(M)$ and these are positive integers. This implies that $\ell(M) \leq e(M)$. □

EXAMPLE 4.12. The A_n-module $R := k[x_1, \ldots, x_n] \simeq A_n/A_n(\partial_1, \ldots, \partial_n)$ is holonomic. Indeed, the Bernstein filtration on A_n induces a filtration on R given by $B_pR = \{f \in R \mid \deg(f) \leq p\}$, hence

$$\mathrm{Gr}^B_\bullet(R) \simeq \mathrm{Gr}^B_\bullet(A_n)/(y_1, \ldots, y_n) = k[x_1, \ldots, x_n].$$

Therefore R is holonomic and $e(R) = 1$.

EXAMPLE 4.13. The A_1-module $M = k[x, x^{-1}]/k[x]$ is isomorphic to A_1/A_1x. It is easy to see that if we take on M the filtration induced by the Bernstein filtration on A_1, then $\mathrm{Gr}^B_\bullet(M) \simeq k[x,y]/(x) \simeq k[y]$, hence M is holonomic and $e(M) = 1$. We then conclude using Theorem 6.37 that $k[x, x^{-1}]$ is a holonomic A_1-module, of multiplicity 2.

EXAMPLE 4.14. Let $P \in A_n$ be nonzero, where $n \geq 1$, and let $M = A_n/A_n \cdot P$. Let $d \geq 0$ be such that $P \in B_d A_n \smallsetminus B_{d-1}A_n$. If $d = 0$, then $M = 0$, hence from now on we assume $d \geq 1$. If we write $P = P_0 + Q$, where $P_0 \in B_{d-1}A_n$ and $Q \in \bigoplus_{|\alpha|+|\beta|=d} k \cdot x^\alpha \partial^\beta$, and if we consider on M the filtration induced from A_n, then $\mathrm{Gr}^B_\bullet(M) \simeq k[x_1, \ldots, x_n, y_1, \ldots, y_n]/(Q(x, y))$, hence $\dim(M) = 2n - 1$ and $e(M) = d$. In particular, we see that M is holonomic if and only if $n = 1$.

The first interesting example is provided by the localization of R at a nonzero element (note that in this case, we don't know yet that this is a finitely generated A_n-module).

THEOREM 4.15. *If $f \in R$ is nonzero, then R_f is a holonomic R-module.*

PROOF. Recall that the A_n-module structure on R induces an A_n-module structure on R_f (see Example 3.15). Let $d = \deg(f)$ and let us consider for every $q \geq 0$

$$B_q R_f = \left\{ \frac{g}{f^q} \mid \deg(g) \leq (d+1)q \right\}.$$

It is clear that this is a k-linear subspace of R_f. Note that $\bigcup_{q\geq 0} B_q R_f = R_f$. Indeed, given $\frac{g}{f^p} \in R_f$, we can rewrite this as $\frac{gf^m}{f^{p+m}}$ and we have

$$\deg(gf^m) = md + \deg(g) \leq (d+1)(p+m) \quad \text{for} \quad m \gg 0.$$

Moreover, we have $B_p A_n \cdot B_q R_f \subseteq B_{p+q}R_f$. In order to see this, it is enough to show that $x_i \cdot B_q R_f \subseteq B_{q+1}R_f$ and $\partial_i \cdot B_q R_f \subseteq B_{q+1}R_f$ for all i. Indeed, if $\frac{g}{f^q} \in B_q R_f$, then

$$x_i \tfrac{g}{f^q} = \tfrac{x_i g f}{f^{q+1}} \quad \text{and} \quad \deg(x_i g f) = d + 1 + \deg(g) \leq (d+1)(q+1)$$

and

$$\partial_i \cdot \tfrac{g}{f^q} = \frac{f\frac{\partial g}{\partial x_i} - qg\frac{\partial f}{\partial x_i}}{f^{q+1}} \quad \text{and} \quad \deg\left(f\frac{\partial g}{\partial x_i} - qg\frac{\partial f}{\partial x_i}\right) \leq d+\deg(g)-1 \leq (d+1)(q+1).$$

Therefore $B_\bullet R_f$ is a filtration on R_f (with respect to the Bernstein filtration on A_n).

It follows from the definition that $\dim_k F_q R_f = \binom{(d+1)q+n}{n}$ for every $q \geq 0$. Note that this is a polynomial of degree n in q, with the top degree term $\frac{(d+1)^n q^n}{n!}$. If M is a finitely generated A_n-submodule of R_f and $B_\bullet M$ is a good filtration on M, then it follows from Proposition 3.24 that there is $\ell \geq 0$ such that $B_q M \subseteq B_{q+\ell} R_f \cap M$, hence there is $C > 0$ such that $\dim_k B_q M \leq \frac{(d+1)^n q^n}{n!} + Cq^{n-1}$ for all $q \gg 0$. This implies that M is holonomic and $e(M) \leq (d+1)^n$.

This implies that R_f is a finitely generated A_n-module: otherwise there is an infinite sequence

$$M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq R_f$$

of finitely generated A_n-submodules. Since they are all holonomic, of multiplicity $\leq (d+1)^n$, it follows from Theorem 6.37 that

$$e(M_1) < e(M_2) \ldots \leq (d+1)^n$$

is a bounded strictly increasing sequence of positive integers, a contradiction. Once we know that R_f is finitely generated, it follows from what we have already proved that R_f is holonomic. $\square$

The theory of holonomic modules over the Weyl algebra was developed by Bernstein in [**Ber71**], in order to prove the existence of what is nowadays called the Bernstein-Sato polynomial of a nonzero polynomial $f \in R = k[x_1, \ldots, x_n]$. This follows by making use of the fact that holonomic $A_n\big(k(s)\big)$-modules have finite length. Because of time constraints, we do not discuss this now, since we will prove the existence of the Bernstein-Sato polynomial in general in Chapter 6. In this general setting, the existence of the Bernstein-Sato polynomial is a key ingredient in proving the fact that the localization of a holonomic $\mathcal{D}$-module is again holonomic.

CHAPTER 5

Dimension and codimension of $\mathcal{D}$-modules and the Sato-Kashiwara filtration

Our goal in this chapter is to prove some basic cohomological properties of finitely generated modules over rings of differential properties. In particular, we will obtain a cohomological description of dimension, which will imply that the two notions of dimension that we defined in Chapters 3 and 4 agree. These results will be useful in the proof of the Bernstein inequality in the next chapter and in setting up duality theory of holonomic $\mathcal{D}$-modules.

5.1. First cohomological results

It is convenient to set up the theory in the following general context. We work in the affine case and consider a ring R with a filtration $F_\bullet R$. We assume that $S := \mathrm{Gr}_\bullet^F(R)$ is commutative and[1] generated over R_0, which is Noetherian, by finitely many elements of degree 1 (in order to be in the setting of Section 3.4, we may take $X = \mathrm{Spec}(R_0)$, but X will not play any role in what follows). In particular, the argument in Corollary 2.22 implies that R is both left and right Noetherian.

Note that $F_\bullet R$ can be considered also as a filtration on the opposite ring R^{op}. Since $\mathrm{Gr}_\bullet^F(R)$ is commutative, this can be identified with $\mathrm{Gr}_\bullet^F(R^{\mathrm{op}})$. All our modules will be *left* R-modules; the right R-modules will be treated as left R^{op}-modules.

From now on, we make one extra assumption: we assume that S is a regular ring, with all maximal ideals of codimension d. Note that this condition is satisfied if $R = D_A$ is the ring of differential operators of an affine n-dimensional variety $\mathrm{Spec}(A)$, with the order filtration (in which case $d = 2n$) or if R is the Weyl algebra $A_n(k)$, with the Bernstein filtration (in which case again $d = 2n$). The relevant consequence for us is that the global dimension $\mathrm{gldim}(S)$ is equal to d: this is the content of the result of Auslander-Buchsbaum-Serre (see [**Mat89**, Theorem 19.2]). Recall that if T is an arbitrary ring, then the (left) global dimension of T is

$$\begin{aligned}\mathrm{gldim}(T) &:= \sup\big\{\mathrm{pd}_T(M) \mid M \text{ left module over } T\big\}\\ &= \sup\big\{i \geq 0 \mid \mathrm{Ext}^i_T(M,N) \neq 0 \text{ for some left modules } M, N \text{ over } T\big\}.\end{aligned}$$

Moreover, in this definition we may assume that M is finitely generated and, if T is left Noetherian, that N is finitely generated too (see [**Mat89**, Lemma 2, p.155]). Standard arguments (using, for example, Cartan resolutions) imply that, keeping the assumption that T is left Noetherian, for every bounded complex of left T-modules $A^\bullet$, with finitely generated cohomology modules, there is a bounded complex of finitely generated projective R-modules $P^\bullet$ and a quasi-isomorphism $P^\bullet \to A^\bullet$.

[1]This condition is only added since we used it, for the sake of simplicity, in Section 3.4.

We will make use of the basic results on good filtrations on finitely generated R-modules from Section 3.4. In order to simplify the notation, whenever a good filtration $F_\bullet M$ is chosen on M, we will write $\mathrm{Gr}(M)$ instead of $\mathrm{Gr}_\bullet^F(M)$.

The basic idea is to import cohomological results from the commutative setting via the use of filtrations. The key tool for doing this is the following. Suppose that M is a finitely generated R-module and let us choose a good filtration $F_\bullet M$ on M. A *filtered free resolution* $P^\bullet \to M$ is a complex

$$\ldots \to P^j \to \ldots \to P^{-1} \to P^0 \to M \to 0 \tag{5.1}$$

where each $P = P^j$ is a *filtered finitely generated free* R-module, that is, it is free with certain generators $e_1, \ldots, e_m$ and with filtration given by $F_pP = \bigoplus_i F_{p-d_i}R\cdot e_i$ for all i (for some $d_1, \ldots, d_m \in \mathbf{Z}$), all maps in the complex are morphisms of filtered R-modules[2], and for every $i \in \mathbf{Z}$, the complex

$$\ldots \to \mathrm{Gr}_i^F(P^j) \to \ldots \to \mathrm{Gr}_i^F(P^{-1}) \to \mathrm{Gr}_i^F(P^0) \to \mathrm{Gr}_i^F(M) \to 0$$

is exact. This last condition is equivalent to the fact that for every i, the complex

$$\ldots \to F_iP^j \to \ldots \to F_iP^{-1} \to F_iP^0 \to F_iM \to 0$$

is exact; by taking the direct limit of these complexes, we see that also (5.1) is exact.

LEMMA 5.1. *Given an R-module M with a good filtration $F_\bullet M$, there is a filtered free resolution of M.*

PROOF. Note first that if $u_i \in F_{d_i}M$ for $1 \leq i \leq N$ are such that the $\overline{u_i} \in \mathrm{Gr}_{d_i}^F(M)$ generate $\mathrm{Gr}_\bullet^F(M)$ over S, then $F_pM \subseteq F_{p-1}M + \sum_{i=1}^N F_{p-d_i}R \cdot F_{d_i}M$ for all p. Arguing by induction on p, we deduce that

$$F_pM = \sum_{i=1}^N F_{p-d_i}R \cdot F_{d_i}M \quad \text{for all} \quad p \in \mathbf{Z}.$$

Let $P_0 = \bigoplus_{i=1}^N Re_i$ be a free R-module, with a filtration given by $F_pP_0 = \bigoplus_i F_{p-d_i}e_i$, and let $f\colon P_0 \to M$ be the R-linear map given by $f(e_i) = u_i$ for all i. Note that we have $F_pM = f(F_pP_0)$ for all p. Let $M_1 = \mathrm{Ker}(f)$, with the filtration given by $F_pM_1 = F_pP_0 \cap M_1$. In this case we have a short exact sequence

$$0 \to \mathrm{Gr}(M_1) \to \mathrm{Gr}(P_0) \to \mathrm{Gr}(M) \to 0.$$

In particular, this implies that the filtration on M_1 is good. We can then repeat the process, constructing $P_1 \to M_1$ as above and continuing in this way we obtain a filtered resolution of M. □

PROPOSITION 5.2. Given two finitely generated R-modules M and N endowed with good filtrations, for every i we get a (noncanonical) filtration[3] on $\mathrm{Ext}_R^i(M,N)$ such that the following properties hold:

(Ext1) $\mathrm{Gr}\big(\mathrm{Ext}_R^i(M,N)\big)$ is isomorphic to a subquotient of $\mathrm{Ext}_S^i\big(\mathrm{Gr}(M), \mathrm{Gr}(N)\big)$.

(Ext2) If $\mathrm{Ext}_S^\ell\big(\mathrm{Gr}(M), \mathrm{Gr}(N)\big) = 0$ for all $\ell < i$, then $\mathrm{Gr}\big(\mathrm{Ext}_A^i(M,N)\big)$ is isomorphic to a subobject of $\mathrm{Ext}_S^i\big(\mathrm{Gr}(M), \mathrm{Gr}(N)\big)$.

[2]If $(M, F_\bullet M)$ and $(N, F_\bullet N)$ are filtered R-modules, then an R-linear map $f\colon M \to N$ is a *morphism of filtered modules* if $f(F_pM) \subseteq F_pN$ for all $p \in \mathbf{Z}$.

[3]Note that each $\mathrm{Ext}_R^i(M,N)$ is just an Abelian group, hence a filtration is a family of subgroups that satisfies properties i)-iii) in Definition 3.19.

PROOF. Let $P^\bullet \to M$ be a filtered free resolution as in (5.1). We get an induced good filtration on the terms of the complex $\mathrm{Hom}_R(P^\bullet, N)$ by putting

$$F_p\mathrm{Hom}_R(P^j, N) = \big\{\varphi \in \mathrm{Hom}_R(P^j, N) \mid \varphi(F_iP^j) \subseteq F_{i+p}N \text{ for all } i \in \mathbf{Z}\big\}.$$

Note that if $P^j = Re_1 \oplus \ldots \oplus Re_m$ and the filtration is given by $F_pP^j = \bigoplus_i F_{p-d_i}R \cdot e_i$, then

$$F_p\mathrm{Hom}_R(P^j, N) \simeq \bigoplus_i F_{p+d_i}N.$$

This immediately implies that $F_\bullet\mathrm{Hom}_R(P^j, N)$ is indeed a filtration and that the maps in $\mathrm{Hom}_R(P^\bullet, N)$ are morphisms of filtered R-modules. Since we also have $\mathrm{Gr}^F_\bullet(P^j) \simeq \bigoplus_i S(-d_j)$, it is then clear that we have an isomorphism of complexes

$$\mathrm{Gr}\big(\mathrm{Hom}_R(P^\bullet, N)\big) \simeq \mathrm{Hom}_S\big(\mathrm{Gr}(P^\bullet), \mathrm{Gr}(N)\big).$$

Note that the i-th cohomology of the right-hand side complex is $\mathrm{Ext}^i_S\big(\mathrm{Gr}(M), \mathrm{Gr}(N)\big)$.

The spectral sequence associated to the filtered complex $T^\bullet = \mathrm{Hom}_R(P^\bullet, N)$ has

$$E_1^{p,q} = \mathcal{H}^{p+q}\big(\mathrm{Gr}^F_{-p}(T^\bullet\big) = \mathrm{Ext}^{p+q}_S\big(\mathrm{Gr}(M), \mathrm{Gr}(N)\big)_{-p}$$

and we have a filtration on the cohomology of $T^\bullet$ with the property that

$$E_\infty^{p,q} \simeq \mathrm{Gr}^F_{-p}\big(\mathrm{Ext}^{p+q}_R(M, N)\big)$$

(the properties ii) and iii) in Definition 3.19 are satisfied since this is the case for the filtration on each $\mathrm{Hom}_R(P^j, N)$). Note that for every $r \geq 1$ and every i, the graded S-module $E^i_{r+1} := \bigoplus_{q-p=i} E_r^{-p,q}$ is a subquotient of E^i_r. We thus have S-submodules

$$\ldots B^i_r \subseteq B^i_{r-1} \subseteq \ldots \subseteq B^i_2 \subseteq B^i_1 \subseteq A^i_1 \subseteq A^i_2 \subseteq \ldots \subseteq A^i_{r-1} \subseteq A^i_r \subseteq \ldots \subseteq E^i_1$$

such that $E^i_r = A^i_r/B^i_r$ for all $r \geq 1$. Since $E^i_1 \simeq \mathrm{Ext}^i_S\big(\mathrm{Gr}(M), \mathrm{Gr}(N)\big)$ is a Noetherian S-module, it follows that there is r_0 such that $B^i_r = B^i_{r_0}$ and thus $E^i_{r+1} \hookrightarrow E^i_r$ for all $r \geq r_0$. Since our filtrations are start at 0 and are exhaustive, it follows that

$$E^i_\infty := \bigoplus_{q-p=i} E_\infty^{-p,q} = \bigcap_{r \geq r_0} E^i_r.$$

Therefore

$$\mathrm{Gr}^F_\bullet\big(\mathrm{Ext}^i_R(M, N)\big) \simeq E^i_\infty$$

is an S-submodule of $E^i_{r_0}$, which is a subquotient of E^i_1. This gives the assertion in (Ext1).

On the other hand, under the assumption in (Ext2), we have $E_1^{p,q} = 0$ whenever $p + q < i$. This implies that $E_r^{p,q} = 0$ for all $r \geq 1$ and $p + q < i$, and we conclude that E^i_{r+1} is a subobject of E^i_r for all $r \geq 1$. By the previous discussion, it follows that $E^i_\infty = E^i_{r_0}$ and this is an S-submodule of E^i_1. This gives the assertion in (Ext1). □

REMARK 5.3. In the setting of Proposition 5.2, we will be especially interested in the case when $N = R$, in which case each $\mathrm{Ext}^i_R(M, R)$ has a natural structure of left R^{op}-module. In this case the filtration constructed in the proposition is a filtration with respect to the filtration $F_\bullet R^{\mathrm{op}}$ on R^{op}. Note that the assertion in (Ext1) implies that the filtration constructed on each $\mathrm{Ext}^i_R(M, R)$ is good.

As a first consequence, we obtain the following

THEOREM 5.4. *The rings R and R^{op} have finite global dimension; in fact, we have* $\mathrm{gldim}(R) \leq d$ *and* $\mathrm{gldim}(R^{\mathrm{op}}) \leq d$.

PROOF. Our hypothesis on R implies that $\mathrm{Ext}^i_S(M', N') = 0$ for all finitely generated S-modules M' and N' and every $i > d$. Condition (Ext1) above then implies that $\mathrm{Ext}^i_R(M, N) = 0$ for all finitely generated R-modules M and N. We thus obtain $\mathrm{gldim}(R) \leq d$ and the same argument applies for R^{op}. $\square$

Let $D^b_{\mathrm{fg}}(R)$ be the triangulated subcategory of the derived category of the category of all R-modules consisting of those u such that $\mathcal{H}^i(u) = 0$ for $|i| \gg 0$ and $\mathcal{H}^i(u)$ is a finitely generated R-module for all $i \in \mathbf{Z}$. Every object in $D^b_{\mathrm{fg}}(R)$ can be represented by a finitely bounded complex of finitely generated R-modules. For a quick overview of notions related to derived categories, see Appendix A.

Since R is an $R-R$-bimodule, it follows that we have an exact functor $\mathbf{D} = \mathbf{D}_R \colon D^b_{\mathrm{fg}}(R) \to D^b_{\mathrm{fg}}(R^{\mathrm{op}})$ given by $\mathbf{D}(M) = \mathbf{R}\mathrm{Hom}_R(M, R)$. Note that by Theorem 5.4, every object $u \in D^b_{\mathrm{fg}}(R)$ can be represented by a bounded complex $P^\bullet$ of finitely generated projective R-modules. In this case $\mathbf{D}_R(u) = \mathrm{Hom}_R(P^\bullet, R)$. Since this is a complex of projective R^{op}-modules, we see that $\mathbf{D}_{R^{\mathrm{op}}}\big(\mathbf{D}_R(u)\big) \in D^b_{\mathrm{fg}}(R)$ is computed by $\mathrm{Hom}_{R^{\mathrm{op}}}\big(\mathrm{Hom}_R(P^\bullet, R), R^{\mathrm{op}}\big)$. The natural transformation given on R-modules by evaluation:

$$M \to \mathrm{Hom}_{R^{\mathrm{op}}}\big(\mathrm{Hom}_R(M, R), R^{\mathrm{op}}\big), u \mapsto \big(\varphi \mapsto \varphi(u)\big) \tag{5.2}$$

extends thus to a natural transformation of functors $\alpha_R \colon \mathrm{Id} \to \mathbf{D}_{R^{\mathrm{op}}} \circ \mathbf{D}_R$.

THEOREM 5.5. *The contravariant functor* $\mathbf{D}_R \colon D^b_{\mathrm{fg}}(R) \to D^b_{\mathrm{fg}}(R^{\mathrm{op}})$ *is an anti-equivalence of categories with inverse* $\mathbf{D}_{R^{\mathrm{op}}}$.

PROOF. The assertion follows if we show that both α_R and $\alpha_{R^{\mathrm{op}}}$ are isomorphisms of functors. In fact, it is enough to treat α_R, since the assertion for $\alpha_{R^{\mathrm{op}}}$ follows by replacing R with R^{op}.

It is clear that $\alpha_R(R)$ is an isomorphism and it is also clear that if M_1 and M_2 are R-modules, then $\alpha_R(M_1 \oplus M_2)$ is an isomorphism if and only if $\alpha_R(M_1)$ and $\alpha_R(M_2)$ are isomorphisms. We conclude from this that $\alpha_R(M)$ is an isomorphism for every finitely generated projective R-module M.

If $P^\bullet$ is a bounded complex of finitely generated R-modules, we conclude that we have an isomorphism of complexes $P^\bullet \to \mathrm{Hom}_{R^{\mathrm{op}}}\big(\mathrm{Hom}_R(P^\bullet, R), R^{\mathrm{op}}\big)$. By the discussion preceding the statement of the theorem, we see that $\alpha_R(u)$ is an isomorphism for every $u \in D^b_{\mathrm{fg}}(R)$.

$\square$

5.2. Dimension and codimension

As in the case of $\mathcal{D}$-modules, we can use good filtrations in order to associate a subset of $\mathrm{Spec}(S)$ to every finitely generated R-module M. Indeed, we choose a good filtration $F_\bullet M$ of M and define the *characteristic variety* of M to be the closed subset of $\mathrm{Spec}(S)$ given by the support of $\mathrm{Gr}(M)$. Arguing as in Proposition 3.30, we see that $\mathrm{Char}(M)$ is independent of the choice of good filtration. The *dimension* of M is $\dim(M) := \dim\big(\mathrm{Char}(M)\big)$. Note that we have $\dim(M) \leq d$.

REMARK 5.6. Arguing as in Proposition 3.33, we see that if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of finitely generated R-modules, then $\mathrm{Char}(M) = \mathrm{Char}(M') \cup \mathrm{Char}(M'')$. In particular, we have $\dim(M) = \max\big\{\dim(M'), \dim(M'')\big\}$.

We now extend the notion of characteristic variety to the objects of the derived category.

DEFINITION 5.7. For a an object $u \in D^b_{\mathrm{fg}}(R)$, we put

$$\mathrm{Char}(u) = \bigcup_{i\in\mathbf{Z}} \mathrm{Char}\big(\mathcal{H}^i(u)\big).$$

REMARK 5.8. Given an exact triangle

$$u \to v \to w \to u[1]$$

in $D^b_{\mathrm{fg}}(R)$, it follows from the long exact sequence in cohomology and Remark 5.6 that $\mathrm{Char}(v) \subseteq \mathrm{Char}(u) \cup \mathrm{Char}(w)$.

We show that this notion is preserved by duality.

COROLLARY 5.9. For every $u \in D^b_{\mathrm{fg}}(R)$, we have $\mathrm{Char}(u) = \mathrm{Char}\big(\mathbf{D}_R(u)\big)$.

PROOF. It follows from Theorem 5.5 that it is enough to prove the inclusion

$$\mathrm{Char}\big(\mathbf{D}_R(u)\big) \subseteq \mathrm{Char}(u). \tag{5.3}$$

Note that, by definition, we have $\mathcal{H}^i\big(\mathbf{D}_R(u)\big) = \mathrm{Ext}^i_R(u, R)$. Furthermore, using Remark 5.6 and the hypercohomology spectral sequence computing the $\mathrm{Ext}^i_R(u, R)$ in terms of the $\mathrm{Ext}^p_R\big(\mathcal{H}^q(u), R\big)$ (see Remark A.24), we see that it is enough to prove (5.3) when $u = M$ is a finitely generated R-module. We choose a good filtration on M and for every i, we use property (Ext1) in Proposition 5.2 to get a good filtration on $\mathrm{Ext}^i_R(M, R)$ such that the corresponding graded module is a subquotient of $\mathrm{Ext}^i_S\big(\mathrm{Gr}(M), S\big)$. Since it is clear that

$$\mathrm{Supp}\big(\mathrm{Ext}^i_S\big(\mathrm{Gr}(M), S\big) \subseteq \mathrm{Supp}\big(\mathrm{Gr}(M)\big) = \mathrm{Char}(M),$$

we deduce

$$\mathrm{Char}\big(\mathbf{D}_R(M)\big) = \bigcup_i \mathrm{Char}\big(\mathrm{Ext}^i_R(M, R)\big) \subseteq \mathrm{Char}(M).$$

$\square$

We next introduce a cohomological notion of codimension.

DEFINITION 5.10. Let M be a finitely generated nonzero R-module. The *codimension* of M is

$$j(M) := \min\big\{i \mid \mathrm{Ext}^i_R(M, R) \neq 0\big\}.$$

Note that by Theorem 5.5, we have $\mathbf{D}_R(M) \neq 0$, hence $\mathrm{Ext}^i_R(M, R) \neq 0$ for some i.

REMARK 5.11. Note that if R is a commutative Noetherian ring, then it follows from the cohomological description of depth that

$$j(M) = \mathrm{depth}\big(\mathrm{Ann}_R(M), R\big)$$

(see [**Mat89**, Theorem 16.6]). Furthermore, if we assume that R is regular (hence Cohen-Macaulay), with all maximal ideals of the same codimension, it follows that

$$j(M) = \mathrm{codim}\big(\mathrm{Ann}_R(M)\big) = \dim(R) - \dim(M)$$

(see [**Mat89**, Theorem 17.4]).

DEFINITION 5.12. A finitely generated nonzero R-module M is *pure* if for every nonzero submodule N of M, we have $j(N) = j(M)$.

REMARK 5.13. If R is a regular commutative ring, with all maximal ideals of the same codimension, then it follows from Remark 5.11 that M is pure if and only if for every nonzero submodule N of M, we have $\dim(N) = \dim(M)$. This is equivalent to the fact that $\dim(R/\mathfrak{p})$ is constant, when $\mathfrak{p}$ varies over the associated primes of M (equivalently, M has no embedded associated primes and $\mathrm{Supp}(M)$ has pure dimension). Indeed, if $\mathfrak{p}$ is an associated prime of M, then we have an embedding $R/\mathfrak{p} \hookrightarrow M$, hence M pure implies $\dim(R/\mathfrak{p}) = \dim(M)$. Conversely, if $\dim(R/\mathfrak{p}) = \dim(M)$ for all associated primes of M and N is a nonzero submodule of M, then for every minimal prime $\mathfrak{p}$ in $\mathrm{Supp}(N)$, $\mathfrak{p}$ is an associated prime of N, hence of M, and thus $\dim(R/\mathfrak{p}) = \dim(M)$.

REMARK 5.14. It is clear that a nonzero submodule of a pure module is again pure.

THEOREM 5.15. *If M is a nonzero finitely generated R-module, then the following hold:*

1) *We have* $\dim(M) + j(M) = d$. *In particular, we see that if $F_\bullet M$ is a good filtration on M, then* $j(M) = j\big(\mathrm{Gr}(M)\big)$.
2) *For every i such that* $\mathrm{Ext}^i_R(M,R) \neq 0$, *we have*
$$j\big(\mathrm{Ext}^i_R(M,R)\big) \geq i.$$
3) *If $j(M) = \ell$, then $\mathrm{Ext}^\ell_R(M,R)$ is a pure R-module with* $j\big(\mathrm{Ext}^\ell_R(M,R)\big) = \ell$.

PROOF. We first assume that R is a commutative regular ring, with all maximal ideals of codimension d. The equality in 1) follows from Remark 5.11. By 1), in order to prove 2), it is enough to show that
$$\dim\big(\mathrm{Ext}^i_R(M,R)\big) \leq d - i.$$
Suppose that $\mathfrak{p}$ is a prime in $\mathrm{Supp}\big(\mathrm{Ext}^i_R(M,R)\big)$ with $\dim(R/\mathfrak{p}) = \dim\big(\mathrm{Ext}^i_R(M,R)\big)$. Since $R_\mathfrak{p}$ is a regular ring and $\mathrm{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}) \neq 0$, it follows that
$$i \leq \dim(R_\mathfrak{p}) = d - \dim(R/\mathfrak{p}) = d - \dim\big(\mathrm{Ext}^i_R(M,R)\big).$$

Suppose now that $j(M) = \ell$. By Remark 5.13 and using the assertion in 2), in order to prove 3) it is enough to show that for every associated prime $\mathfrak{p}$ of $\mathrm{Ext}^\ell_R(M,R)$, we have $\mathrm{codim}(\mathfrak{p}) \leq \ell$ (indeed, this first gives $\dim\big(\mathrm{Ext}^\ell_R(M,R)\big) \geq d - \ell$, so we have equality by 2), and then we get the purity of $\mathrm{Ext}^\ell_R(M,R)$). If we consider a resolution $P^\bullet$ of M by finitely generated projective R-modules and $Q^\bullet = \mathrm{Hom}_R(P^\bullet, R)$, then we have an exact complex
$$0 \to Q^0 \to \ldots \to Q^{\ell-1} \to Q^\ell \to T \to 0$$
and $\mathrm{Ext}^\ell_R(M,R) \subseteq T$. Therefore $\mathfrak{p}$ is an associated prime of T and since $\mathrm{pd}_{R_\mathfrak{p}}(T_\mathfrak{p}) \leq \ell$, it follows from the Auslander-Buchsbaum theorem (see [**Mat89**, Theorem 19.1]) that
$$\mathrm{codim}(\mathfrak{p}) = \dim(R_\mathfrak{p}) = \mathrm{depth}(R_\mathfrak{p}) = \mathrm{depth}(T_\mathfrak{p}) + \mathrm{pd}_{R_\mathfrak{p}}(T_\mathfrak{p}) \leq \ell.$$

We next turn to the general (noncommutative) case. Let $F_\bullet M$ be a good filtration on M. We first show that $j(M) = j\big(\mathrm{Gr}(M)\big)$. It follows from property (Ext1) in Proposition 5.2 that

$$j(M) \geq j\big(\mathrm{Gr}(M)\big) \quad \text{and} \quad \dim\big(\mathrm{Ext}^i_R(M,R)\big) \leq \dim\big(\mathrm{Ext}^i_S\big(\mathrm{Gr}(M),S\big)\big). \tag{5.4}$$

Suppose, arguing by contradiction, that $j(M) > j\big(\mathrm{Gr}(M)\big)$. Note that by Corollary 5.9, we have

$$\mathrm{Char}(M) = \bigcup_i \mathrm{Char}\big(\mathrm{Ext}^i_R(M,R)\big),$$

hence

$$\dim(M) = \max_i \dim\big(\mathrm{Ext}^i_R(M,R)\big),$$

where the maximum is over those i with $\mathrm{Ext}^i_R(M,R) \neq 0$. We thus need to have $i \geq j(M) > j\big(\mathrm{Gr}(M)\big)$, in which case it follows from the second inequality in (5.4) and assertions 1) and 2) for the ring S that

$$\dim\big(\mathrm{Ext}^i_R(M,R)\big) \leq d - j\big(\mathrm{Ext}^i_S(\mathrm{Gr}(M),S)\big) \leq d - i < d - j\big(\mathrm{Gr}(M)\big) = \dim\big(\mathrm{Gr}(M)\big).$$

We thus conclude that $\dim(M) < \dim\big(\mathrm{Gr}(M)\big)$, a contradiction.

The assertion in 1) now follows immediately from the commutative case. Then in order to prove 2), it is enough to show that $\dim\big(\mathrm{Ext}^i_R(M,R)\big) \leq d - i$ and this follows from the corresponding inequality in the commutative case and the second inequality in (5.4).

Finally, in order to prove 3), we first note that if we have a good filtration on M such that $\mathrm{Gr}(M)$ is pure, then M is pure. Indeed, if N is a nonzero submodule of M and we consider on N the induced filtration, then $\mathrm{Gr}(N)$ is a nonzero submodule of $\mathrm{Gr}(M)$, hence

$$j(N) = j\big(\mathrm{Gr}(N)\big) = j\big(\mathrm{Gr}(M)\big) = j(M).$$

Note now that by property (Ext2) in Proposition 5.2, since $\ell = j(M)$, we have a good filtration on $\mathrm{Ext}^\ell_R(M,R)$, with $\mathrm{Gr}\big(\mathrm{Ext}^\ell_R(M,R)\big)$ a subobject of $\mathrm{Ext}^\ell_S\big(\mathrm{Gr}(M),S\big)$. On the other hand, this latter object is pure of codimension ℓ by 3) in the commutative case, since $j\big(\mathrm{Gr}(M)\big) = \ell$. We thus conclude that $\mathrm{Gr}\big(\mathrm{Ext}^\ell_R(M,R)\big)$ is pure of codimension ℓ, which implies, as we have seen, that $\mathrm{Ext}^\ell_R(M,R)$ is pure, of codimension ℓ. This completes the proof of the theorem. $\square$

We record one assertion that was proved in the proof of the above theorem (we will prove the converse statement in Proposition 5.27 below):

Proposition 5.16. If M is a finitely generated nonzero R-module and $F_\bullet M$ is a good filtration such that $\mathrm{Gr}(M)$ is pure of codimension j, then M is pure of codimension j.

The following corollary follows from assertion 1) in Theorem 5.15 and Remark 5.6.

Corollary 5.17. Given a short exact sequence of finitely generated nonzero R-modules

$$0 \to M' \to M \to M'' \to 0,$$

we have $j(M) = \min\big\{j(M'), j(M'')\big\}$.

Corollary 5.18. If M_1 and M_2 are nonzero submodules of the finitely generated R-module M, then $j(M_1 + M_2) = \min\big\{j(M_1), j(M_2)\big\}$.

PROOF. Let $j = \min\big\{j(M_1), j(M_2)\big\}$. Note that we have $(M_1 + M_2)/M_1 \simeq M_2/(M_1 \cap M_2)$ and it follows from Corollary 5.17 that $j\big((M_1 + M_2)/M_1\big) \geq j(M_2) \geq j$. Another application of the same corollary given $j(M_1 + M_2) = \min\big\{j(M_2), j((M_1+M_2)/M_2)\big\} \geq j$. The fact that $j(M_1+M_2) \leq j$ follows directly from the corollary. $\square$

REMARK 5.19. An important consequence of assertion 1) in Theorem 5.15 is that the dimension of a finitely generated R-module M does not depend on the filtration we consider on R (as long as the graded rings have the same dimension). In particular, we see that if M is a finitely generated module over the Weyl algebra $A_n(k)$, for an algebraically closed field k of characteristic 0, the two notions of dimension that we defined in Chapter 3 (via the order filtration) and Chapter 4 (via the Bernstein filtration) coincide.

5.3. The Sato-Kashiwara filtration

We next discuss a useful decreasing filtration on an arbitrary finitely generated AR-module M, by suitable R-submodules. We first give the definition of this filtration due to Gabber.

DEFINITION 5.20. If M is a finitely generated R-module, let

$$C^i(M) = \sum_{j(N)\geq i} N \subseteq M,$$

where the sum is over all nonzero submodules $N \subseteq M$, with $j(N) \geq i$.

REMARK 5.21. Note that if M is nonzero and $j = j(M)$, then

$$M = C^j(M) \supseteq C^{j+1}(M) \supseteq \ldots \supseteq C^d(M) \supseteq C^{d+1}(M) = 0.$$

REMARK 5.22. It follows from Corollary 5.18 that if $C^i(M) \neq 0$, then $j\big(C^i(M)\big) \geq i$, hence $C^i(M)$ is the unique largest submodule of M of codimension $\geq i$.

REMARK 5.23. It is clear that if M is nonzero and $j = j(M)$, then M is pure if and only if $C^{j+1}(M) = 0$. In general, for every i, if $C^i(M)/C^{i+1}(M)$ is nonzero, then it is pure of codimension i. Indeed, on one hand we have $j\big(C^i(M)/C^{i+1}(M)\big) \geq i$ by Corollary 5.17. On the other hand, every nonzero submodule of $C^i(M)/C^{i+1}(M)$ is of the form $N/C^{i+1}M$, for some N with $C^{i+1}(M) \subsetneq N \subseteq C^i(M)$, hence $j(N) = i$ and $j\big(N/C^{i+1}(M)\big) \leq i$ by Corollary 5.17.

We next give a cohomological description of this filtration due to Sato and Kashiwara. Let M be a finitely generated nonzero R-module. For every i, we have an exact triangle in $D^b_{\mathrm{fg}}(R^{\mathrm{op}})$:

$$\tau^{\leq i-1}\mathbf{D}_R(M) \longrightarrow \mathbf{D}_R(M) \longrightarrow \tau^{\geq i}\mathbf{D}_R(M) \xrightarrow{+1}$$

(see Example A.20). We obtain an exact sequence

$$\mathrm{Ext}^{-1}_{R^{\mathrm{op}}}\big(\tau^{\leq i-1}\mathbf{D}_R(M), R^{\mathrm{op}}\big) \to \mathrm{Ext}^{0}_{R^{\mathrm{op}}}\big(\tau^{\geq i}\mathbf{D}_R(M), R^{\mathrm{op}}\big) \to \mathrm{Ext}^{0}_{R^{\mathrm{op}}}\big(\mathbf{D}_R(M), R^{\mathrm{op}}\big)$$

where the third term is canonically isomorphic to M by Theorem 5.5. We claim that the first term in this sequence is 0: indeed, for every q, we have either $\mathcal{H}^q\big(\tau^{\leq i-1}\mathbf{D}_R(M)\big) \simeq \mathrm{Ext}^q_R(M, R)$ or $\mathcal{H}^q\big(\tau^{\leq i-1}\mathbf{D}_R(M)\big) = 0$. In particular, we see that if $\mathcal{H}^q\big(\tau^{\leq i-1}\mathbf{D}_R(M)\big)$ is nonzero, then its codimension is $\geq q$ by assertion 2) in Theorem 5.15. Therefore

$$\operatorname{Ext}^{-1}_{R^{\rm op}}\big(\operatorname{Ext}^q_R(M,R)[-q],R^{\rm op}\big)=\operatorname{Ext}^{q-1}_{R^{\rm op}}\big(\operatorname{Ext}^q_R(M,R),R^{\rm op}\big)=0.$$

This gives our claim by successively considering the various truncation functors provided in Example A.20.

Definition 5.24. With the above notation, for every i, we let $S^i(M)$ be the image of the injective map

$$\alpha_i\colon \operatorname{Ext}^0_{R^{\rm op}}\big(\tau^{\geq i}\mathbf{D}_R(M),R^{\rm op}\big)\to M.$$

Applying the truncation functors in Example A.20 to $\tau^{\geq i}\mathbf{D}_R(M)$, we get an exact triangle

$$\operatorname{Ext}^i_R(M,R)[-i]\longrightarrow\tau^{\geq i}\mathbf{D}_R(M)\longrightarrow\tau^{\geq i+1}\mathbf{D}_R(M)\xrightarrow{+1}.$$

This gives an exact sequence
(5.5)

$$0\to S^{i+1}(M)\xrightarrow{\beta_i}S^i(M)\to\operatorname{Ext}^i_{R^{\rm op}}\big(\operatorname{Ext}^i_R(M,R),R^{\rm op}\big)\to\operatorname{Ext}^1_{R^{\rm op}}\big(\tau^{\geq i+1}\mathbf{D}_R(M),R^{\rm op}\big).$$

The injectivity of β_i follows from the fact that $\alpha_i\circ\beta_i=\alpha_{i+1}$.

Remark 5.25. It is clear that the $S^i(M)$ are functorial: if $f\colon M_1\to M_2$ is a morphism of finitely generated R-modules, then $f\big(S^i(M_1)\big)\subseteq S^i(M_2)$ for all i.

Theorem 5.26. *For every finitely generated R-module M, we have $C^i(M)=S^i(M)$ for all i.*

Proof. Assertion 2) in Theorem 5.15 implies that if $\operatorname{Ext}^i_{R^{\rm op}}\big(\operatorname{Ext}^i_R(M,R),R^{\rm op})\big)\neq 0$, then

$$j\big(\operatorname{Ext}^i_{R^{\rm op}}(\operatorname{Ext}^i_R(M,R),R^{\rm op})\big)\geq i.$$

Using the exact sequence (5.5), we thus obtain by descending induction on i that if $S^i(M)\neq 0$, then

$$j\big(S^i(M)\big)\geq i.$$

Therefore $S^i(M)\subseteq C^i(M)$.

In order to prove the reverse inclusion, we may and will assume that $C^i(M)\neq 0$. Note first that since $j\big(C^i(M)\big)\geq i$, we have by definition of codimension that the canonical morphism

$$\mathbf{D}_R\big(C^i(M)\big)\to\tau^{\geq i}\mathbf{D}_R\big(C^i(M)\big)$$

is an isomorphism. We thus have $S^i\big(C^i(M)\big)=C^i(M)$. It follows from the functoriality of S^i (see Remark 5.25) applied for $C^i(M)\hookrightarrow M$ that $C^i(M)=S^i\big(C^i(M)\big)\subseteq S^i(M)$, which completes the proof of the theorem. □

We end this chapter with the following converse to Proposition 5.16:

Proposition 5.27. If M is a pure nonzero R-module, then $\operatorname{Char}(M)$ has pure dimension. In fact, there is a good filtration on M such that $\operatorname{Gr}(M)$ is pure.

Proof. Let $\ell=j(M)$ and let $N=\operatorname{Ext}^\ell_R(M,R)$, so that assertion 3) in Theorem 5.15 implies that $j(N)=\ell$ and $\operatorname{Ext}^\ell_{R^{\rm op}}(N,R^{\rm op})$ is pure, of codimension ℓ. On the other hand, since M is pure, we have $C^\ell(M)=M$ and $C^{\ell+1}(M)=0$. We thus conclude using Theorem 5.26 and the exact sequence (5.5) that we have an embedding

$$M \hookrightarrow \mathrm{Ext}^{\ell}_{R^{\mathrm{op}}}(N, R^{\mathrm{op}}).$$

We choose a good filtration on N and then apply property (Ext2) in Proposition 5.2 to get a good filtration on $\mathrm{Ext}^{\ell}_{R^{\mathrm{op}}}(N, R^{\mathrm{op}})$ such that $\mathrm{Gr}\big(\mathrm{Ext}^{\ell}_{R^{\mathrm{op}}}(N, R^{\mathrm{op}})\big)$ is a subobject of $\mathrm{Ext}^{\ell}_{S}\big(\mathrm{Gr}(N), S\big)$. This latter object is pure by assertion 3) in Theorem 5.15 (in the commutative case), hence also $\mathrm{Gr}\big(\mathrm{Ext}^{\ell}_{R^{\mathrm{op}}}(N, R^{\mathrm{op}})\big)$ is pure. If we take on M the induced filtration from that on $\mathrm{Ext}^{\ell}_{R^{\mathrm{op}}}(N, R^{\mathrm{op}})$, we see that $\mathrm{Gr}(M)$ is a submodule of $\mathrm{Gr}\big(\mathrm{Ext}^{\ell}_{R}(N, R^{\mathrm{op}})\big)$, hence it is pure. In particular, $\mathrm{Char}(M)$ has pure dimension (see Remark 5.13). $\square$

CHAPTER 6

Holonomic $\mathcal{D}$-modules

We begin by introducing the main functors on $\mathcal{D}$-modules: the pull-back and the push-forward functors. We then prove Kashiwara's theorem, showing that if Z is a smooth subvariety of X, then there is an equivalence of categories between $\mathcal{D}_Z$-modules and $\mathcal{D}_X$-modules supported on Z. By combining Kashiwara's theorem with the Sato-Kashiwara's filtration, we give a proof of Bernstein's inequality and then discuss the basic properties of the category of holonomic $\mathcal{D}$-modules on a given smooth variety. Following this, we prove one of our key results, the existence of b-functions for elements of a holonomic $\mathcal{D}$-module and discuss the classical application of b-functions to meromorphic extension of complex powers. We end this chapter by proving preservation of holonomicity for the inverse and direct image functors and setting up the 6-functor formalism for the derived category of holonomic $\mathcal{D}$-modules.

6.1. Pull-back and push-forward of $\mathcal{D}$-modules

We begin by discussing the pull-back of $\mathcal{D}$-modules. As usual, we work over an algebraically closed field k, with $\mathrm{char}(k) = 0$. Let $f\colon X \to Y$ be a morphism of smooth irreducible algebraic varieties over k.

We will denote by $\mathcal{D}(\mathcal{O}_X)$ and $\mathcal{D}(\mathcal{D}_X)$ the derived categories associated to the Abelian categories $\mathcal{M}od(\mathcal{O}_X)$ and $\mathcal{M}od(\mathcal{D}_X)$ of $\mathcal{O}_X$-modules and left $\mathcal{D}_X$-modules, respectively. We will write $\mathcal{D}_{\mathrm{qc}}(\mathcal{D}_X)$ (respectively, $\mathcal{D}_{\mathrm{coh}}(\mathcal{D}_X)$) for the full subcategory of $\mathcal{D}(\mathcal{D}_X)$ consisting of those u with $\mathcal{H}^i(u)$ quasi-coherent (respectively, coherent) for all $i \in \mathbf{Z}$ (note that these are triangulated subcategories). We also have triangulated subcategories such as $\mathcal{D}^b(\mathcal{D}_X)$, $\mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X)$, $\mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_X)$, etc. We will denote by $\mathcal{D}(\mathcal{D}_X^{\mathrm{op}})$, $\mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X^{\mathrm{op}})$, etc. the corresponding derived categories of right $\mathcal{D}$-modules.

We first show that if $\mathcal{M}$ is a left $\mathcal{D}_Y$-module, then the natural $\mathcal{O}_X$-module structure on $f^*(\mathcal{M}) = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{M})$ can be extended to a structure of left $\mathcal{D}_X$-module. For this, it is convenient to think of left $\mathcal{D}$-modules as $\mathcal{O}$-modules with an integrable connection (see Chapter 3.2). Given the connection

$$\nabla\colon \mathcal{M} \to \Omega_Y \otimes_{\mathcal{O}_Y} \mathcal{M},$$

after pulling-back via f^{-1} and using the canonical morphism $f^*(\Omega_Y) \to \Omega_X$, we obtain a map

$$f^{-1}(\mathcal{M}) \xrightarrow{f^{-1}(\nabla)} f^{-1}(\Omega_Y) \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{M}) \to f^*(\Omega_Y) \otimes_{\mathcal{O}_X} f^*(\mathcal{M}) \to \Omega_X \otimes_{\mathcal{O}_X} f^*(\mathcal{M}),$$

which has a unique extension to a k-linear map ∇ that satisfies the Leibniz rule:

$$f^*(\mathcal{M}) \to \Omega_X \otimes_{\mathcal{O}_X} f^*(\mathcal{M}).$$

Explicitly, if $u \in \mathcal{M}$ is such that $\nabla(u) = \sum_i \eta_i \otimes u_i$ for some $\eta_i \in \Omega_X$ and $u_i \in \mathcal{M}$, then for $h \in \mathcal{O}_X$, we have

$$\nabla(h \otimes u) = dh \otimes u + \sum_i hf^*(\eta_i) \otimes (1 \otimes u_i).$$

It is straightforward to check that since the connection on $\mathcal{M}$ is integrable, the connection on $f^*(\mathcal{M})$ is integrable as well. Therefore we have a (right-exact) functor

$$f^* \colon \mathcal{M}od(\mathcal{D}_Y) \to \mathcal{M}od(\mathcal{D}_X).$$

REMARK 6.1. We can describe the $\mathcal{D}_X$-module structure on $f^*(\mathcal{M})$ in local coordinates, as follows. After possibly restricting to a suitable open subset of X, we may assume that we have coordinates $y_1, \ldots, y_n$ on Y and $x_1, \ldots, x_m$ on X. Let $f_j = y_j \circ f \in \mathcal{O}_X(X)$. Since we have $f^*(dy_j) = \sum_{i=1}^m \frac{\partial f_j}{\partial x_i} dx_i$, it is easy to see that for every $u \in \mathcal{M}$ and $h \in \mathcal{O}_X$, we have

$$\partial_{x_i}(h \otimes u) = \tfrac{\partial h}{\partial x_i} \otimes u + \sum_{j=1}^n h \frac{\partial f_j}{\partial x_i} \otimes \partial_{y_j} u. \tag{6.1}$$

REMARK 6.2. The following alternative point of view on pull-back of $\mathcal{D}$-modules is very useful. By considering $\mathcal{D}_Y$ as a $\mathcal{D}_Y - \mathcal{D}_Y$-bimodule, we see that the *transfer module* $\mathcal{D}_{X\to Y} := f^*(\mathcal{D}_Y) = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{D}_Y)$ has a structure of $\mathcal{D}_X - f^{-1}(\mathcal{D}_Y)$-bimodule, where the left structure is the one described above. For every left $\mathcal{D}_Y$-module $\mathcal{M}$, the obvious isomorphism $\mathcal{M} \simeq \mathcal{D}_Y \otimes_{\mathcal{D}_Y} \mathcal{M}$ induces an isomorphism of left $\mathcal{D}_X$-modules $f^*(\mathcal{M}) \simeq \mathcal{D}_{X\to Y} \otimes_{f^{-1}(\mathcal{D}_Y)} f^{-1}(\mathcal{M})$. We note that f^{-1} is an exact functor between the corresponding Abelian categories.

REMARK 6.3. We have a unique morphism of left $\mathcal{D}_X$-module $\mathcal{D}_X \to \mathcal{D}_{X\to Y}$ that maps 1 to $1 \in \mathcal{O}_X = f^*(\mathcal{O}_Y) \subseteq f^*(\mathcal{D}_X)$. The restriction to $\mathcal{T}_X$ is the morphism $\mathcal{T}_X \to f^*(\mathcal{T}_Y)$, the dual of $f^*(\Omega_Y) \to \Omega_X$.

DEFINITION 6.4. If $f \colon X \to Y$ is a morphism between smooth varieties over k, the (derived) pull-back functor is

$$\mathbf{L}f^* \colon \mathcal{D}^-(\mathcal{D}_Y) \to \mathcal{D}^-(\mathcal{D}_X) \quad \text{given by} \quad \mathbf{L}f^*(-) = \mathcal{D}_{X\to Y} \otimes^L_{f^{-1}(\mathcal{D}_Y)} f^{-1}(-).$$

Note that this is an exact functor (in the sense of triangulated categories).

REMARK 6.5. It is a consequence of Remark 6.2 that if $\mathcal{M}$ is a left $\mathcal{D}_Y$-module, then we have an isomorphism of left $\mathcal{D}$-modules $\mathcal{H}^0\big(\mathbf{L}f^*(\mathcal{M})\big) \simeq f^*(\mathcal{M})$.

REMARK 6.6. It is clear that we have a commutative diagram of functors (up to equivalence)

$$\begin{tikzcd}
\mathcal{D}^-(\mathcal{D}_Y) \arrow[r, "\mathbf{L}f^*"] \arrow[d] & \mathcal{D}^-(\mathcal{D}_X) \arrow[d] \\
\mathcal{D}^-(\mathcal{O}_Y) \arrow[r, "\mathbf{L}f^*"] & \mathcal{D}^-(\mathcal{O}_X),
\end{tikzcd}$$

where the vertical functors are the natural ones associating $\mathcal{O}$-modules to $\mathcal{D}$-modules. This implies that we have induced functors $\mathbf{L}f^* \colon \mathcal{D}^b(\mathcal{D}_Y) \to \mathcal{D}^b(\mathcal{D}_X)$ and $\mathbf{L}f^* \colon \mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_Y) \to \mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X)$.

PROPOSITION 6.7. If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of smooth varieties, we have canonical isomorphisms

$$\mathcal{D}_{X\to Z} \simeq \mathcal{D}_{X\to Y} \otimes^L_{f^{-1}(\mathcal{D}_Y)} f^{-1}(\mathcal{D}_{Y\to Z}) \simeq \mathcal{D}_{X\to Y} \otimes_{f^{-1}(\mathcal{D}_Y)} f^{-1}(\mathcal{D}_{Y\to Z}).$$

PROOF. Note that $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes^L_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{D}_Y)$ since $\mathcal{D}_Y$ is a flat $\mathcal{O}_Y$-module. Using associativity of (derived) tensor product, we get canonical isomorphisms

$$\mathcal{D}_{X\to Y} \otimes^L_{f^{-1}(\mathcal{D}_Y)} f^{-1}(\mathcal{D}_{Y\to Z}) \simeq \big(\mathcal{O}_X \otimes^L_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{D}_Y)\big) \otimes^L_{f^{-1}(\mathcal{D}_Y)} f^{-1}(\mathcal{D}_{Y\to Z})$$
$$\simeq \mathcal{O}_X \otimes^L_{f^{-1}(\mathcal{O}_Y)} f^{-1}\big(\mathcal{O}_Y \otimes^L_{g^{-1}(\mathcal{O}_Z)} g^{-1}(\mathcal{D}_Z)\big)$$
$$\simeq \mathcal{O}_X \otimes^L_{f^{-1}(\mathcal{O}_Y)} \big(f^{-1}(\mathcal{O}_Y) \otimes_{(gf)^{-1}(\mathcal{O}_Z)} (gf)^{-1}(\mathcal{D}_Z) \simeq \mathcal{O}_X \otimes^L_{(gf)^{-1}(\mathcal{O}_Z)} (gf)^{-1}(\mathcal{D}_Z)$$
$$\simeq \mathcal{D}_{X\to Z}.$$

The second isomorphism in the statement follows from the fact that $\mathcal{H}^i(\mathcal{D}_{X\to Z}) = 0$ for all $i \neq 0$. $\square$

COROLLARY 6.8. If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of smooth varieties, then we have a canonical isomorphism of functors on $\mathcal{D}^-(\mathcal{D}_Z)$:

$$\mathbf{L}(gf)^* \simeq \mathbf{L}f^* \circ \mathbf{L}g^*.$$

PROOF. The assertion follows from the description of $\mathcal{D}_{X\to Z}$ in the proposition and associativity of the (derived) tensor product. $\square$

REMARK 6.9. Every morphism $f\colon X \to Y$ factors as $X \overset{i}{\hookrightarrow} X \times Y \xrightarrow{p} Y$, where $i(x) = \big(x, f(x)\big)$ and p is the projection onto the second component. Since i is a closed immersion and p is smooth and we have $\mathbf{L}f^* \simeq \mathbf{L}i^* \circ \mathbf{L}p^*$ by Corollary 6.8, it follows that in order to describe the pull-back by arbitrary morphisms, it is enough to describe the pull-back corresponding to closed immersions and smooth morphisms. We treat these now.

EXAMPLE 6.10. Let $i\colon X \to Y$ be a closed immersion. After restricting to suitable open subsets, we may assume that we have coordinates $y_1, \ldots, y_n, x_1, \ldots, x_m$ on Y such that X is defined by $(y_1, \ldots, y_n)$. In this case $\overline{x_1}, \ldots, \overline{x_m}$ give coordinates on X and it follows from (6.1) that if $\mathcal{M}$ is a left $\mathcal{D}_Y$-module, then each ∂_{x_i} induces the action of $\partial_{\overline{x_i}}$ on $i^*(\mathcal{M}) = \mathcal{M}/(y_1, \ldots, y_n)\mathcal{M}$. In particular, since $\mathcal{D}_Y = \bigoplus_{\alpha,\beta} \mathcal{O}_Y \partial_x^\alpha \partial_y^\beta$, we see that

$$\mathcal{D}_{X\to Y} = \mathcal{D}_X/(y_1, \ldots, y_n)\mathcal{D}_X = \bigoplus_{\beta \in \mathbf{Z}^n_{\geq 0}} \mathcal{D}_X \partial_y^\beta,$$

where the right multiplication by ∂_{y_i} and y_i is given by

$$P\partial_y^\beta \cdot \partial_{y_i} = P\partial_y^{\beta+e_i} \quad \text{and} \quad P\partial_y^\beta y_i = \beta_i P \partial_y^{\beta - e_i}.$$

In particular, we see that it can happen that $\mathcal{M}$ is a coherent $\mathcal{D}_Y$-module, but $i^*(\mathcal{M})$ is not a coherent $\mathcal{D}_X$-module.

EXAMPLE 6.11. Let $p\colon X \to Y$ be a smooth morphism. After restricting to suitable open subsets, we may assume that we have coordinates $y_1, \ldots, y_n$ on Y and $x_1, \ldots, x_m$ on X, with $n \leq m$, such that $y_i \circ p = x_i$ for $1 \leq i \leq n$. In this case the functor $p^*(-)$ is exact. It follows from (6.1) that for every left $\mathcal{D}_X$-module $\mathcal{M}$, the action of ∂_{x_i} on $p^{-1}(\mathcal{M})$ is induced by the action of ∂_{y_i} on $\mathcal{M}$ for all $i \leq n$ and it is is 0 for $i > n$. Since $\mathcal{D}_Y = \bigoplus_{\alpha\in\mathbf{Z}^n_{\geq 0}} \mathcal{O}_Y\partial_y^\alpha$, we have an isomorphism of $\mathcal{O}_X$-modules $\mathcal{D}_{X\to Y} \simeq \bigoplus_{\alpha\in\mathbf{Z}^n_{\geq 0}} \mathcal{O}_X\partial_y^\alpha$. The description of the action of the ∂_{x_i} implies that the canonical morphism $\mathcal{D}_X \to \mathcal{D}_{X\to Y}$ induces an isomorphism of left $\mathcal{D}_X$-modules $\mathcal{D}_X/\mathcal{D}_X(\partial_{x_{n+1}},\ldots,\partial_{x_m}) \simeq \mathcal{D}_{X\to Y}$. Furthermore, via this isomorphism, the right action of ∂_{y_i} is given by the right action of ∂_{x_i} for $1\leq i\leq n$.

We next turn to the push-forward of $\mathcal{D}$-modules. This is more naturally defined for right $\mathcal{D}$-modules. Indeed, note that if $f\colon X\to Y$ is a morphism of smooth, irreducible varieties over k and if $\mathcal{M}$ is a right $\mathcal{D}_X$-module, then $\mathcal{M}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$ is a right $f^{-1}(\mathcal{D}_Y)$-module, and thus $f_*\big(\mathcal{M}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}\big)$ is a right $\mathcal{D}_Y$-module. An important issue to note is that we are composing a left exact functor with a right exact functor, so the resulting functor does not have any good exactness properties. The correct approach is to work at the level at derived categories. We thus define

$$f_+\colon \mathcal{D}^b(\mathcal{D}_X^{\rm op})\to\mathcal{D}^b(\mathcal{D}_Y^{\rm op}),\quad f_+(u) = \mathbf{R}f_*(u\otimes^L_{\mathcal{D}_X}\mathcal{D}_{X\to Y}).$$

We note that both $\mathbf{R}f_*(-)$ and $-\otimes^L_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$ preserve the bounded derived category, hence this composition is well-defined (in fact, we can be more precise: if $\dim(X)=n$ and all fibers of f have dimension $\leq d$, then for all $u\in\mathcal{D}^b(\mathcal{D}_X^{\rm op})$ such that $\mathcal{H}^i(u)=0$ unless $i\in[a,b]$, we have $\mathcal{H}^i\big(f_+(u)\big)=0$ unless $i\in[a-n,b+d]$). We emphasize that this is not a derived functor of the corresponding functor defined between the Abelian categories of right $\mathcal{D}$-modules.

In order to define the corresponding functor for left $\mathcal{D}$-modules, we use the equivalence with the category of right $\mathcal{D}$-modules. More precisely, we define the functor $f_+\colon\mathcal{D}^b(\mathcal{D}_X)\to\mathcal{D}^b(\mathcal{D}_Y)$ as the composition

$$\mathcal{D}^b(\mathcal{D}_X)\xrightarrow{\tau_X}\mathcal{D}^b(\mathcal{D}_X^{\rm op})\xrightarrow{f_+}\mathcal{D}^b(\mathcal{D}_Y^{\rm op})\xrightarrow{\tau_Y^{-1}}\mathcal{D}^b(\mathcal{D}_Y),$$

where for any smooth variety Z, we denote by $\tau_Z\colon\mathcal{D}^b(\mathcal{D}_Z)\longrightarrow\mathcal{D}^b(\mathcal{D}_Z^{\rm op})$ the equivalence induced at the level of derived categories by the one in Proposition 3.58, and we denote its inverse by τ_Z^{-1}.

Proposition 6.12. If $f\colon X\to Y$ and $g\colon Y\to Z$ are two morphisms as above, then we have a natural isomorphism of functors $\mathcal{D}^b(\mathcal{D}_X)\to\mathcal{D}^b(\mathcal{D}_Z)$:

$$(g\circ f)_+\simeq g_+\circ f_+.$$

Proof. It follows immediately from the definition that it is enough to prove the corresponding statement for the functors $\mathcal{D}^b(\mathcal{D}_X^{\rm op})\to\mathcal{D}^b(\mathcal{D}_Z^{\rm op})$. Using Proposition 6.7, we see that for every $u\in\mathcal{D}^b(\mathcal{D}_X^{\rm op})$, we have

$$(g\circ f)_+(u)=\mathbf{R}(g\circ f)_*(u\otimes^L_{\mathcal{D}_X}\mathcal{D}_{X\to Z})\simeq\mathbf{R}g_*\big(\mathbf{R}f_*(u\otimes^L_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}\otimes^L_{f^{-1}(\mathcal{D}_Y)}f^{-1}(\mathcal{D}_{Y\to Z})))\big)$$
$$\simeq\mathbf{R}g_*\big(\mathbf{R}f_*(u\otimes^L_{\mathcal{D}_X}\mathcal{D}_{X\to Y})\otimes^L_{\mathcal{D}_Y}\mathcal{D}_{Y\to X}\big)=g_+\big(f_+(u)\big),$$

where the second isomorphism is a consequence of the projection formula[1] □

[1]We are using the fact that for every morphism of smooth varieties $f\colon X\to Y$, every $u\in\mathcal{D}^b\big(f^{-1}(\mathcal{D}_Y)^{\rm op}\big)$, and every $v\in\mathcal{D}^b(\mathcal{D}_Y)$, we have a canonical isomorphism $\mathbf{R}f_*(u)\otimes^L_{\mathcal{D}_Y}v\simeq\mathbf{R}f_*(u\otimes^L_{f^{-1}(\mathcal{D}_Y)}f^{-1}(v)\big)$. Indeed, using the adjoint property of the pair $(f^{-1},\mathbf{R}f_*)$ and the canonical morphism $f^{-1}\big(\mathbf{R}f_*(u)\big)\to u$, we obtain a canonical morphism $\mathbf{R}f_*(u)\otimes^L_{\mathcal{D}_Y}v\to\mathbf{R}f_*(u\otimes^L_{f^{-1}(\mathcal{D}_Y)}f^{-1}(v)\big)$. It is enough to check this is an isomorphism locally on Y, hence we may assume that Y is affine. In this case, it follows from Theorem 5.4 that we can represent v by a bounded complex of projective left $\mathcal{D}_Y$-modules, in which case the assertion is clear.

REMARK 6.13. Note that for every $f\colon X \to Y$, the functor f_+ induces a functor $f_+\colon \mathcal{D}^b_{\rm qc}(\mathcal{D}_X) \to \mathcal{D}^b_{\rm qc}(\mathcal{D}_X)$, since τ_X, τ_Y, $\mathbf{R}f_*$ and $- \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ preserve the subcategories with quasi-coherent cohomology.

EXAMPLE 6.14. Suppose that $i\colon X \hookrightarrow Y$ is a closed immersion. We have seen in Example 6.10 that in this case $\mathcal{D}_{X\to Y}$ is a locally free left $\mathcal{D}_X$-module (of infinite rank). Therefore in this case both functors i_* and $-\otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ are exact functors. We thus conclude that in this case i_+ is the functor induced at the level of derived categories by an exact functor $i_+\colon \mathcal{M}od(\mathcal{D}_X) \to \mathcal{M}od(\mathcal{D}_Y)$. Let's describe this functor.

After restricting to suitable open subsets of Y, we may assume that we have coordinates $y_1, \ldots, y_n, x_1, \ldots, x_m$ on Y as in Example 6.10. Suppose that $\mathcal{M}$ is a left $\mathcal{D}_X$-module. Using the explicit description of the equivalence of categories in Proposition 3.58, as well as the description of $\mathcal{D}_{X\to Y}$ in Example 6.10, we see that we have an isomorphism $i_+(\mathcal{M}) \simeq \mathcal{M} \otimes_k k[\partial_{y_1}, \ldots, \partial_{y_n}]$, where the left multiplication by ∂_{y_i} and y_i is given by

$$\partial_{y_i} \cdot (u \otimes \partial_y^\beta) = u \otimes \partial_y^{\beta+e_i} \quad \text{and} \quad y_i \cdot (u \otimes \partial_y^\beta) = -\beta_i u \otimes \partial_y^{\beta-e_i}.$$

Note that we have an isomorphism of $\mathcal{O}_Y$-modules $\mathcal{M} \otimes_{\mathcal{O}_X} i_+(\mathcal{O}_X) \simeq i_+(\mathcal{M})$ and it is straightforward to check that this is independent of the choice of coordinates.

EXAMPLE 6.15. Suppose that $f\colon X \to Y$ is an étale morphism. Note that in this case the morphism of left $\mathcal{D}_X$-modules $\mathcal{D}_X \to \mathcal{D}_{X\to Y}$ (see Remark 6.3) is an isomorphism and $\omega_X = f^*(\omega_Y)$, hence in this case $f_+(u) = \mathbf{R}f_*(u)$ for every $u \in \mathcal{D}^b(\mathcal{D}_X)$. If f is also an affine morphism, then the functor f_* is exact on quasi-coherent $\mathcal{O}_X$-modules, hence the functor $f_+\colon \mathcal{D}^b_{\rm qc}(X) \to \mathcal{D}^b_{\rm qc}(Y)$ is induced by the exact functor f_* on the category of quasi-coherent $\mathcal{D}_X$-modules.

EXAMPLE 6.16. Let X be a smooth variety and let $p\colon X \to Y = \mathrm{Spec}(k)$ be the morphism to a point. Let us show that in this case, for every left $\mathcal{D}_X$-module $\mathcal{M}$, we have $p_+(\mathcal{M}) = \mathbf{R}\Gamma\big(\mathrm{DR}_X(\mathcal{M})\big)$. We denote by $\tau\colon \mathcal{M}od(\mathcal{D}_X) \to \mathcal{M}od(\mathcal{D}_X^{\rm op})$ the equivalence of categories between left and right $\mathcal{D}_X$-modules. Note that we have $\mathcal{D}_{X\to Y} = \mathcal{O}_X$, hence

$$p_+(\mathcal{M}) = \mathbf{R}\Gamma\big(\tau(\mathcal{M}) \otimes^L_{\mathcal{D}_X} \mathcal{O}_X\big).$$

We have seen in Example 3.65 that a free resolution of $\mathcal{O}_X$ as a $\mathcal{D}_X$-module is given by $\tau^{-1}(\mathrm{DR_X}(\mathcal{D}_X))$, hence

$$\tau(\mathcal{M}) \otimes^L_{\mathcal{D}_X} \mathcal{O}_X \simeq \tau(\mathcal{M}) \otimes_{\mathcal{D}_X} \tau^{-1}(\mathrm{DR_X}(\mathcal{D}_X)) \simeq \mathrm{DR}_X(\mathcal{D}_X) \otimes_{\mathcal{D}_X} \mathcal{M} \simeq \mathrm{DR}_X(\mathcal{M}),$$

where the second isomorphism follows from Exercise 3.60.

6.2. Kashiwara's Equivalence Theorem

We begin with some general comments regarding the support of coherent $\mathcal{D}$-modules. As usual, we work on a smooth variety X over k. Recall that if $\mathcal{M}$ is a sheaf on X, then its support is

$$\mathrm{Supp}(\mathcal{M}) := \{x \in X \mid \mathcal{M}_x \neq 0\}.$$

Even if $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module, its support is not necessarily a closed subset of X. What is easy to see is that if $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$ module and $U \subseteq X$ is an affine open subset, then $\mathrm{Supp}(\mathcal{M}) \cap U$ is contained in the closed subset of U defined by the annihilator of $\mathcal{M}(U)$ in $\mathcal{O}_X(U)$. This inclusion is well-known to be an equality if $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module, but not in general.

REMARK 6.17. If $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module, then for a coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ we have $\mathrm{Supp}(\mathcal{M}) \subseteq V(\mathcal{I})$ if and only if

$$\mathcal{M} = \bigcup_{m \geq 1} \{u \in \mathcal{M} \mid \mathcal{I}^m u = 0\}.$$

Indeed, this can be checked on affine open subsets and it follows immediately from the fact that the support of a module M is the union of the supports of its finitely generated submodules.

The next proposition shows that the support is closed also for coherent $\mathcal{D}_X$-modules.

PROPOSITION 6.18. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, with characteristic variety $\mathrm{Char}(\mathcal{M}) \subseteq T^*X$. If $\pi\colon T^*X \to X$ is the canonical projection and $i_0\colon X \to T^*X$ is the zero-section, then

$$\mathrm{Supp}(\mathcal{M}) = \pi\big(\mathrm{Char}(\mathcal{M})\big) = i_0^{-1}\big(\mathrm{Char}(\mathcal{M})\big).$$

In particular, $\mathrm{Supp}(\mathcal{M})$ is closed in X.

PROOF. It is enough to check the assertion in the proposition locally, hence we may and will assume that X is an affine variety, with $R = \mathcal{O}_X(X)$, and let $S = \mathcal{O}(T^*X)$. Let $M = \mathcal{M}(X)$ and let $F_\bullet M$ be a good filtration on M, with $N = \mathrm{Gr}^F_\bullet(M)$. Note that for every prime ideal $\mathfrak{p}$ in R, we have $M_\mathfrak{p} \neq 0$ if and only if $(F_iM)_\mathfrak{p} \neq 0$ for some i (since the filtration is exhaustive), which is the case if and only if $N_\mathfrak{p} \neq 0$ (since $F_iM = 0$ for $i \ll 0$). This in turn holds if and only if there is a prime ideal $\mathfrak{q}$ in S lying over $\mathfrak{p}$ such that $N_\mathfrak{q} \neq 0$. In other words, we have

$$\mathrm{Supp}(\mathcal{M}) = \pi\big(\mathrm{Char}(\mathcal{M})\big)$$

(recall that, by definition, $\mathrm{Char}(\mathcal{M})$ is the support of the finitely generated S-module N). On the other hand, the characteristic variety $\mathrm{Char}(\mathcal{M}) \subseteq T^*(X)$ is a conical subvariety, which implies that we have

$$\pi\big(\mathrm{Char}(\mathcal{M})\big) = i_0^{-1}\big(\mathrm{Char}(\mathcal{M})\big).$$

Since the right-hand side is clearly closed in X, it follows that $\mathrm{Supp}(\mathcal{M})$ is closed in X. □

One can also prove directly that the support of a coherent $\mathcal{D}$-module is closed, as follows:

EXERCISE 6.19. Let X be a smooth variety and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. If $U \subseteq X$ is an affine open subset, $u_1, \ldots, u_r \in \mathcal{M}(U)$ are generators over $\mathcal{D}_X(U)$, and $I = \bigcap_{i=1}^r \mathrm{Ann}_{\mathcal{O}_X(U)}(u_i) \subseteq \mathcal{O}_X(U)$, then $\mathrm{Supp}(\mathcal{M}) \cap U = V(I)$.

The main result of this section is the following theorem due to Kashiwara:

THEOREM 6.20. *If Z is a smooth, irreducible, closed subvariety of the smooth variety X and $i\colon Z \hookrightarrow X$ is the inclusion map, then $i_+\colon \mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_Z) \to \mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_X)$ gives an equivalence of categories between $\mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_Z)$ and the full subcategory of $\mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_X)$ consisting of those $\mathcal{M}$ with* $\mathrm{Supp}(\mathcal{M}) \subseteq Z$.

REMARK 6.21. Note that the assertion in the above theorem stands in stark contrast with when happens for $\mathcal{O}$-modules: if $\mathcal{I}_Z$ is the ideal sheaf defining Z in X, then a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ has support contained in Z if and only if $\mathcal{F} = \bigcup_{m\geq 1}\{u \in \mathcal{F} \mid \mathcal{I}_Z^m u = 0\}$. On the other hand, we have $\mathcal{F} \simeq i_*(\mathcal{G})$, for some quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$, if and only if $\mathcal{I}_Z \cdot \mathcal{F} = 0$.

PROOF OF THEOREM 6.20. We first treat the case when X is affine, with coordinates $x_1, \ldots, x_n, z$ such that Z is defined by (z). Let $A = \mathcal{O}_X(X)$ and $B = A/(z) = \mathcal{O}_Z(Z)$. Recall from Example 6.14 that if M is a D_B-module, then $i_+(M) = M \otimes_k k[\partial_z]$, with the action of z and ∂_z being given by

$$z \cdot (u \otimes \partial_z^m) = -mu \otimes \partial_z^{m-1} \quad \text{and} \quad \partial_z \cdot (u \otimes \partial_z^m) = u \otimes \partial_z^{m+1}.$$

It is then clear that $\mathrm{Supp}(i_+M) \subseteq Z$: indeed, we have $z^{m+1} \cdot (u \otimes \partial_z^m) = 0$ for every $u \in M$ and $m \geq 0$.

For every D_A-module N, we put $G(N) = \{u \in N \mid zu = 0\}$. Since $G(N)$ has an induced structure of B-module and it is preserved by $\partial_{x_1}, \ldots, \partial_{x_n}$, it follows that $G(N)$ has a natural structure of D_B-module. In this way we get a functor from D_A-modules to D_B-modules.

Note first that we have an isomorphism of functors $G \circ i_+ \simeq \mathrm{Id}$. Indeed, if M is a D_B-module and $u = \sum_m u_m \partial_z^m \in i_+(M)$, then $z \cdot u = -\sum_{i\geq 1} m u_m \partial_z^{m-1}$. We thus have a canonical isomorphism $G\big(i_+(M)\big) \simeq M$.

Suppose now that N is a D_A-module with $\mathrm{Supp}(N) \subseteq Z$. Let D be the Euler operator $\partial_z z = z\partial_z + 1$ and let $N_i := \{u \in N \mid Du = iu\}$ for $i \in \mathbf{Z}$. We proceed to prove some easy properties of the N_i.

Note first that we have

$$z \cdot N_i \subseteq N_{i+1} \quad \text{for} \quad i \in \mathbf{Z}. \tag{6.2}$$

Indeed, we have $D(zu) = \partial_z z^2 u = z\partial_z zu + zu = z(D+1)u$, which gives (6.2). We similarly have

$$\partial_z \cdot N_i \subseteq N_{i-1} \quad \text{for} \quad i \in \mathbf{Z}. \tag{6.3}$$

Indeed, we have $D(\partial_z u) = \partial_z z \partial_z u = \partial_z^2 zu - \partial_z u = \partial_z(D-1)u$, which gives (6.3).

It follows directly from the definition that multiplication by $\partial_z z$ gives a bijective map $N_i \to N_i$ for all $i \neq 0$. We also see that multiplication by $z\partial_z = (\partial_z z - 1)$ is bijective on N_i for $i \neq 1$. By (6.2) and (6.3), we have maps $N_i \xrightarrow{z} N_{i+1}$ and $N_{i+1} \xrightarrow{\partial_z} N_i$ and by looking at the two compositions we see that these are bijective for all $i \neq 0$.

By assumption, every element of N is annihilated by some z^N, hence $N_i = 0$ for all $i \geq 1$. It follows from (6.2) that $zN_0 = 0$, hence $N_0 \subseteq G(N)$. On the other hand, if $zu = 0$, then clearly $\partial_z zu = 0$, hence in fact we have $G(N) = N_0$. Consider the morphism of D_A-modules

$$\varphi\colon i_+(N_0) = N_0 \otimes_k k[\partial_z] \to N, \quad \varphi\left(\sum_{i\geq 0} u_i \otimes \partial_z^i\right) = \sum_{i\geq 0} \partial_z^i u_i.$$

Note that since $\partial_z^i u_i \in N_{-i}$ and these are distinct eigenspaces of the operator D, it follows that φ is injective (we are also using the fact that multiplication by ∂_z^i on N_0 is injective for all $i \geq 0$). In order to show that φ is an isomorphism, it is thus enough to show that if $z^m u = 0$ for some $u \in N$ and some $m \geq 1$, then

$u \in N_0 \oplus \ldots \oplus N_{-m+1}$ (recall that the map $\partial_z^i \colon N_0 \to N_{-i}$ is surjective for every $i \geq 0$). This is a consequence of the following formula

$$\partial_z^m z^m u = \prod_{i=0}^{m-1} (D+i),$$

which follows by an easy induction on m. We thus conclude that $i_+\big(G(N)\big) \simeq N$, completing the proof of Kashiwara's theorem in this case.

We also note that it follows from the explicit description of $i_+(M)$, where M is a D_B-module, that the support of M is contained in a smooth closed subvariety W of Z if and only if the same holds for $i_+(M)$. We then immediately deduce, by induction on r, that if X is affine, with coordinates $x_1, \ldots, x_n$ such that Z is defined by $(x_1, \ldots, x_r)$, then i_+ gives an equivalence of categories from the category of quasi-coherent $\mathcal{D}_Z$-modules to the category of $\mathcal{D}_X$-modules with support in Z. Moreover, the inverse equivalence takes N to the subset of N annihilated by the ideal of Z in X.

The general case now follows for formal reasons: the fact that i_+ is a fully faithful functor follows by restricting to suitable affine open subsets. Indeed, if we write $X = U_1 \cup \ldots \cup U_N$ and for every i and j we write $U_i \cap U_j = \bigcup_k V_{ijk}$, then for every $\mathcal{D}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we have

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{F}, \mathcal{G}) = \mathrm{Ker}\left(\bigoplus_i \mathrm{Hom}_{\mathcal{D}_{U_i}}(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i}) \to \bigoplus_{i,j,k} \mathrm{Hom}_{\mathcal{D}_{V_{ijk}}}(\mathcal{F}|_{V_{ijk}}, \mathcal{G}_{V_{ijk}})\right)$$

and a similar description holds for morphisms of $\mathcal{D}_Z$-modules. Once we have fully faithfulness, the fact that i_+ is essentially surjective follows again by restricting to suitable affine open subsets. This completes the proof of the theorem. $\square$

Remark 6.22. Let $i \colon Z \hookrightarrow X$ be as in Theorem 6.20. If $\mathcal{M}$ is a quasi-coherent $\mathcal{D}_Z$-module, then it follows from the explicit local description of $i_+(\mathcal{M})$ that $\mathcal{M}$ is a coherent $\mathcal{D}_Z$-module if and only if $i_+(\mathcal{M})$ is a coherent $\mathcal{D}_X$-module. Therefore i_+ induces an equivalence of categories between $\mathcal{M}od_{\mathrm{coh}}(\mathcal{D}_Z)$ and the full subcategory of $\mathcal{M}od_{\mathrm{coh}}(\mathcal{D}_X)$ consisting of objects supported on Z.

Remark 6.23. Using truncation functors (see Example A.20), it is easy to deduce from Theorem 6.20 that in the setting of the theorem, i_+ gives an equivalence of categories between $\mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_Z)$ and the subcategory $\mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_{X,Z})$ of $\mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X)$ consisting of objects u such that $\mathcal{H}^i(u)$ is supported on Z for all $i \in \mathbf{Z}$.

Given a closed immersion $i \colon Z \hookrightarrow X$ of smooth, irreducible varieties, with $\mathrm{codim}_X(Z) = r$, it is convenient to introduce the following shift of $\mathbf{L}i^*$: we put $i^\dagger = \mathbf{L}i^*[-r] \colon \mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X) \to \mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_Z)$.

Proposition 6.24. With the above notation, the inverse of the equivalence $i_+ \colon \mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_Z) \simeq \mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_{X,Z})$ is given by $i^\dagger$ and the same assertion holds at the level of the corresponding Abelian categories.

Proof. We first treat the case when X is affine, with $R = \mathcal{O}_X(X)$, $r = 1$, and we have coordinates $x_1, \ldots, x_n, z$ on X such that Z is defined by the ideal (z). If N is a D_R-module with support on Z and $N_0 = \{u \in N \mid zu = 0\}$, then we have seen in the proof of Theorem 6.20 that $N \simeq i_+(N_0)$. Note that by the explicit description of the action of z on $i_+(N_0)$, it follows that $\mathrm{Tor}_1^R\big(N, R/(z)\big) \simeq N/zN = 0$. On the other hand, we have $N_0 \simeq \mathrm{Tor}_1^R\big(N, R/(z)\big)$, and thus $N_0 \simeq i^\dagger(N)$. Since we have seen that $N \simeq i_+(N_0)$, this gives the assertion in the proposition in this case. Moreover, this isomorphism is independent of the choice of coordinates: we leave checking this as an exercise.

It is clear that if $Z \overset{i_1}{\hookrightarrow} Y \overset{i_2}{\hookrightarrow} X$ are two such closed immersions, then we have $(i_2 \circ i_1)^\dagger \simeq i_1^\dagger \circ i_2^\dagger$. Using this, we deduce the assertion in the proposition whenever X is affine and we have coordinates $x_1, \ldots, x_n, z_1, \ldots, z_r$ such that Z is defined by $(z_1, \ldots, z_r)$. Furthermore, the isomorphism $i_+\big(i^\dagger(N)\big) \simeq N$ for every D_R-module N with support contained in Z is independent of the choice of coordinates. We may thus glue these isomorphisms to get a corresponding isomorphism in the global case. This gives the assertion in the proposition at the level of Abelian categories and the one for derived categories is an immediate consequence. $\square$

We end this section by describing the behavior of the characteristic variety under the push-forward by a closed immersion. Recall that if $i\colon Z \hookrightarrow X$ is a closed immersion of smooth, irreducible varieties, then the injective homomorphism $\mathcal{T}_Z \hookrightarrow \mathcal{T}_X|_Z$ induces a morphism $\varphi\colon T^*X|_Z \to T^*Z$. This is a surjective morphism of geometric vector bundles on Z, hence it is locally trivial, with fiber $\mathbf{A}_k^r$, where $r = \mathrm{codim}_X(Z)$. We also have a closed immersion $\psi\colon T^*X|_Z \hookrightarrow T^*X$, of codimension r.

Proposition 6.25. With the above notation, for every coherent $\mathcal{D}_Z$-module $\mathcal{M}$, we have
$$\mathrm{Char}(i_+\mathcal{M}) = \psi\big(\varphi^{-1}(\mathrm{Char}(\mathcal{M}))\big).$$
In particular, we have $\dim(i_+\mathcal{M}) = \dim(\mathcal{M}) + r$.

Proof. The second assertion follows the first one and the fact that φ is locally trivial, with fiber $\mathbf{A}_k^r$. The first assertion can be checked locally, hence we may assume that we have algebraic coordinates $x_1, \ldots, x_n, y_1, \ldots, y_r$ on X such that Z is defined by $(y_1, \ldots, y_r)$. Note that these coordinates give isomorphisms $T^*X \simeq X \times \mathbf{A}_k^{n+r}$ and $T^*Z \simeq Z \times \mathbf{A}_k^n$ such that $\varphi\colon T^*X|_Z \simeq Z \times \mathbf{A}_k^{n+r} \to T^*Z \simeq Z \times \mathbf{A}_k^n$ gets identified with the projection onto the first components, while ψ gets identified with $i \times 1_{\mathbf{A}_k^{n+r}}$.

It follows from Example 6.14 that $i_+\mathcal{M} = \mathcal{M} \otimes_k k[\partial_{y_1}, \ldots, \partial_{y_n}]$. If $F_\bullet\mathcal{M}$ is a good filtration on $\mathcal{M}$, we put
$$F_p(i_+\mathcal{M}) = \bigoplus_{\beta \in \mathbf{Z}^r} F_{p-|\beta|}\mathcal{M} \otimes \partial_y^\beta \quad \text{for every} \quad p \in \mathbf{Z}.$$
It is clear that this is a filtration on $i_+\mathcal{M}$ compatible with $F_\bullet\mathcal{D}_X$ and we have $\mathrm{Gr}_\bullet^F(i_+\mathcal{M}) = \psi_*\varphi^*\big(\mathrm{Gr}_\bullet^F(\mathcal{M})\big)$. Since the support of this sheaf on T^*X is $\psi\big(\varphi(W)\big)$, where $W = \mathrm{Supp}\big(\mathrm{Gr}_\bullet^F(\mathcal{M})\big)$, we obtain the assertion in the proposition. $\square$

6.3. The proof of Bernstein's inequality

We can now give the proof of Bernstein's inequality for the dimension of the characteristic variety.

Proof of Theorem 3.38. We argue by induction on $n = \dim(X)$, the case $n = 0$ being trivial. The assertion is local, so after covering X by finitely many affine open subsets, we may and will assume that X is affine with $A = \mathcal{O}_X(X)$,

$R = D_A$, and $M = \mathcal{M}(X)$. Recall that the finite decreasing filtration on M by D_R-submodules

$$M = C^0(M) \supseteq C^1(M) \supseteq \ldots \supseteq C^{2n}(M) \supseteq C^{2n+1}(M) = 0$$

given in Chapter 5.3 has the property that for every i such that $C^i(M)/C^{i+1}(M)$ is nonzero, this quotient is a pure R-module of codimension i (see Remark 5.23). Using Proposition 5.27, it follows that in this case $\mathrm{Char}\big(C^i(M)/C^{i+1}(M)\big)$ has pure dimension $2n - i$. Since we have

$$\mathrm{Char}(M) = \bigcup_{i=0}^{2n} \mathrm{Char}\big(C^i(M)/C^{i+1}(M)\big),$$

it follows that it is enough to show that if M is a nonzero pure R-module with $\dim(M) < n$, then we get a contradiction.

Since $\dim\big(\mathrm{Char}(M)\big) < n$, it follows that if $Z = \pi\big(\mathrm{Char}(M)\big)$, where $\pi\colon T^*X \to X$ is the canonical projection, then $Z \neq X$ (recall that Z is a closed subset of X, equal to $\mathrm{Supp}(M)$, by Proposition 6.18). After replacing X by an open subset U such that $U \cap Z$ is nonempty, smooth, and irreducible, we may and will assume that Z is smooth and irreducible. Let $r = \mathrm{codim}_X(Z) \geq 1$. We deduce from Kashiwara's Theorem 6.20 that if $i\colon Z \hookrightarrow X$, then there is a coherent $\mathcal{D}_Z$-module $\mathcal{N}$ such that $\mathcal{M} \simeq i_+(\mathcal{N})$. Since we clearly have $\mathcal{N} \neq 0$, the inductive hypothesis gives $\dim(\mathcal{N}) \geq n - r$. On the other hand, we get from Proposition 6.25 $\dim(\mathcal{M}) = \dim(\mathcal{N}) + r \geq n$, a contradiction. This completes the proof of the theorem. $\square$

Remark 6.26. The lower bound in Theorem 3.38 on the dimension of the irreducible components of the characteristic variety also works for right $\mathcal{D}$-modules. This is a consequence of the fact that the equivalence between left and right $\mathcal{D}$-modules preserves the characteristic variety (see Remark 3.64).

Remark 6.27. For a coherent left $\mathcal{D}_X$-module $\mathcal{M}$ and $i \in \mathbf{Z}$, we will consider the right $\mathcal{D}_X$-module $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$. Note that this is a quasi-coherent (in fact, coherent) right $\mathcal{D}_X$-module and for every affine open subset $U \subseteq X$, its sections over U are given by $\mathrm{Ext}^i_{\mathcal{D}_X(U)}\big(\mathcal{M}(U), \mathcal{D}_X(U)\big)$. This follows by computing the Ext modules using finitely generated free resolutions in the first argument, which implies that computing the Ext modules commutes with localizing at a given element. A similar statement holds if we start with a coherent right $\mathcal{D}_X$-module.

Corollary 6.28. If $\mathcal{M}$ is a coherent left $\mathcal{D}_X$-module on the smooth, irreducible n-dimensional variety X, then $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$ for $i > n$; a similar result holds for right $\mathcal{D}_X$-modules.

Proof. The assertion is local, hence we may assume that X is affine, with $A = \mathcal{O}_X(X)$ and let $R = D_A$. We deduce from Theorem 3.38 and assertion 1) in Theorem 5.15 that $j(M) \leq n$ for every finitely generated nonzero R-module M and the same bound holds for R^{op}-modules by Remark 6.26. On the other hand, by assertion 2) in Theorem 5.15, if M is a finitely generated nonzero R-module and i is such that the R^{op}-module $\mathrm{Ext}^i_R(M, R) \neq 0$, then $j\big(\mathrm{Ext}^i_R(M, R)\big) \geq i$. Therefore $i \leq n$, giving the assertion in the corollary. The assertion for right $\mathcal{D}_X$-modules follows in the same way. $\square$

6.4. Holonomic $\mathcal{D}$-modules: first properties

In this section we begin the discussion of holonomic $\mathcal{D}$-modules. Let X be a smooth, irreducible, n-dimensional variety over k. Recall that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is holonomic if $\mathcal{M} = 0$ or $\dim(\mathcal{M}) = n$.

EXAMPLE 6.29. If $\mathcal{E}$ is a $\mathcal{D}_X$-module which is coherent as an $\mathcal{O}_X$-module (and thus locally free by Proposition 3.18), then $\mathcal{E}$ is holonomic. Indeed, we saw in Example 3.34 that $\mathrm{Char}(\mathcal{E})$ is the 0-section of T^*X.

We also have the following converse: if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module such that $\mathrm{Char}(\mathcal{M})$ is the 0-section of T^*X, then $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module. Since this is a local assertion, we may assume that X is affine, with coordinates $x_1, \ldots, x_n \in R = \mathcal{O}_X(X)$, and $M = \mathcal{M}(X)$ has a good filtration $F_\bullet M$ such that the support of $N = \mathrm{Gr}_\bullet^F(M)$ is the subset defined by $(y_1, \ldots, y_n) \subseteq S = R[y_1, \ldots, y_n]$, where $y_i = \overline{\partial_{x_i}}$ for $1 \leq i \leq n$. In this case there is m such that $(y_1, \ldots, y_n)^m \cdot N = 0$. This implies that N is finitely generated as an R-module: if $u_1, \ldots, u_\ell$ generate N as an S-module, it follows that the $y^\alpha u_i$, where $1 \leq i \leq \ell$ and $|\alpha| \leq m-1$ generate N as an R-module. In this case it is clear that there is d such that $F_iM = F_{i+1}M$ for all $i \geq d$, hence $M = F_dM$ is a finitely generated R-module.

REMARK 6.30. If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module on X, then there is a nonempty open subset $U \subseteq X$, such that $\mathcal{M}|_U$ is a coherent $\mathcal{O}_U$-module. Indeed, every irreducible component of $\mathrm{Char}(\mathcal{M})$ has dimension n. If such a component dominates X, since it is a conical subvariety of T^*X, it is equal to the 0-section. Therefore, as we have seen in Example 6.29, it is enough to take U to be the complement in X of the union of the images of the irreducible components of $\mathrm{Char}(\mathcal{M})$ that do not dominate X.

EXAMPLE 6.31. If $X = \mathbf{A}_k^1$ and $M = k[x, x^{-1}]/k[x]$, then $M \simeq A_1(k)/A_1(k) \cdot (x)$. If we consider on M the induced filtration by the order filtration on $A_1(k)$, we see that $\mathrm{Gr}_\bullet^F(M) \simeq k[x, y]/(x)$, hence the characteristic variety of M is the filber of T^*X over 0. In particular, M is holonomic.

REMARK 6.32. It follows from Remark 3.64 that if $\mathcal{M}^r$ is the right $\mathcal{D}_X$-module corresponding to the left $\mathcal{D}_X$-module $\mathcal{M}$, then $\mathcal{M}^r$ is holonomic if and only if $\mathcal{M}$ is holonomic.

PROPOSITION 6.33. Given an exact sequence

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

of coherent left (right) $\mathcal{D}_X$-modules on X, we have that $\mathcal{M}$ is holonomic if and only if $\mathcal{M}'$ and $\mathcal{M}''$ are holonomic.

PROOF. The assertion follows from the fact that

$$\dim(\mathcal{M}) = \max\left\{\dim(\mathcal{M}'), \dim(\mathcal{M}'')\right\}$$

and the lower bound on the dimension of coherent $\mathcal{D}_X$-modules in Theorem 3.38. $\square$

REMARK 6.34. It follows from the above proposition that the category of holonomic left (right) $\mathcal{D}_X$-modules is an Abelian category, closed under extensions, that we will denote by $\mathcal{M}od_{\mathrm{hol}}(\mathcal{D}_X)$. We conclude from the long exact sequence in cohomology associated to an exact triangle that if $\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ is the subcategory of $D^b_{\rm coh}(\mathcal{D}_X)$ consisting of those u such that $\mathcal{H}^i(u)$ are holonomic for all i, then $D^b_{\rm hol}(\mathcal{D}_X)$ is a triangulated subcategory.

PROPOSITION 6.35. For every coherent left $\mathcal{D}_X$-module $\mathcal{M}$, the right $\mathcal{D}_X$-module $\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)$ is holonomic. The analogous assertion holds if $\mathcal{M}$ is a right $\mathcal{D}_X$-module.

PROOF. The assertion is local, hence we may assume that X is affine. In this case, by combining assertions 1) and 2) in Theorem 5.15, we see that if $\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)$ is nonzero, then its dimension is $\leq n$, hence it is holonomic. $\square$

PROPOSITION 6.36. A coherent left $\mathcal{D}_X$-module $\mathcal{M}$ is holonomic if and only if $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)=0$ for all $i\neq n$ and a similar assertion holds for right $\mathcal{D}_X$-modules. Moreover, the functor $\mathcal{E}xt^n_{\mathcal{D}_X}(-,\mathcal{D}_X)$ gives an antiequivalence between the categories of of left and right holonomic $\mathcal{D}_X$-modules, with inverse $\mathcal{E}xt^n_{\mathcal{D}_X^{\rm op}}(-\mathcal{D}_X^{\rm op})$.

PROOF. We have already seen in Corollary 6.28 that $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)=0$ for all $i>n$. For the first assertion, we may assume that X is affine. In this case, if $\mathcal{M}\neq 0$, then $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)=0$ for all $i<n$ if and only if $j(\mathcal{M})\geq n$, which is equivalent to $\dim(\mathcal{M})\leq n$, by assertion 1) in Theorem 5.15. The last statement in the proposition follows now by simply globalizing the assertion in Theorem 5.5 (note that since we have canonical natural transformations from the two identity functors to the two compositions, checking that these are isomorphisms of functors can be done in affine charts). $\square$

PROPOSITION 6.37. In the Abelian category of holonomic left (right) $\mathcal{D}_X$-modules, every object has finite length.

PROOF. It is enough to show that every object in these categories has no strictly increasing or strictly decreasing infinite sequences of subobjects. For increasing sequences, the assertion follows from the fact that coherent $\mathcal{D}_X$-modules are locally finitely generated over $\mathcal{D}_X$,which is both left and right Noetherian. The assertion for decreasing sequences follows from the previous one using the anti-equivalence in Proposition 6.36. $\square$

REMARK 6.38. If X is affine and M is a finitely generated module over $\mathcal{D}_X(X)$, we have the Sato-Kashiwara filtration on M discussed in Chapter 5.3. Note that if $f\in\mathcal{O}_X(X)$ is nonzero, then $C^i(M_f)=C^i(M)_f$ (this follows, for example, from the cohomological description of the filtration in Theorem 5.26). This implies that this filtration globalizes: for every coherent $\mathcal{D}_X$-module $\mathcal{M}$ on a smooth, irreducible, n-dimensional variety X we have a decreasing filtration on $\mathcal{M}$ by $\mathcal{D}_X$-submodules

$$C^0(\mathcal{M})=\mathcal{M}\supseteq\ldots\supseteq C^n(\mathcal{M})\supseteq C^{n+1}(\mathcal{M})=0$$

(the fact that $C^{n+1}(\mathcal{M})=0$ is a consequence of Theorem 3.38). Note that $C^n(M)$ is a holonomic submodule of $\mathcal{M}$. In fact, every holonomic submodule $\mathcal{M}'\subseteq\mathcal{M}$ is contained in $C^n(\mathcal{M})$: this can be checked on affine open subsets, in which case the assertion follows from Gabber's description of $C^n(M)$.

Finally, we note that it follows from (the globalized version of) Theorem 5.26 that $C^n(\mathcal{M})$ is the image of the canonical injective map

$$\mathcal{E}xt^0_{\mathcal{D}_X^{\rm op}}\big(\tau^{\geq n}\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X),\mathcal{D}_X^{\rm op}\big)\hookrightarrow\mathcal{M}.$$

Since $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)=0$ for $i>n$ by Corollary 6.28, the above map becomes

$$\mathcal{E}xt^n_{\mathcal{D}_X^{\rm op}}\big(\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X),\mathcal{D}_X^{\rm op}\big)\hookrightarrow\mathcal{M}.$$

The following technical result will be used in the next section in the proof of the existence of b-functions.

PROPOSITION 6.39. Let X be a smooth, irreducible variety and $U\subseteq X$ an open subset. If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module and $\mathcal{N}$ is a holonomic $\mathcal{D}_U$-submodule of $\mathcal{M}|_U$, then there is a holonomic $\mathcal{D}_X$-submodule $\widetilde{\mathcal{N}}$ of $\mathcal{M}$ such that $\widetilde{\mathcal{N}}|_U=\mathcal{N}$.

PROOF. Let $n=\dim(X)$. Since $\mathcal{N}$ is a holonomic submodule of $\mathcal{M}|_U$, it follows from Remark 6.38 that $\mathcal{N}\subseteq C^n(\mathcal{M}|_U)=C^n(\mathcal{M})|_U$. After replacing $\mathcal{M}$ by $C^n(\mathcal{M})$, we may assume that $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module. Arguing as in the proof of Proposition 3.25, we choose a coherent $\mathcal{O}_U$-module $\mathcal{N}_0\subseteq\mathcal{N}$ such that $\mathcal{N}=\mathcal{D}_U\cdot\mathcal{N}_0$. By [**Har77**, Exercise II.5.15], we can find a coherent $\mathcal{O}_X$-submodule $\widetilde{\mathcal{N}}_0\subseteq\mathcal{M}$ such that $\widetilde{\mathcal{N}}_0|_U=\mathcal{N}_0$. If we take $\widetilde{\mathcal{N}}=\mathcal{D}_X\cdot\widetilde{\mathcal{N}}_0$, then we clearly have $\widetilde{\mathcal{N}}|_U=\mathcal{N}$. Moreover, $\widetilde{\mathcal{N}}$ is a coherent $\mathcal{D}_X$-module, hence it is holonomic being a $\mathcal{D}_X$-submodule of a holonomic module. □

We end this section with a discussion of a refinement of the characteristic variety. We will not need it in what follows, but it is a very important invariant of a coherent $\mathcal{D}_X$-module, so it is worth mentioning it.

DEFINITION 6.40. Let X by a smooth variety and $\pi\colon Y=T^*X\to X$ the canonical projection. Given a coherent $\mathcal{D}_X$-module $\mathcal{M}$ on X, consider a good filtration $F_\bullet\mathcal{M}$ on $\mathcal{M}$ and let $\mathcal{F}$ be the coherent sheaf on T^*X such that $\pi_*(\mathcal{F})\simeq{\rm Gr}^F_\bullet(\mathcal{M})$. The *characteristic cycle* ${\rm CC}(\mathcal{M})$ is given by

$${\rm CC}(\mathcal{M}):=\sum_Z\ell_{\mathcal{O}_{Y,Z}}(\mathcal{F}_Z)Z,$$

where the sum is over the irreducible components of ${\rm Supp}(\mathcal{F})={\rm Char}(\mathcal{M})$.

PROPOSITION 6.41. The definition of the characteristic cycle is independent of the choice of good filtration.

PROOF. This assertion is a bit trickier than in the case of the characteristic variety. The argument that we give is based on a trick due to Bernstein. We first treat the case when we have two good filtrations $F_\bullet\mathcal{M}$ and $G_\bullet\mathcal{M}$ such that $G_\bullet\mathcal{M}$ is *close* to $F_\bullet\mathcal{M}$, in the sense that

$$F_i\mathcal{M}\subseteq G_i\mathcal{M}\subseteq F_{i+1}\mathcal{M}\quad\text{for all}\quad i\in\mathbf{Z}.$$

In this case, for every $i\in\mathbf{Z}$, we get an induced map

$$\varphi_i\colon F_i\mathcal{M}/F_{i-1}\mathcal{M}\to G_i\mathcal{M}/G_{i-1}\mathcal{M}$$

and we have

$${\rm Ker}(\varphi)=G_{i-1}\mathcal{M}/F_{i-1}\mathcal{M}\quad\text{and}\quad{\rm Coker}(\varphi)=G_i\mathcal{M}/F_i\mathcal{M}.$$

Consider the graded ${\rm Gr}^F_\bullet(\mathcal{D}_X)$-module $\mathcal{N}=\bigoplus_{i\in\mathbf{Z}}G_i\mathcal{M}/F_i\mathcal{M}$. We see that if $\varphi=\bigoplus_i\varphi_i$, we get an exact sequence of graded ${\rm Gr}^F_\bullet(\mathcal{D}_X)$-modules

$$0\to\mathcal{N}(-1)\to{\rm Gr}^F_\bullet(\mathcal{M})\xrightarrow{\varphi}{\rm Gr}^G_\bullet(\mathcal{M})\to\mathcal{N}\to 0.$$

Since $\mathcal{N}$ and $\mathcal{N}(-1)$ are isomorphic as (nongraded) modules over $\mathrm{Gr}^F_\bullet(\mathcal{D}_X)$, we see that the corresponding coherent $\mathcal{O}_Y$-modules have the same length at the generic point of every irreducible component of $\mathrm{Char}(\mathcal{M})$. This proves the equality of the characteristic cycles associated to $F_\bullet\mathcal{M}$ and $G_\bullet\mathcal{M}$ in this case.

Suppose now that $F_\bullet\mathcal{M}$ and $G_\bullet$ are two arbitrary good filtrations on $\mathcal{M}$. For every $k\in\mathbf{Z}$, we define a filtration $F^{(k)}_\bullet\mathcal{M}$ on $\mathcal{M}$ by

$$F^{(k)}_i\mathcal{M}=F_i\mathcal{M}+G_{i+k}\mathcal{M}\quad\text{for all}\quad i\in\mathbf{Z}.$$

It is straightforward to see that this is, indeed, a good filtration on $\mathcal{M}$. Since both $F_\bullet\mathcal{M}$ and $G_\bullet\mathcal{M}$ are good filtrations, it follows from Proposition 3.24 that for $k\ll 0$ we have $F^{(k)}_\bullet\mathcal{M}=F_\bullet\mathcal{M}$ and for $k\gg 0$, we have $F^{(k)}_\bullet\mathcal{M}=G_{\bullet+k}\mathcal{M}$. On the other hand, note that we have

$$F^{(k)}_i\mathcal{M}\subseteq F^{(k+1)}_i\mathcal{M}\subseteq F^{(k)}_{i+1}\mathcal{M}\quad\text{for all}\quad i\in\mathbf{Z}.$$

Therefore $F^{(k+1)}_\bullet\mathcal{M}$ is close to $F^{(k)}_\bullet\mathcal{M}$ and, by what we have already proved, it follows that the characteristic cycles constructed with respect to $F^{(k)}_\bullet\mathcal{M}$ and $F^{(k+1)}_\bullet\mathcal{M}$ agree for all $k\in\mathbf{Z}$. Since clearly $G_\bullet\mathcal{M}$ and $G_{\bullet+k}\mathcal{M}$ give the same characteristic cycle, we obtain the conclusion in the proposition. $\square$

For holonomic $\mathcal{D}_X$-modules, we obtain the following additivity property of characteristic cycles:

PROPOSITION 6.42. If X is a smooth, irreducible variety and we have a short sequence

$$0\to\mathcal{M}'\to\mathcal{M}\to\mathcal{M}''\to 0$$

of holonomic $\mathcal{D}_X$-modules on X, then

$$\mathrm{CC}(\mathcal{M})=\mathrm{CC}(\mathcal{M}')+\mathrm{CC}(\mathcal{M}'').$$

PROOF. The assertion follows from the equality

$$\mathrm{Char}(\mathcal{M})=\mathrm{Char}(\mathcal{M}')\cup\mathrm{Char}(\mathcal{M}''),$$

the fact that all irreducible components of these varieties have the same dimension, and the additivity of length in short exact sequences. $\square$

6.5. Existence of b-functions

In the next section we will prove the preservation of holonomicity by push-forward and pull-back functors. The most interesting result in this direction is the preservation of holonomicity by push-forward under open immersions, that we treat now. The key ingredient here is the existence of b-functions for holonomic $\mathcal{D}$-modules. As we will see later, b-functions also provide interesting invariants of singularities.

Given a smooth, irreducible algebraic variety X over k and $f\in\mathcal{O}_X(X)$ nonzero, let $U=\{x\in X\mid f(x)\neq 0\}$ and let $j\colon U\hookrightarrow X$ be the inclusion. We consider the sheaf of $k[s]$-algebras $\mathcal{D}_X[s]:=k[s]\otimes_k\mathcal{D}_X$ on X (here s is a new variable). Given a quasi-coherent $\mathcal{D}_X$-module $\mathcal{M}$ which is an $\mathcal{O}_X[1/f]$-module (equivalently, the canonical morphism $\mathcal{M}\to j_+(\mathcal{M}|_U)$ is an isomorphism), we consider the $\mathcal{O}_X[s]$-module

$$\mathcal{M}[s]f^s:=\mathcal{M}\otimes_k k[s]f^s,$$

where $k[s]f^s$ is a free $k[s]$-module with basis the symbol f^s. For $P(s) \in k[s]$ and $u \in M$, we write $P(s)uf^s$ instead of $u \otimes P(s)f^s$.

LEMMA 6.43. *The $\mathcal{O}_X[s]$-module structure on $\mathcal{M}[s]f^s$ can be extended to a $\mathcal{D}_X[s]$-module structure such that every $D \in \mathcal{T}_X$ acts by the "expected" rule*

$$D \cdot P(s)uf^s = P(s)(Du)f^s + sP(s)\frac{D(f)}{f}uf^s.$$

PROOF. The assertion can be checked locally on affine charts, in which case the presentation of $\mathcal{D}_X$ in Proposition 2.25 induces a similar presentation of $\mathcal{D}_X[s]$. The only nontrivial things to check are that $D \cdot P(s)huf^s - h\big(D \cdot P(s)uf^s\big) = P(s)D(h)uf^s$ if $h \in \mathcal{O}_X$ and $D \in \mathcal{T}_X$ (this is clear) and that

$$D_1 \cdot \big(D_2 \cdot P(s)uf^s\big) - D_2 \cdot \big(D_1 \cdot P(s)uf^s\big) = [D_1, D_2] \cdot P(s)uf^s$$

if $D_1, D_2 \in \mathcal{T}_X$. This follows easily from the definition:

$$\begin{aligned}
&D_1 \cdot \big(D_2 \cdot P(s)uf^s\big) - D_2 \cdot \big(D_1 \cdot P(s)uf^s\big)\\
&= D_1 \cdot \Big(P(s)(D_2u)f^s + sP(s)\frac{D_2(f)}{f}uf^s\Big) - D_2 \cdot \Big(P(s)(D_1u)f^s + sP(s)\frac{D_1(f)}{f}uf^s\Big)\\
&= P(s)D_1(D_2u)f^s + sP(s)\frac{D_1(f)}{f}(D_2u)f^s + sP(s)\frac{D_2(f)}{f}(D_1u)f^s\\
&+sP(s)\cdot\frac{D_1\big(D_2(f)\big) - D_2(f)D_1(f)}{f^2}uf^s + s^2P(s)\frac{D_1(f)D_2(f)}{f^2}uf^s - P(s)D_2(D_1u)f^s\\
&-sP(s)\frac{D_2(f)}{f}(D_1u)f^s - sP(s)\frac{D_1(f)}{f}(D_2u)f^s - sP(s)\cdot\frac{D_2\big(D_1(f)\big) - D_1(f)D_2(f)}{f^2}uf^s\\
&-s^2P(s)\frac{D_1(f)D_2(f)}{f^2}uf^s = P(s)\big([D_1, D_2]u\big)f^s + sP(s)\frac{[D_1, D_2](f)}{f}uf^s\\
&= [D_1, D_2] \cdot P(s)uf^s.
\end{aligned}$$

□

REMARK 6.44. Note that we have a $\mathcal{D}_X$-linear automorphism given by $s \mapsto s+1$, that is

$$T \colon \mathcal{M}[s]f^s \to \mathcal{M}[s]f^s \quad \text{given by} \quad T\big(P(s)uf^s\big) = P(s+1)fuf^s,$$

with inverse given by $T^{-1}\big(P(s)uf^s\big) = P(s-1)\frac{1}{f}uf^s$. Note that T is not $k[s]$-linear, but satisfies $T\big(Q(s)w\big) = Q(s+1)w$ for every $w \in \mathcal{M}[s]f^s$ and $Q \in k[s]$.

The following is the main result of this section:

THEOREM 6.45. *With the above notation, if $\mathcal{M}|_U$ is a holonomic $\mathcal{D}_U$-module, then for every $u \in \Gamma(X, \mathcal{M}) = \Gamma(U, \mathcal{M}|_U)$, there is a nonzero $b(s) \in k[s]$ such that*

$$b(s)uf^s \in \mathcal{D}_X[s] \cdot (fu)f^s. \tag{6.4}$$

PROOF. Let $n = \dim(X)$. If we consider an open cover $X = U_1 \cup \ldots \cup U_r$ and if for every i we have a nonzero $b_i(s) \in k[s]$ that satisfies the condition in the theorem on U_i, then the least common multiple $b(s)$ of $b_1(s), \ldots, b_r(s)$ satisfies the condition in the theorem on X. Therefore from now on we may and will assume that X is affine, with $R = \mathcal{O}_X(X)$ and $M = \mathcal{M}(X)$.

Let $K = \overline{k(s)}$ be an algebraic closure of $k(s)$ (the fact that we have to go to the algebraic closure is due to the fact that in developing the theory we always assumed that the ground field is algebraically closed). In addition to $M[s]f^s$, we will also consider

$$M(s)f^s := M \otimes_k k(s)f^s = M[s]f^s \otimes_{k[s]} k(s) \quad \text{and}$$

$$M_K f^s := M \otimes_k K f^s = M[s]f^s \otimes_{k[s]} K.$$

Let $U_K \hookrightarrow X_K$ be the morphism induced by $U \hookrightarrow X$ via extension of scalars to K, so $\overline{R} := \mathcal{O}_{X_K}(X_K) = R \otimes_k K$ and $\mathcal{O}_{X_K}(U_K) = R_f \otimes_k K = \overline{R}_f$. Note that $D_R \otimes_k K = D_{\overline{R}/K}$, which we simply denote by $D_{\overline{R}}$, and $D_{R_f} \otimes_k K = D_{\overline{R}_f/K}$, which again we simply denote by $D_{\overline{R}_f}$ (this follows, for example from Proposition 2.25). Since $M[s]f^s$ is a $D_R[s]$-module (in fact, a $D_{R_f}[s]$-module), it is clear that $M_K f^s$ is a $D_{\overline{R}}$ (in fact, a $D_{\overline{R}_f}$-module).

We first show that as a $D_{\overline{R}_f}$-module, $M_K f^s$ is holonomic. Indeed, by assumption M is holonomic as a D_{R_f}-module. Let $F_\bullet M$ be a good filtration of M as a D_{R_f}-module, so each $F_p M$ is an R_f-module. For every $p \in \mathbf{Z}$, we put

$$F_p M_K f^s = F_p M \otimes_k K f^s.$$

Note that if $u \in F_p M$ and D is a derivation in D_R, then for all $P \in K$, we have

$$D \cdot P u f^s = P(Du)f^s + sP\tfrac{D(f)}{f} u f^s \in F_{p+1} M_K f^s + F_p M_K f^s \subseteq F_{p+1} M_K f^s.$$

It is then clear that $F_\bullet M_K f^s$ is a filtration of $M_K f^s$ as a $D_{\overline{R}_f}$-module and furthermore, we see that $\mathrm{Gr}_\bullet^F(M_K f^s) \simeq \mathrm{Gr}_\bullet^F(M) \otimes_k K$ as a module over

$$\mathrm{Sym}^\bullet_{\overline{R}_f}\big(\mathrm{Der}_K(\overline{R}_f)\big) \simeq \mathrm{Sym}^\bullet_{R_f}\big(\mathrm{Der}_k(R_f)\big) \otimes_k K.$$

In particular, this is a finitely generated module (hence the filtration on $M_K f^s$ is good) and its dimension is equal to the dimension of $\mathrm{Gr}_\bullet^F(M)$, which by assumption is n. Since $\dim(\overline{R}_f) = \dim(R_f) = n$, we conclude that indeed $M_K f^s$ is a holonomic $D_{\overline{R}_f}$-module.

In particular, we know that $M_K f^s$ is a finitely generated $D_{\overline{R}_f}$-module, hence we can find a finitely generated $D_{\overline{R}}$-submodule $N \subseteq M_K f^s$ such that $N_f = M_K f^s$. Using the fact that $M_K f^s$ is a holonomic $D_{\overline{R}_f}$-module and Proposition 6.39, we conclude that, in fact, we may assume that N is a holonomic $D_{\overline{R}}$-module.

Since $uf^s \in M_K f^s$, it follows that there is $d \geq 0$ such that $f^d u f^s \in N$. Consider now the following decreasing sequence of $D_R[s]$-submodules of $M[s]f^s$:

$$D_R[s] \cdot f^d u f^s \supseteq D_R[s] \cdot f^{d+1} u f^s \supseteq \ldots.$$

Tensoring with K over $k[s]$, we obtain a decreasing sequence of submodules of N:

$$D_{\overline{R}} \cdot f^d u f^s \supseteq D_{\overline{R}} \cdot f^{d+1} u f^s \supseteq \ldots. \tag{6.5}$$

Since N is a holonomic $D_{\overline{R}}$-module, it follows from Proposition 6.37 that this sequence stabilizes. Moreover, if (6.5) stabilizes after tensoring with K over $k[s]$, then it stabilizes after tensoring with $k(s)$ over $k[s]$. Therefore there are $\ell \geq 0$ and $Q \in D_R \otimes_k k(s)$ such that

$$f^{d+\ell} u f^s = Q \cdot f^{d+\ell+1} u f^s.$$

If $q(s) \neq 0$ is such that $P(s) := q(s)Q(s) \in D_R[s]$, then we have

$$q(s) f^{d+\ell} u f^s = P \cdot f^{d+\ell} u f^s.$$

Applying $T^{-d-\ell}$, where T is the automorphism in Remark 6.44, we see that if $b(s)=q(s-d-\ell)$, then $b(s)uf^s\in D_R[s]\cdot fuf^s$. This completes the proof of the theorem. $\square$

REMARK 6.46. In the setting of Theorem 6.45, it is clear that the set of polynomials $b(s)\in k[s]$ that satisfy the conclusion of the theorem is a (nonzero) ideal in $k[s]$. The monic generator of this ideal is the *b-function* $b_w(s)$, where it is common to label this by the element $w=uf^s\in\Gamma\big(X,\mathcal{M}[s]f^s\big)$. An important special case is that when $\mathcal{M}=\mathcal{O}_X[1/f]$ and $u=1$, so $w=f^s$, in which the corresponding b-function is denoted by $b_f(s)$, and it is also called the *Bernstein-Sato polynomial* of f. The existence of $b_f(s)$ for certain polynomials f was proved and used by Sato, while the existence for arbitrary polynomials f was proved by Bernstein in [**Ber71**]. For arbitrary f (also for holomorphic functions), the existence of $b_f(s)$ is due to Kashiwara [**Kas77**].

REMARK 6.47. For future reference, we note that, with the notation in the proof of Theorem 6.45, the $D_{\overline{R}}$-module $D_{\overline{R}}\cdot uf^s$ is holonomic. Indeed, since N is holonomic, it is enough to show that $D_{\overline{R}}\cdot uf^s\subseteq N$. Recall that we have a nonnegative integer d such that $f^duf^s\in N$. On the other hand, we have a nonzero $b(s)$ such that $b(s)uf^s\in D_R[s]\cdot fuf^s$. Using this and applying the automorphisms $T,\ldots,T^{d-1}$, we see that if $p(s)=\prod_{i=0}^{d-1}b(s+i)$, then $p(s)uf^s\in D_R[s]\cdot f^duf^s$. Since $p(s)$ is invertible in K, it follows that $D_{\overline{R}}\cdot uf^s\subseteq N$.

REMARK 6.48. In the setting of Theorem 6.45, we may specialize s to integers. More precisely, for every $m\in\mathbf{Z}$, we have a morphism of sheaves of rings $\alpha_m\colon\mathcal{D}_X[s]\to\mathcal{D}_X$ that maps $P(s)$ to $P(m)$ for $P(s)\in\mathcal{D}_X[s]$ and a surjective morphism of sheaves of $\mathcal{D}_X$-modules $\beta_m\colon\mathcal{M}[s]f^s\to\mathcal{M}$ where $\beta_m\big(Q(s)uf^s\big)=Q(m)f^mu$ for every $Q(s)\in k[s]$ and $u\in\mathcal{M}$ (we here consider $\mathcal{M}$ as a $\mathcal{D}_X[s]$-module via α_m). In particular, by specializing s to $m\in\mathbf{Z}$ in (6.4), we conclude that $b(m)f^mu\in\mathcal{D}_X\cdot f^{m+1}u$. Note also that since multiplication by f is invertible on $\mathcal{M}$, the kernel of β_m is $(s-m)\cdot\mathcal{M}[s]f^s$.

REMARK 6.49. In the setting of Theorem 6.45, if $\mathcal{M}|_U$ is generated over $\mathcal{D}_U$ by $u_1,\ldots,u_r\in\Gamma(U,\mathcal{M}_U)$, and if $m_1,\ldots,m_r\in\mathbf{Z}$ are such that for every i, the polynomial $b_{u_i}(s)$ has no integer roots $<-m_i$, then $\mathcal{M}$ is generated over $\mathcal{D}_X$ by $\frac{1}{f^{m_1}}u_1,\ldots,\frac{1}{f^{m_r}}u_r$. Indeed, we know that $\mathcal{M}$ is generated over $\mathcal{D}_X$ by $\frac{1}{f^m}u_i$, where $m\in\mathbf{Z}$ and $1\le i\le r$. On the other hand, we know that for every i, we have

$$b_{u_i}(-m)\tfrac{1}{f^m}u_i\in\mathcal{D}_X\cdot\tfrac{1}{f^{m-1}}u_i\quad\text{for all}\quad m\in\mathbf{Z}$$

(see Remark 6.48). If $m>m_i$, then by assumption $b_{u_i}(-m)\neq 0$, and thus $\frac{1}{f^m}u_i$ lies in the $\mathcal{D}_X$-submodule generated by $\frac{1}{f^{m-1}}u_i$. This implies our assertion. In particular, we see that $\mathcal{M}$ is coherent (we will see in the next result that $\mathcal{M}$ is, in fact, holonomic).

We make use of the existence of b-functions in order to prove preservation of holonomicity under direct image via open immersions.

THEOREM 6.50. *If X is a smooth, irreducible variety, U is an open subset of X, and $j\colon U\hookrightarrow X$ is the inclusion, then j_+ maps $\mathcal{D}^b_{\rm hol}(\mathcal{D}_U)$ to $\mathcal{D}^b_{\rm hol}(\mathcal{D}_X)$.*

PROOF. Standard arguments using the truncation functors in Example A.20 imply that it is enough to show that for every holonomic $\mathcal{D}_U$-module $\mathcal{N}$, we have

$\mathcal{H}^i(j_+\mathcal{N})$ holonomic for all $i \in \mathbf{Z}$. Since this is a local assertion on X, after taking a suitable affine cover, we may and will assume that X is affine, with $R = \mathcal{O}_X(X)$, in which case we need to prove that $H^q(U, \mathcal{O}_U)$ is a holonomic D_R-module for every $q \in \mathbf{Z}$.

Furthermore, it is enough to show that this is the case when $U = X \smallsetminus V(f)$, when $f \in R$ is nonzero. Indeed, let us assume that we know the assertion in this case, and let us write $U = X \smallsetminus V(f_1, \ldots, f_r)$, for suitable nonzero $f_1, \ldots, f_r \in R$. If $U_i = X \smallsetminus V(f_i)$, then we can compute $H^q(U, \mathcal{O}_U)$ as the q-th cohomology of the Čech complex

$$0 \to \bigoplus_{i_1} \Gamma(U_{i_1}, \mathcal{O}_X) \to \bigoplus_{i_1<i_2} \Gamma(U_{i_1} \cap U_{i_2}, \mathcal{O}_X) \to \ldots \to \Gamma(U_1 \cap \ldots \cap U_r, \mathcal{O}_X) \to 0,$$

placed in cohomological degrees $0, \ldots, r-1$. Since the holonomic D_R-modules form an Abelian category (see Remark 6.34), and since each intersection $U_{i_1} \cap \ldots \cap U_{i_\ell}$ can be written as $X \smallsetminus V(f_{i_1} \cdots f_{i_\ell})$, we conclude that $H^q(U, \mathcal{O}_U)$ is a holonomic D_R-module for all $q \in \mathbf{Z}$.

Therefore from now on we assume that $U = X \smallsetminus V(f)$ for some nonzero $f \in R$. We need to show that if M is a holonomic D_{R_f}-module, then M is holonomic also as a D_R-module. Arguing by induction on the number of generators, we reduce to the case when M is generated over D_{R_f} by one nonzero element $u \in M$. Let us consider the finitely generated $D_R[s]$-module $D_R[s]uf^s \subseteq M[s]f^s$. On $D_R[s]$ we have the filtration given by $F_p\big(D_R[s]\big) = F_pD_R \otimes_k k[s]$ for $p \in \mathbf{Z}$ and on $D_R[s]uf^s$ we consider the corresponding good filtration given $F_p\big(D_R[s]uf^s\big) = F_p\big(D_R[s]\big) \cdot uf^s$ for all $p \in \mathbf{Z}$. Let $W = \mathrm{Gr}^F_\bullet\big(D_R[s]uf^s\big)$, which is a finitely generated graded module over $\mathrm{Gr}^F_\bullet(D_R)[s]$. Note that if $K = \overline{k(s)}$ and $\overline{R} = R \otimes_k K$, then $W \otimes_{k[s]} K$ is the graded module over $\mathrm{Gr}^F_\bullet(D_{\overline{R}})$ corresponding to a good filtration on $D_{\overline{R}} \cdot uf^s$, which is a holonomic $D_{\overline{R}}$-module by Remark 6.47. Therefore the dimension of the support of $W \otimes_{k[s]} K$ over $\mathrm{Gr}^F_\bullet(D_R) \otimes_k K$ is $\dim(X)$.

Recall now that for every $m \in \mathbf{Z}$ we have a $D_R[s]$-linear map

$$D_R[s] \cdot uf^s \hookrightarrow M[s]f^s \xrightarrow{\beta_m} M,$$

where $D_R[s]$ acts on M via the map $D_R[s] \to D_R$ that maps s to m (see Remark 6.48). Note that the composition is surjective for $m \ll 0$, as follows from the existence of the b-function (see Remark 6.49). If we consider the filtration $F_\bullet M$ on M induced by the filtration on $D_R[s] \cdot uf^s$, we see that $\mathrm{Gr}^F_\bullet(M)$ is a quotient of $W \otimes_{k[s]} k[s]/(s-m)$. It is a consequence of Nakayama's Lemma that the support of $W \otimes_{k[s]} k[s]/(s-m)$ over $\mathrm{Gr}^F_\bullet(R)$ is the fiber of $\mathrm{Supp}(W)$ over $s = m$. We deduce using the behavior of behavior of fibers of morphisms that for all m but a finite set, the dimension of the support of $W \otimes_{k[s]} k[s]/(s-m)$ is equal to the dimension of the support of $W \otimes_{k[s]} K$, hence to $\dim(X)$. We thus conclude that M is a holonomic D_R-module. $\square$

6.6. Preservation of holonomicity

Our goal in this section is to prove the following two important results concerning preservation of holonomicity for the functors that we defined for $\mathcal{D}$-modules.

THEOREM 6.51. *If $f\colon X \to Y$ is a morphism between smooth, irreducible varieties over k, then $\mathbf{L}f^*\colon \mathcal{D}^b_{\rm qc}(\mathcal{D}_Y) \to \mathcal{D}^b_{\rm qc}(\mathcal{D}_X)$ induces a functor $\mathcal{D}^b_{\rm hol}(\mathcal{D}_Y) \to \mathcal{D}^b_{\rm hol}(\mathcal{D}_X)$.*

THEOREM 6.52. *If $f\colon X \to Y$ is a morphism between smooth, irreducible varieties over k, then $f_+\colon \mathcal{D}^b(\mathcal{D}_X) \to \mathcal{D}^b(\mathcal{D}_Y)$ induces a functor $\mathcal{D}^b_{\rm hol}(\mathcal{D}_X) \to \mathcal{D}^b_{\rm hol}(\mathcal{D}_Y)$.*

REMARK 6.53. A special case of Theorem 6.52 says that if $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module on a smooth, irreducible, n-dimensional variety X, then its q-de Rham cohomology

$$H^q_{\rm dR}(\mathcal{M}) := R^{q-n}\Gamma\big(\mathrm{DR}_X(\mathcal{M})\big)$$

is a finitely generated k-vector space (note that the shift by n is due to the fact that we consider the de Rham complex to be placed in cohomological degrees $-n,\ldots,0$). Indeed, it follows from Example 6.16 that this is $\mathcal{H}^{q-n}\big(p_+(\mathcal{M})\big)$, where p is the canonical morphism $X \to \mathrm{Spec}(k)$. Note that by taking $\mathcal{M} = \mathcal{O}_X$, we obtain the q-de Rham cohomology $H^q_{\rm dR}(X)$ of the variety X, and we see that $\dim_k H^q_{\rm dR}(X) < \infty$ for all q.

Before giving the proofs of these theorems, we need some preparations. We begin with a general result concerning the preservation of coherence under proper morphisms.

THEOREM 6.54. *If $f\colon X \to Y$ is a proper morphism between smooth irreducible varieties over k, then $f_+\colon \mathcal{D}^b(\mathcal{D}_X) \to \mathcal{D}^b(\mathcal{D}_Y)$ induces a functor $\mathcal{D}^b_{\rm coh}(\mathcal{D}_X) \to \mathcal{D}^b_{\rm coh}(\mathcal{D}_Y)$.*

PROOF. We need to show that if $u \in \mathcal{D}^b_{\rm coh}(\mathcal{D}_X)$, then $\mathcal{H}^q\big(f_+(u)\big)$ is a coherent $\mathcal{D}_Y$-module for all q. A standard inductive argument using the truncation functors in Example A.20 shows that it is enough to prove this in the case when $u = \mathcal{M}$ is a coherent $\mathcal{D}_X$-module. Furthermore, since the equivalence of categories between left and right $\mathcal{D}_X$-modules that we discussed in Chapter 3.6 preserves coherence, it is enough to show that for every *right* coherent $\mathcal{D}_X$-module $\mathcal{M}$, we have that $\mathcal{H}^q\big(f_+(\mathcal{M})\big)$ is a coherent right $\mathcal{D}_X$-module for all q.

We will use the fact that $\mathcal{M}$ admits a surjective morphism of $\mathcal{D}_X$-modules $\mathcal{F} \to \mathcal{M}$, with $\mathcal{F}$ an *induced coherent* right $\mathcal{D}_X$-module: this means that $\mathcal{F} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X$, where $\mathcal{E}$ is a coherent $\mathcal{O}_X$-module and the right $\mathcal{D}_X$-module structure on the tensor product comes from $\mathcal{D}_X$. Indeed, note the we have a canonical surjective morphism of right $\mathcal{D}_X$-modules

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \mathcal{M}, \quad v \otimes P \mapsto vP.$$

Furthermore, since $\mathcal{M}$ is locally finitely generated over $\mathcal{D}_X$, it follows as in the proof of Proposition 3.25 that there is a coherent $\mathcal{O}_X$-submodule $\mathcal{E} \subseteq \mathcal{M}$ such that $\mathcal{M} = \mathcal{E} \cdot \mathcal{D}_X$. Therefore the composition

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X \hookrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \mathcal{M}$$

is surjective.

Suppose first that we know that $f_+(\mathcal{F}) \in \mathcal{D}^b_{\rm coh}(\mathcal{D}_Y^{\rm op})$ for every induced coherent right $\mathcal{D}_X$-module. We show that $\mathcal{H}^q\big(f_+(\mathcal{M})\big)$ is a coherent right $\mathcal{D}_Y$-module for every $\mathcal{M}$ and every q by descending induction on q. This is clear for $q \gg 0$, since in this case $\mathcal{H}^q\big(f_+(\mathcal{M})\big)=0$. If we know the assertion for $q+1$ and for all coherent right $\mathcal{D}_X$-modules, then given $\mathcal{M}$, we consider a short exact sequence

$$0\to\mathcal{N}\to\mathcal{F}\to\mathcal{M}\to 0,$$

with $\mathcal{F}$ an induced coherent right $\mathcal{D}_X$-module. Since we have an exact sequence

$$\mathcal{H}^q\big(f_+(\mathcal{F})\big)\to\mathcal{H}^q\big(f_+(\mathcal{M})\big)\to\mathcal{H}^{q+1}\big(f_+(\mathcal{N})\big),$$

we conclude that $\mathcal{H}^q\big(f_+(\mathcal{M})\big)$ is coherent using the inductive hypothesis for $\mathcal{N}$ and the assertion for coherent induced $\mathcal{D}$-modules.

In order to conclude the proof, it is thus enough to consider the case when $\mathcal{M}=\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{D}_X$. In this case, by definition we have

$$f_+(\mathcal{M})=\mathbf{R}f_*\big((\mathcal{E}\otimes^L_{\mathcal{O}_X}\mathcal{D}_X)\otimes^L_{\mathcal{D}_X}\mathcal{D}_{X\to Y}\big)\simeq\mathbf{R}f_*\big(\mathcal{E}\otimes^L_{\mathcal{O}_X}(\mathcal{O}_X\otimes^L_{f^{-1}(\mathcal{O}_Y)}f^{-1}(\mathcal{D}_Y))\big)$$
$$\simeq\mathbf{R}f_*\big(\mathcal{E}\otimes^L_{f^{-1}(\mathcal{O}_Y)}f^{-1}(\mathcal{D}_Y)\big)\simeq\mathbf{R}f_*(\mathcal{E})\otimes^L_{\mathcal{O}_Y}\mathcal{D}_Y,$$

where the last isomorphism is a consequence of the projection formula. Since f is proper, we know that $R^qf_*(\mathcal{E})$ is a coherent $\mathcal{O}_X$-module for all q, and since $\mathcal{D}_Y$ is a flat $\mathcal{O}_Y$-module, it follows that

$$\mathcal{H}^q\big(f_+(\mathcal{M})\big)\simeq R^qf_*(\mathcal{E})\otimes_{\mathcal{O}_Y}\mathcal{D}_Y,$$

hence it is a coherent right $\mathcal{D}_Y$-module. This completes the proof of the theorem. $\square$

Next, we introduce a slightly modified version of the antiequivalence discussed in Chapter 6.4: for a smooth, irreducible n-dimensional variety, we consider the exact functor

$$\mathbf{D}_X\colon\mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_X)\to\mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_X),\quad u\mapsto\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(u,\mathcal{D}_X)\otimes_{\mathcal{O}_X}\omega_X^{-1}[n].$$

Note that $-\otimes_{\mathcal{O}_X}\omega_X^{-1}$ is (the derived version of) the equivalence from $\mathcal{D}_X^{\mathrm{op}}$-modules to $\mathcal{D}_X$-modules discussed in Chapter 3.6.

Exercise 6.55. Show that if $\mathcal{M}$ is a quasi-coherent $\mathcal{D}_X$-module, then we have a canonical isomorphism of $\mathcal{D}_X$-modules

$$\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\omega_X\otimes_{\mathcal{O}_X}\mathcal{M},\mathcal{D}_X^{\mathrm{op}})\simeq\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)\otimes_{\mathcal{O}_X}\omega_X^{-1}.$$

Deduce that for every $u\in\mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X)$, we have an isomorphism in $\mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_X)$:

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\omega_X\otimes_{\mathcal{O}_X}u,\mathcal{D}_X^{\mathrm{op}})\simeq\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(u,\mathcal{D}_X)\otimes_{\mathcal{O}_X}\omega_X^{-1}.$$

Remark 6.56. Note that we have a canonical transformation of functors on $\mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_X)$:

$$\mathrm{Id}\to\mathbf{R}\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}\big(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(-,\mathcal{D}_X),\mathcal{D}_X^{\mathrm{op}}\big)$$

and a similar one for the corresponding functors on $\mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_X^{\mathrm{op}})$. In order to check that these are isomorphisms of functors, we can argue locally, in which case the assertion follows from Theorem 5.5. It is then clear that $\mathbf{D}_X$ is an antiequivalence of categories and, using the above exercise, we see that its inverse is isomorphic to $\mathbf{D}_X$.

Remark 6.57. With this notation, Proposition 6.36 says that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is holonomic if and only if $\mathbf{D}_X(\mathcal{M})$ is supported in cohomological degree 0, in which case $\mathbf{D}_X(\mathcal{M})$ is holonomic by Proposition 6.35. It is then standard to check, using the truncation functors in Example A.20 that if $u\in\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$, then $\mathbf{D}_X(u)\in\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$.

EXERCISE 6.58. Show that for every X, we have $\mathbf{D}_X(\mathcal{O}_X) \simeq \mathcal{O}_X$. Show also that if $X = \mathbf{A}^1$ and $M = k[x, x^{-1}]/k[x]$, then $\mathbf{D}_X(M) \simeq M$.

The following theorem gives a compatibility between duality and proper push-forward. For a proof, see [**HTT08**, Theorem 2.7.2].

THEOREM 6.59. *If $f\colon X \to Y$ is a proper morphism between smooth, irreducible varieties, then for every $u \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$, we have a functorial isomorphism*

$$\alpha_f(u)\colon \mathbf{D}_Y\big(f_+(u)\big) \simeq f_+\big(\mathbf{D}_X(u)\big).$$

Moreover, this satisfies the following three properties:

i) *α_f is compatible with restriction to open subsets, in the sense that if V is an open subset of Y and $g\colon f^{-1}(V) \to V$ is the induced morphism, then $\alpha_g(u|_{f^{-1}(V)}) = \alpha_f(u)|_V$.*

ii) *α_f is compatible with composition of morphisms: if $g\colon Y \to Z$ is another proper morphism, then*

$$\alpha_{g\circ f}(u) = g_+\big(\alpha_f(u)\big) \circ \alpha_g\big(f_+(u)\big).$$

iii) *α_f is compatible with duality: for every $u \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$, we have $\alpha_f\big(\mathbf{D}_X(u)\big) = \mathbf{D}_Y\big(\alpha_f(u)\big)$.*

We now introduce a shifted version of the pull-back functor: if $f\colon X \to Y$ is a morphism of smooth, irreducible varieties, with $\dim(X) = d_X$ and $\dim(Y) = d_Y$, then

$$f^\dagger\colon \mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_Y) \to \mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X), \quad f^\dagger(u) = \mathbf{L}f^*(u)[d_X - d_Y].$$

Note that it follows from Corollary 6.8 that if $g\colon Y \to Z$ is another such morphism, then we have an isomorphism of functors

$$f^\dagger \circ g^\dagger \simeq (g \circ f)^\dagger. \tag{6.6}$$

We have already seen the functor $f^\dagger$ in the case of a closed immersion in Section 6.2.

The following proposition gives an adjointness property of the pair $(f_+, f^\dagger)$ when f is a proper morphism.

PROPOSITION 6.60. If $f\colon X \to Y$ is a proper morphism, then for every $u \in \mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_X)$ and $v \in \mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_Y)$, we have a canonical isomorphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}\big(f_+(u), v\big) \simeq \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}\big(u, f^\dagger(v)\big).$$

In particular, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_Y)}\big(f_+(u), v\big) \simeq \mathrm{Hom}_{\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)}\big(u, f^\dagger(v)\big).$$

For the proof, we will use the following

LEMMA 6.61. *For every smooth, irreducible variety X, if $u \in \mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_X)$ and $v \in \mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X)$, then we have a canonical isomorphism*

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(u, v) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(u, \mathcal{D}_X) \otimes^L_{\mathcal{D}_X} v.$$

PROOF. We have a canonical morphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(u, \mathcal{D}_X) \otimes^L_{\mathcal{D}_X} v \to \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(u, v)$$

(this is the derived version of the morphism $\mathrm{Hom}_{D_R}(M, D_R) \otimes_{D_R} N \to \mathrm{Hom}_{D_R}(M, N)$ for two left D_R-modules M and N). To check that this is an isomorphism, we may argue locally, hence we may assume that X is affine. In this case, by Theorem 5.4, we can represent u by a bounded complex of projective modules, in which case the assertion is clear. $\square$

Proof of Proposition 6.60. Let $d_X = \dim(X)$ and $d_Y = \dim(Y)$. Using Lemma 6.61 and the definition of $\mathbf{D}_Y$, we have canonical isomorphisms

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}\big(f_+(u), v\big) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}\big(f_+(u), \mathcal{D}_Y\big) \otimes^L_{\mathcal{D}_Y} v \simeq \omega_Y \otimes \mathbf{D}_Y\big(f_+(u)\big) \otimes^L_{\mathcal{D}_Y} v[-d_Y]$$
$$\simeq \omega_Y \otimes f_+\big(\mathbf{D}_X(u)\big) \otimes^L_{\mathcal{D}_Y} v[-d_Y],$$

where the last isomorphism follows from Theorem 6.59. By definition of f_+ and $\mathbf{D}_X$, the last term is further canonically isomorphic to

$$f_+\big(\omega_X \otimes \mathbf{D}_X(u)\big) \otimes^L_{\mathcal{D}_Y} v[-d_Y] \simeq \mathbf{R}f_*\big(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(u, \mathcal{D}_X) \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}\big) \otimes^L_{\mathcal{D}_Y} v[d_X - d_Y]$$
$$\simeq \mathbf{R}f_*\big(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(u, \mathcal{D}_X) \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \otimes^L_{f^{-1}\mathcal{D}_Y} f^{-1}(v)\big)[d_X - d_Y],$$

where the last isomorphism follows from the projection formula. By definition of $\mathbf{L}f^*$ and $f^\dagger$, this last term is canonically isomorphic to

$$\mathbf{R}f_*\big(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(u, \mathcal{D}_X) \otimes_{\mathcal{D}_X} \mathbf{L}f^*(v)\big)[d_X - d_Y] \simeq \mathbf{R}f_*\big(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(u, \mathcal{D}_X) \otimes^L_{\mathcal{D}_X} f^\dagger(v)\big)$$
$$\simeq \mathbf{R}f_*\mathbf{R}\mathcal{H}om\big(u, f^\dagger(v)\big),$$

with the last isomorphism given by Lemma 6.61. This completes the proof of the first assertion in the proposition. The second assertion follows by taking $\mathcal{H}^0\big(\mathbf{R}\Gamma(-)\big)$. $\square$

Given any closed subvariety Z of the smooth variety X, recall that we have the functor $\Gamma_Z(-)$ defined on the category of $\mathcal{O}_X$-modules, where $\Gamma_Z(\mathcal{F})$ is the subsheaf of $\mathcal{F}$ consisting of those sections of $\mathcal{F}$ with support inside Z. This is a left exact functor and we get a corresponding derived functor $\mathbf{R}\Gamma_Z(-)$; we usually write $\mathcal{H}^q_Z(-)$ for $R^i\Gamma_Z(-)$. Note that if $\mathcal{M}$ is a $\mathcal{D}_X$-module, then $\Gamma_Z(\mathcal{M})$ is a $\mathcal{D}_X$-submodule of $\mathcal{M}$: if $\mathcal{I}_Z$ is the ideal of Z and a section u of $\mathcal{M}$ is annihilated by $\mathcal{I}_Z^m$, then for every derivation D, the section Du of $\mathcal{M}$ is annihilated by $\mathcal{I}_Z^{m+1}$. Therefore Γ_Z gives a left exact functor on $\mathcal{M}od(\mathcal{D}_X)$ and we get a corresponding derived functor

$$\mathbf{R}\Gamma_Z \colon \mathcal{D}^b(\mathcal{D}_X) \to \mathcal{D}^b(\mathcal{D}_X)$$

that preserves the subcategory $\mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X)$. Note that for every $u \in \mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X)$, we have a functorial exact triangle

$$\mathbf{R}\Gamma_Z(u) \to u \to j_+(u|_{X\smallsetminus Z}) \xrightarrow{+1}, \tag{6.7}$$

where $j \colon X \smallsetminus Z \hookrightarrow X$ is the inclusion of the complement of Z. This follows by representing u by a complex $\mathcal{I}^\bullet$ of injective modules and by noting that for every q and every open subset V of X, the sequence

$$0 \to \Gamma_Z(V, \mathcal{I}^q) \to \Gamma(V, \mathcal{I}^q) \to \Gamma(V \cap U, \mathcal{I}^q) \to 0$$

is exact, since $\mathcal{I}^q$ is flasque.

Example 6.62. An important special case is when Z is a smooth, irreducible subvariety of X and $u \in \mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X)$, in which case the exact triangle (6.7) becomes

$$i_+\big(i^\dagger(u)\big) \to u \to j_+(u|_{X\smallsetminus Z}) \xrightarrow{+1} . \tag{6.8}$$

Indeed, it is enough to show that we have a functorial isomorphism

$$\mathbf{R}\Gamma_Z(u) \simeq i_+\big(i^\dagger(u)\big) \quad \text{for every} \quad u \in \mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X).$$

Since the cohomology sheaves of $\mathbf{R}\Gamma_Z(u)$ have support in Z, it follows from Kashiwara's equivalence theorem and Proposition 6.24 that $\mathbf{R}\Gamma_Z(u) \simeq i_+(v)$, where $v = i^\dagger\big(\mathbf{R}\Gamma_Z(u)\big)$. Therefore it is enough to show that the canonical morphism $\mathbf{R}\Gamma_Z(u) \to u$ induces an isomorphism after applying $i^\dagger$. By the exact triangle (6.7), it is enough to show that for every $w \in \mathcal{D}^b_{\mathrm{qc}}(U)$, we have $i^\dagger\big(j_+(w)\big) = 0$. In fact, this is a general fact that holds for arbitrary quasi-coherent $\mathcal{O}_U$-modules and arbitrary closed subsets Z. Indeed, note first that using the truncation functors in Example A.20, we see that it is enough to prove this when $u = \mathcal{M}$ is a quasi-coherent $\mathcal{D}_U$-module (or $\mathcal{O}_U$-module). Furthermore, the assertion can be checked locally, hence we may assume that X is affine, with $R = \mathcal{O}_X(X)$ and generators $f_1, \ldots, f_r$ for the ideal I_Z defining Z. Let $M = \Gamma(U, \mathcal{M})$. Note that if we denote by $C^\bullet$ the complex

$$0 \to \bigoplus_{i_1 \le r} R_{f_{i_1}} \to \bigoplus_{1 \le i_1 < i_2 \le r} R_{f_{i_1} f_{i_2}} \to \ldots \to R_{f_1 \cdots f_r} \to 0,$$

placed in cohomological degrees $0, \ldots, r-1$, then $C^\bullet \otimes^L_R M$ is represented by the complex $C^\bullet \otimes_R M$ (since $C^\bullet$ is a complex of flat R-modules) and this also represents $Rj_*(\mathcal{M})$. Therefore we need to show that

$$R/I_Z \otimes^L_R C^\bullet \otimes^L_R M = 0.$$

Of course, it is enough to show that $R/I_Z \otimes^L_R C^\bullet = 0$, and this is clear: since $C^\bullet$ is a complex of flat R-modules, this element is represented by the complex $R/I_Z \otimes^L_R C^\bullet$, which is the 0 complex since $R/I_Z \otimes_R R_{f_i} = 0$ for $1 \le i \le r$. This completes the proof of our assertion.

Remark 6.63. It follows from the exact triangle in Example 6.62 that if Z is a smooth, irreducible, closed subvariety of X and $U = X \smallsetminus Z$, and $i \colon Z \hookrightarrow X$ and $j \colon U \hookrightarrow X$ are the inclusions, then for every $u \in \mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_X)$ such that $i^\dagger(u) \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_Z)$ and $j^\dagger(u) \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_U)$, we have $u \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$. Indeed, note that in this case we have $i_+ i^\dagger(u) \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ by Proposition 6.25 and $j_+ j^\dagger(u) \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ by Theorem 6.50.

We can now give the proof of preservation of holonomicity under inverse image.

Proof of Theorem 6.51. Since we can factor f as $X \overset{i}{\hookrightarrow} X \times Y \xrightarrow{p} Y$, where i is a closed immersion and p is the projection onto the second component, and $\mathbf{L}f^* \simeq \mathbf{L}i^* \circ \mathbf{L}p^*$, it is enough to prove the assertion in the theorem when f is a closed immersion or a projection.

Suppose first that $f \colon X \to Y$ is a closed immersion and $v \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_Y)$. Let $j \colon U \hookrightarrow Y$ be the inclusion map from the complement of X in Y. In this case, it follows from Example 6.62 that we have an exact triangle

$$f_+\big(f^\dagger(v)\big) \to v \to j_+\big(v|_U\big) \xrightarrow{+1}.$$

Since $j_+\big(v|_U\big) \in \mathcal{D}^b_{\mathrm{hol}}(Y)$ by Theorem 6.50, it follows from the exact triangle that $f_+\big(f^\dagger(v)\big) \in \mathcal{D}^b_{\mathrm{hol}}(Y)$. Using Proposition 6.25, we conclude that $\mathbf{L}f^*(v)$, which is a translate of $f^\dagger(v)$, lies in $\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$.

Suppose now that $f \colon X \times Y \to Y$ is the projection onto the second component. By a standard argument using the truncation functors in Example A.20, we see that it is enough to show that if $\mathcal{M}$ is a holonomic $\mathcal{D}_Y$-module, then $\mathbf{L}f^*(\mathcal{M}) = f^*(\mathcal{M})$ is a holonomic $\mathcal{D}_{X \times Y}$-module. The assertion is local, hence we may assume that both X and Y are affine, with $R = \mathcal{O}(X)$ and $S = \mathcal{O}(Y)$, so $\mathcal{O}(X \times Y) = R \otimes_k S$. If M is the S-module corresponding to $\mathcal{M}$, then the $R \otimes_k S$-module corresponding that $f^*(\mathcal{M})$ is $R \otimes_k M$. If $F_\bullet M$ is a good filtration on M, then we have a filtration on $R \otimes_k M$ given by

$$F_p(R \otimes_k M) = R \otimes_k F_p M \quad \text{for all} \quad p \in \mathbf{Z}.$$

It is clear that we have $\mathrm{Gr}_\bullet^F(R \otimes_k M) = R \otimes_k \mathrm{Gr}_\bullet^F(M)$, so this is a good filtration of $R \otimes_k M$ and

$$\mathrm{Char}\big(f^*(\mathcal{M})\big) = X \times \mathrm{Char}(\mathcal{M}) \subseteq X \times T^*Y \subseteq T^*(X \times Y).$$

In particular, we see that if $\mathcal{M}$ is holonomic, then so is $f^*(\mathcal{M})$. This completes the proof of the theorem. $\square$

We now turn to preservation of holonomicity under push-forward. We first state the following base-change theorem.

THEOREM 6.64. *Given a Cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd}$$

in which X, Y, X', and Y' are smooth and irreducible, we have an isomorphism of functors

$$g^\dagger \circ f_+ \simeq f'_+ \circ g'^\dagger \colon \mathcal{D}^b_{\mathrm{qc}}(X) \to \mathcal{D}^b_{\mathrm{qc}}(Y').$$

PROOF. The idea is to factor g as $Y' \overset{i}{\hookrightarrow} Y' \times Y \xrightarrow{p} Y$, where i is a closed immersion and p is the projection onto the second component. The assertion in the theorem is then proved by using the explicit description of $g^\dagger$ when g is a closed immersion or a projection. For a detailed proof, see [**HTT08**, Theorem 1.7.3]. $\square$

Before giving the proof of Theorem 6.52, we need a characterization of objects in $\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$, which is of independent interest. We begin with the following:

LEMMA 6.65. *If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module on the smooth, irreducible variety X, then there is a nonempty affine open subset $U \subseteq X$ such that $\mathcal{M}(U)$ is a free $\mathcal{O}_X(U)$-module (possibly of infinite rank).*

PROOF. After replacing X by an affine open subset, we may assume that X is affine, with $R = \mathcal{O}_X(X)$ and $S = \mathrm{Gr}_\bullet^F(D_R)$. Let $M = \mathcal{M}(X)$ and let $F_\bullet M$ be a good filtration on M. The graded S-module $\mathrm{Gr}_\bullet^F(M)$ is finitely generated and S is a finitely generated R-algebra. In this case, it follows from Generic Freeness that there is a nonzero $f \in R$ such that each $\mathrm{Gr}_i^F(M)_f$ is a free R_f-module (see [**Eis95**, Theorem 14.4]). If we choose elements $u_{i,j} \in F_i M_f$ whose images in $\mathrm{Gr}_i^F(M)_f$ give an R_f-basis, then $(u_{i,j})_{i,j}$ is a basis of $\mathcal{M}(U)$ over $\mathcal{O}_X(U)$, where U is the affine open subset $\{x \mid f(x) \neq 0\}$. $\square$

THEOREM 6.66. *If X is a smooth, irreducible variety and $w \in \mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_X)$, then the following are equivalent:*

i) $w \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$.

ii) *For every* $x \in X$*, if* $i_x \colon \{x\} \hookrightarrow X$ *is the inclusion, we have* $\dim_k \mathcal{H}^q\big(i_x^\dagger(w)\big) < \infty$ *for all* $q \in \mathbf{Z}$*.*

iii) *There is a sequence of closed subsets*

$$X = Z_0 \supseteq Z_1 \supseteq \ldots \supseteq Z_r \supseteq Z_{r+1} = \emptyset$$

such that for all j*, with* $0 \leq j \leq r$*, the locally closed subset* $W_j = Z_j \smallsetminus Z_{j+1}$ *is smooth, and if* $i_j \colon W_j \hookrightarrow X$ *is the inclusion, then all* $\mathcal{H}^q\big(i_j^\dagger(w)\big)$ *are* $\mathcal{O}_{W_j}$*-coherent.*

Proof. The implication i)$\Rightarrow$ii) follows from Theorem 6.51: note that a $\mathcal{D}$-module on a point is just a k-vector space and this is holonomic or coherent if and only if it its dimension over k is finite.

The implication iii)$\Rightarrow$ii) is clear by (6.6), using the fact that $\mathcal{O}$-coherent $\mathcal{D}$-modules are holonomic (see Example 6.29) and the preservation of holonomicity by pull-back to a point (see Theorem 6.51). Therefore, in order to complete the proof of the theorem, it is enough to show ii)$\Rightarrow$iii), i). In other words, it will be enough to construct a sequence of closed subsets as in ii) such that, in addition, the restriction of w to each $X \smallsetminus Z_i$ has holonomic cohomology; for $i = r+1$, we thus get the assertion in i). Arguing by Noetherian induction, we see that it is enough to show that for every nonempty closed subset Z of X such that the restriction of w to $X \smallsetminus Z$ lies in $\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_{X \smallsetminus Z})$, there is a closed subset $Z' \subsetneq Z$ such that the restriction of w to $X \smallsetminus Z'$ is in $\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_{X \smallsetminus Z'})$, $W = Z \smallsetminus Z'$ is smooth, and if $i_W \colon W \hookrightarrow X$ is the inclusion, then $\mathcal{H}^q\big(i_W^\dagger(u)\big)$ is $\mathcal{O}_W$-coherent for all $q \in \mathbf{Z}$.

Let $U \subseteq X$ be an open subset such that $U \cap Z$ is a smooth, irreducible open subset of Z. If $i \colon Z \cap U \hookrightarrow U$ and $j \colon U \smallsetminus Z \hookrightarrow U$ are the inclusions, then it follows from Example 6.62 that we have an exact triangle in $\mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_U)$:

$$i_+ i^\dagger(w|_U) \to w|_U \to j_+(w|_{U \smallsetminus Z}) \xrightarrow{+1} .$$

Since $w|_{U \smallsetminus Z} \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_{U \smallsetminus Z})$, we have $j_+(w|_{U \smallsetminus Z}) \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_U)$ by Theorem 6.50. Since $w|_U \in \mathcal{D}^b_{\mathrm{coh}}(U)$, we conclude from the above exact triangle that $i_+ i^\dagger(w|_U) \in \mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_U)$, and thus $i^\dagger(w|_U) \in \mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_{U \cap Z})$ (see Remark 6.22). We can thus apply Lemma 6.65 to conclude that after possibly replacing U by a smaller open subset, we may assume that $U \cap Z$ is affine and every $\mathcal{H}^q\big(i^\dagger(w|_U)\big)$ is a free $\mathcal{O}_{U \cap Z}$-module. In this case, it is easy to see that for every $x \in U \cap Z$ and every $q \in \mathbf{Z}$, we have

$$\mathcal{H}^q\big(i_x^\dagger(w)\big) \simeq \mathcal{H}^{q+d}\big(i^\dagger(w|_U)\big) \otimes k(x),$$

where $d = \dim(Z \cap U)$. Therefore the hypothesis in ii) implies that in fact every $\mathcal{H}^q\big(i^\dagger(w|_U)\big)$ is a coherent $\mathcal{O}_{Z \cap U}$-module; in particular, it is holonomic, and thus $i^\dagger(w|_U) \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_{Z \cap U})$. If we put $Z' = Z \smallsetminus U$, then $W = Z \cap U$ and $i_W^\dagger(w) = i^\dagger(w|_U) \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_W)$. Moreover, since

$$X \smallsetminus Z' = (X \smallsetminus Z) \sqcup W$$

and since the restriction of w to $X \smallsetminus Z$ is in $\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_{X \smallsetminus Z})$, it follows from Remark 6.63 that the restriction of w to $X \smallsetminus Z'$ is in $\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_{X \smallsetminus Z'})$. This completes the proof of the theorem. $\square$

We can now prove that holonomicity is preserved by $\mathcal{D}$-module push-forward.

Proof of Theorem 6.52. Recall that by a theorem of Nagata (see for example [**Con07**]), every variety W admits an open immersion into a complete variety

$\overline{W}$ (recall that all our varieties are assumed to be separated). Furthermore, it follows from Hironaka's resolution of singularities that if W is smooth and irreducible, we may assume that $\overline{W}$ is smooth and irreducible as well.

Given the morphism $f\colon X \to Y$ in the statement, we may apply this to find open immersions $j_X\colon X \hookrightarrow \overline{X}$ and $j_Y\colon Y \hookrightarrow \overline{Y}$, with $\overline{X}$ and $\overline{Y}$ both smooth and irreducible. Moreover, after possibly replacing $\overline{X}$ by a higher birational model, we may assume that we have a morphism $\overline{f}\colon \overline{X} \to \overline{Y}$ such that $\overline{f} \circ j_X = j_Y \circ f$. For every $u \in \mathcal{D}^b_{\mathrm{hol}}(X)$, we have

$$j_{Y+}\big(f_+(u)\big) \simeq \overline{f}_+\big(j_{X+}(u)\big).$$

Since $f_+(u) = j_{Y+}\big(f_+(u)\big)|_Y$, in order to show that $f_+(u) \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$, it is enough to show that $\overline{f}_+\big(j_{X+}(u)\big)$. Since $j_{X+}(u) \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_{\overline{X}})$ by Theorem 6.50, it is enough to show that $\overline{f}_+(v) \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_{\overline{Y}})$ for every $v \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_{\overline{X}})$. Hence from now on we may assume that both X and Y are complete varieties.

Let us factor f as $X \xrightarrow{i} X \times Y \xrightarrow{p} Y$, where i is the graph embedding associated to f (hence a closed immersion) and p is the projection onto the second component. Since i_+ maps holonomic modules to holonomic modules by Proposition 6.25, we see that it is enough to show that $p_+(v) \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_Y)$ for every $v \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_{X\times Y})$. Since p is proper, it follows from Theorem 6.54 that $p_+(v) \in \mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_Y)$. By Theorem 6.66, In order to show that $p_+(v) \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_Y)$, it is enough to show that for every $y \in Y$, if $i_y\colon \{y\} \hookrightarrow Y$ is the inclusion, then every k-vector space $\mathcal{H}^q\big(i_y^\dagger(p_+(v))\big)$ has finite dimension. Applying Theorem 6.64 for the Cartesian diagram

$$\begin{tikzcd} X \arrow[r, "i'_y"] \arrow[d, "p'"'] & X \times Y \arrow[d, "p"] \\ \{y\} \arrow[r, "i_y"] & Y, \end{tikzcd}$$

where $i'_y(x) = (x, y)$, we see that

$$i_y^\dagger\big(p_+(v)\big) \simeq p'_+\big({i'_y}^\dagger(v)\big).$$

Note that ${i'_y}^\dagger(v) \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ by Theorem 6.51 and we can apply again Theorem 6.54 for the proper morphism p' to conclude that $\mathcal{H}^q\big(i_y^\dagger(p_+(v))\big)$ is a finite-dimensional k-vector space for every q. This completes the proof of the theorem. $\square$

6.7. The functorial formalism for holonomic $\mathcal{D}$-modules

We associate to every morphism of smooth irreducible algebraic varieties $f\colon X \to Y$ four functors f_{D*}, $f_{D!}$, f^*_D, and $f^!_D$ between $\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ and $\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_Y)$ (we use the notation that parallels the one in the topological setting, but we use the D subscript to emphasize the $\mathcal{D}$-module setting).

The functor $f_{D*}\colon \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X) \to \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_Y)$ is simply f_+ (note that this is well-defined by Theorem 6.52). We define $f_{D!}\colon \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X) \to \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_Y)$ by

$$f_{D!} = \mathbf{D}_Y \circ f_{D*} \circ \mathbf{D}_X.$$

Similarly, the functor $f^!_D\colon \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_Y) \to \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ is simply $f^\dagger$ and we define the functor $f^*_D\colon \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_Y) \to \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ by

$$f^*_D = \mathbf{D}_X \circ f^!_D \circ \mathbf{D}_Y.$$

REMARK 6.67. Suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms between smooth, irreducible varieties. Note that by Proposition 6.12, we have an isomorphism of functors $(g\circ f)_{D*} \simeq g_{D*}f_{D*}$ and by Corollary 6.8, we have an isomorphism of functors $(g \circ f)^!_D \simeq f^!_D g^!_D$. Using the fact that $\mathbf{D}_X\mathbf{D}_X \simeq \mathrm{Id}$, we also obtain the isomorphisms $(g \circ f)_{D!} \simeq g_{D!}f_{D!}$ and $(g \circ f)^*_D \simeq f^*_D g^*_D$.

We make a parenthesis to introduce a useful definition concerning exact functors between derived categories.

DEFINITION 6.68. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories, $\mathcal{T}$ a triangulated subcategory of $\mathcal{D}(\mathcal{A})$, and $F\colon \mathcal{T} \to \mathcal{D}(\mathcal{B})$ an exact functor. We say that F is *left t-exact* if for all objects u of $\mathcal{T}$ with $\mathcal{H}^i(u) = 0$ for all $i < 0$, we have $\mathcal{H}^i\big(F(u)\big) = 0$ for all $i < 0$. We say that F is *right t-exact* if for all objects u of $\mathcal{T}$ with $\mathcal{H}^i(u) = 0$ for all $i > 0$, we have $\mathcal{H}^i\big(F(u)\big) = 0$ for all $i > 0$. We say that F is *t-exact* if it is both left t-exact and right t-exact.

EXAMPLE 6.69. If $G\colon \mathcal{A} \to \mathcal{B}$ is a left exact functor between Abelian categories and $\mathcal{A}$ has enough injective objects, then the derived functor $\mathbf{R}G\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ is left t-exact.

REMARK 6.70. Suppose that we have an exact functor $F\colon \mathcal{D}^b_{\mathrm{hol}}(X) \to \mathcal{D}^b_{\mathrm{hol}}(Y)$, where X and Y are two smooth, irreducible varieties. If F is left t-exact (right t-exact), then we have a left (respectively, right) exact functor $\mathcal{M}od_{\mathrm{hol}}(\mathcal{D}_X) \to \mathcal{M}od_{\mathrm{hol}}(\mathcal{D}_Y)$ that maps $\mathcal{M}$ to $\mathcal{H}^0\big(F(\mathcal{M})\big)$. Indeed, given an exact sequence

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

in $\mathcal{M}od_{\mathrm{hol}}(\mathcal{D}_X)$, we have a corresponding exact triangle

$$\mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \xrightarrow{+1}$$

in $\mathcal{D}_{\mathrm{hol}}(\mathcal{D}_X)$, and thus an exact triangle

$$F(\mathcal{M}') \to F(\mathcal{M}) \to F(\mathcal{M}'') \xrightarrow{+1} .$$

By taking the long exact sequence in cohomology, we get

$$\mathcal{H}^{-1}\big(F(\mathcal{M}'')\big) \to \mathcal{H}^0\big(F(\mathcal{M}')\big) \to \mathcal{H}^0\big(F(\mathcal{M})\big) \to \mathcal{H}^0\big(F(\mathcal{M}'')\big) \to \mathcal{H}^1\big(F(\mathcal{M}')\big).$$

If F is left t-exact, then $\mathcal{H}^{-1}\big(F(\mathcal{M}'') = 0$ and if F is right t-exact, then $\mathcal{H}^1\big(F(\mathcal{M}')\big) = 0$.

REMARK 6.71. A useful property of duality is that for every $u \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$, we have $\mathcal{H}^i(u) = 0$ for all $i \geq 0$ if and only $\mathcal{H}^i\big(\mathbf{D}_X(u)\big) = 0$ for all $i \leq 0$ (hence $\mathbf{D}_X$ is a t-exact functor from $\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ to its dual category). Indeed, this follows easily using truncation functors from the fact that if $\mathcal{H}^i(u) = 0$ for all $i \neq 0$, then $\mathcal{H}^i\big(\mathbf{D}_X(u)\big) = 0$ for all $i \neq 0$.

In the following proposition we collect the general basic properties of the functors we introduced:

PROPOSITION 6.72. Given a morphism of smooth irreducible algebraic varieties $f\colon X \to Y$, the following hold:

i) We have an isomorphism of functors $\mathbf{D}_Y \circ f_{D*} \simeq f_{D!} \circ \mathbf{D}_X$.
ii) We have an isomorphism of functors $\mathbf{D}_X \circ f^!_D \simeq f^*_D \circ \mathbf{D}_Y$.
iii) If f is proper, then we have a canonical isomorphism $f_{D!} \simeq f_{D*}$.

iv) The pair $(f_{D!}, f^!_D)$ is an adjoint pair of functors.
v) The pair (f^*_D, f_{D*}) is an adjoint pair of functors.
vi) If f is an affine morphism, then f_{D*} is right t-exact and $f_{D!}$ is left t-exact.
vii) If f is a finite morphism, then $f_{D!} = f_{D*}$ is t-exact.

Proof. The assertions in i) and ii) follow directly from the definitions of $f_{D!}$ and f^*_D and the fact that $\mathbf{D}_X \circ \mathbf{D}_X \simeq \mathrm{Id}$ and $\mathbf{D}_Y \circ \mathbf{D}_Y \simeq \mathrm{Id}$. The assertion in iii) follows from the definition of $f_{D!}$ and the compatibility of f_+ with duality for proper morphisms (see Theorem 6.59).

Note that the assertions in iv) and v) are equivalent: this follows using duality and the isomorphisms in i) and ii). We give all details in this case, in order to illustrate how the argument is carried out. Suppose for example that iv) holds. Using this and the fact that duality is an anti-equivalence of categories, we obtain for every $v \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_Y)$ and $u \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ canonical isomorphisms

$$\mathrm{Hom}\big(f^*_D(v), u\big) \simeq \mathrm{Hom}\big(\mathbf{D}_X(u), \mathbf{D}_X f^*_D(v)\big)$$
$$\simeq \mathrm{Hom}\big(\mathbf{D}_X(u), f^!_D \mathbf{D}_Y(v)\big) \simeq \mathrm{Hom}\big(f_{D!}\mathbf{D}_X(u), \mathbf{D}_Y(v)\big)$$
$$\simeq \mathrm{Hom}\big(\mathbf{D}_Y f_{D*}(u), \mathbf{D}_Y(v)\big) \simeq \mathrm{Hom}\big(v, f_{D*}(u)\big).$$

This gives the assertion in v) and the proof of v)$\Rightarrow$iv) is similar.

In order to prove the assertion in iv), by a theorem of Nagata and Deligne (see [**Con07**]), we may write $f = g \circ j$, where g is proper and j is an open immersion. Therefore it is enough to check separately the cases when f is proper and when f is an open immersion. If f is proper, then this is the assertion in Proposition 6.60 (note that in this case $f_{D!} \simeq f_{D*}$). On the other hand, if f is an open immersion, then we have seen that it is enough to show that the pair of functors (f^*_D, f_{D*}) is an adjoint pair. However, in this case $f^*_D = f^!_D$ is just the quasi-coherent pull-back (this follows from the fact that duality is compatible with restriction to open subsets), while f_{D*} is the quasi-coherent push-forward. The assertion then follows from in the same way as the one for the corresponding pair of functors between the bounded derived categories of $\mathcal{O}$-modules.

If f is an affine morphism, the first assertion in vi) follows from the fact that the quasi-coherent f_* is an exact functor, while $\mathbf{L}f^*$ is a left derived functor. The second assertion follows from this one by duality, using the assertion in Remark 6.71.

Finally, the assertion in vii) follows from the one in vi), since f being finite implies that it is affine and proper, and the latter property gives $f_{D*} \simeq f_{D!}$. $\square$

We next discuss these functors in the case of open and closed immersions.

Proposition 6.73. Let X be a smooth, irreducible variety, $j\colon U \hookrightarrow X$ the inclusion map of an open subset and $i\colon Z \hookrightarrow X$ the inclusion map of the complement Z of U.

i) The functor j_{D*} is left t-exact and the functor $j_{D!}$ is right t-exact.
ii) $j^*_D \simeq j^!_D$.
iii) $j^*_D j_{D*} \simeq \mathrm{Id}$ and $j^!_D j_{D!} \simeq \mathrm{Id}$.

Suppose now, in addition, that Z is smooth.

iv) The functor $i_{D!} = i_{D*}$ is t-exact.
v) $i^*_D i_{D*} \simeq \mathrm{Id}$ and $i^!_D i_{D!} \simeq \mathrm{Id}$.
vi) $i^*_D j_{D!} = 0$ and $i^!_D j_{D*} = 0$.
vii) $j^*_D i_{D*} = 0$.

PROOF. The first assertion in i) follows from the fact that j_+ is given by $\mathbf{R}j_*$ and the second assertion follows by duality, using the assertion in Remark 6.71.

The isomorphism of functors in ii) follows from the fact that $j_D^!$ is given by restriction to U and duality is compatible with restriction to open subsets (we have already used this in the proof of the previous proposition). The first isomorphism in iii) follows from ii) and the definition of j_+ and $j^\dagger$. The second isomorphism follows by duality.

From now on we assume that Z is smooth. The assertion in iv) is clear, since i_+ is induced by an exact functor between Abelian categories. The second isomorphism in v) is a consequence of Proposition 6.24. The first isomorphism again follows by duality.

We have proved the second equality in vi) in Example 6.62. The first one follows by duality.

Finally, note that in order to check the assertion in vii), it is enough to show that for every holonomic $\mathcal{D}_Z$-module $\mathcal{M}$, we have $i_+(\mathcal{M})|_U$. This is clear, since for every quasi-coherent $\mathcal{O}_Z$-module $\mathcal{N}$, we have $i_*(\mathcal{N})|_U = 0$. □

PROPOSITION 6.74. For every morphism $f\colon X \to Y$ between smooth, irreducible varieties, and every $u \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$, we have a unique functorial morphism $\beta_f(u)\colon f_{D!}(u) \to f_{D*}(u)$, with the following properties:

i) If f is proper, then $\beta_f(u)$ is the isomorphism $\alpha_f\big(\mathbf{D}_X(u)\big)$, where α_f is the functorial isomorphism in Theorem 6.59.
ii) If f is an open immersion, then $\beta_f(u)$ is the unique morphism whose restriction to X is the canonical isomorphism.
iii) The natural transformation β_f is compatible with composition of morphisms, in the sense that if $g\colon Y \to Z$ is another morphism between smooth, irreducible varieties, then

$$\beta_{g\circ f} = g_{D*}\big(\beta_f(u)\big) \circ \beta_g\big(f_{D!}(u)\big) = \beta_g\big(f_{D*}(u) \circ g_{D!}\big(\beta_f(u)\big). \tag{6.9}$$

PROOF. Uniqueness is clear: given any $f\colon X \to Y$, we can use the theorem of Nagata and Deligne (see [**Con07**]) to write it as a composition $X \overset{j}{\hookrightarrow} W \overset{p}{\hookrightarrow} Y$, where j is an open immersion and p is proper. Note that β_j and β_p are determined by i) and ii). Regarding β_j, note that by the usual adjoint property of the pair $(j^*, \mathbf{R}j_*)$, given any $v \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_W)$ and a morphism $\varphi\colon j^*(v) \to u$, there is a unique morphism $v \to \mathbf{R}j_*(u)$ whose restriction to X is φ. If we take $v = j_{D!}(u)$ and we take φ to be the canonical isomorphism between $j^*\big(j_{D!}(u)\big)$ and u, this uniquely determines $\beta_j(u)$. The morphism $\beta_f(u)$ is then determined by the condition in ii). Note that the second equality in (6.9) always holds.

In order to complete the proof of the proposition, we need to show that the definition of $\beta_f(u)$ does not depend on the factorization of f and that the condition in iii) holds for any two morphisms f and g. Both these properties follow from conditions i) and ii) in Theorem 6.59. We leave the details as an exercise for the reader. □

REMARK 6.75. It follows from the definition of β_f that it is compatible with duality: with the notation in the above proposition, we have

$$\beta_f\big(\mathbf{D}_X(u)\big) = \mathbf{D}_Y\big(\beta_f(u)\big).$$

Indeed, it is enough to check this separately for proper morphisms and for open immersions. For proper morphisms, the assertion follows from property iii) in Proposition 6.59, while for open immersions it is straightforward to check.

EXAMPLE 6.76. Suppose that X is smooth, irreducible variety, and Z is a smooth, irreducible hypersurface in X. Let $i\colon Z \hookrightarrow X$ and $j\colon U = X \smallsetminus Z \hookrightarrow X$ be the inclusion maps. Note that we have an exact sequence of holonomic $\mathcal{D}_X$-modules

$$0 \to \mathcal{O}_X \to j_+(\mathcal{O}_U) \to \mathcal{H}^1_Z(\mathcal{O}_X) \to 0$$

(cf. Example 6.62). Applying $\mathbf{D}_X$ and using the fact that $\mathbf{D}_X(\mathcal{O}_X) \simeq \mathcal{O}_X$ and $\mathbf{D}_U(\mathcal{O}_U) \simeq \mathcal{O}_U$, we obtain an exact sequence

$$0 \to \mathbf{D}_X\big(\mathcal{H}^1_Z(\mathcal{O}_X)\big) \to j_{D!}(\mathcal{O}_U) \to \mathcal{O}_X \to 0.$$

Note also that since $\mathcal{H}^1_Z(\mathcal{O}_X) \simeq i_{D*}\big(i^!_D(\mathcal{O}_X)\big)[1] \simeq i_{D*}(\mathcal{O}_Z)$ (see Example 6.62), we have

$$\mathbf{D}_X\big(\mathcal{H}^1_Z(\mathcal{O}_X)\big) \simeq \mathbf{D}_X\big(i_+\mathcal{O}_Z\big) \simeq i_+\mathcal{O}_Z.$$

We can also describe locally $j_{D!}(\mathcal{O}_U)$ by generators and relations: suppose that we have algebraic coordinates $x_1, \ldots, x_n$ on X such that Z is defined by (x_1). In this case we have

$$j_+(\mathcal{O}_U) = \mathcal{O}_X[1/x_1] \simeq \mathcal{D}_X/\mathcal{D}_X \cdot (\partial_1 x_1, \partial_2, \ldots, \partial_n).$$

Computing $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X \cdot \partial_1 x_1, \mathcal{D}_X)$ via the Koszul-type complex

$$0 \to \mathcal{D}_X \to \mathcal{D}_X^{\oplus n} \to \ldots \to \mathcal{D}_X^{\oplus n} \xrightarrow{\cdot(\partial_1 x_1, \partial_2, \ldots, \partial_n)} \mathcal{D}_X \to \mathcal{D}_X/\mathcal{D}_X \cdot \partial_1 x_1 \to 0,$$

we obtain

$$j_{D!}(\mathcal{O}_U) \simeq \mathcal{D}_X/\mathcal{D}_X \cdot (x_1\partial_1, \partial_2, \ldots, \partial_n).$$

6.8. The intermediate extension

Let X be a smooth, irreducible variety and $f\colon W \hookrightarrow X$ a locally closed immersion, with W smooth and irreducible. Whenever convenient, we can write f as a composition $W \overset{i}{\hookrightarrow} U \overset{j}{\hookrightarrow} X$, where i is a closed immersion and j is an open immersion.

Recall that for every holonomic $\mathcal{D}_W$-module $\mathcal{M}$, we have the canonical morphism

$$\beta_f(\mathcal{M})\colon f_{D!}(\mathcal{M}) \to f_{D*}(\mathcal{M})$$

given by Proposition 6.9. Note that $\mathcal{H}^i\big(f_{D!}(\mathcal{M})\big) = 0$ for all $i > 0$ and $\mathcal{H}^i\big(f_{D*}(\mathcal{M})\big) = 0$ for all $i < 0$ by assertions i) and iv) in Proposition 6.73. In this case, we have canonical maps induced by truncation functors

$$f_{D!}(\mathcal{M}) \to \mathcal{H}^0\big(f_{D!}(\mathcal{M})\big) \quad \text{and} \quad \mathcal{H}^0\big(f_{D*}(\mathcal{M})\big) \to f_{D*}(\mathcal{M}).$$

It follows from Remark A.21 that we have a unique morphism $\gamma_f(\mathcal{M})$ that makes the following diagram commutative:

$$\begin{tikzcd}
f_{D!}(\mathcal{M}) \arrow[r, "\beta_f(\mathcal{M})"] \arrow[d] & f_{D*}(\mathcal{M}) \\
\mathcal{H}^0\big(f_{D!}(\mathcal{M})\big) \arrow[r, "\gamma_f(\mathcal{M})"] & \mathcal{H}^0\big(f_{D*}(\mathcal{M})\big) \arrow[u]
\end{tikzcd}$$

and in fact, by taking $\mathcal{H}^0$ in this diagram, we see that $\gamma_f(\mathcal{M}) = \mathcal{H}^0\big(\beta_f(\mathcal{M})\big)$.

DEFINITION 6.77. The *intermediate extension* of the holonomic $\mathcal{D}_W$-module $\mathcal{M}$ to X is

$$f_{D!*}(\mathcal{M}) := \mathrm{Im}\big(\gamma_f(\mathcal{M})\big).$$

It is straightforward to see that if $g\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a morphism of holonomic $\mathcal{D}_W$-modules, then the morphism $\mathcal{H}^0\big(f_{D*}(\mathcal{M}_1)\big) \to \mathcal{H}^0\big(f_{D*}(\mathcal{M}_2)\big)$ induces a morphism $f_{D!*}(\mathcal{M}_1) \to f_{D!*}(\mathcal{M}_2)$. We thus get a functor $f_{D!*}\colon \mathcal{M}od_{\mathrm{hol}}(\mathcal{D}_Z) \to \mathcal{M}od_{\mathrm{hol}}(\mathcal{D}_X)$.

REMARK 6.78. Note that if we factor f as a composition $W \overset{i}{\hookrightarrow} U \overset{j}{\hookrightarrow} X$, where i is a closed immersion and j is an open immersion, since $\beta_i(\mathcal{M})$ is an isomorphism, it follows that

$$f_{D!*}(\mathcal{M}) = j_{D!*}\big(i_+(\mathcal{M})\big).$$

Because of this, when discussing the intermediate extension, we can often reduce to the case when f is an open immersion.

We collect in the following theorem the main properties of the minimal extension.

THEOREM 6.79. *Let $f\colon W \to X$ be a locally closed immersion of smooth, irreducible varieties and let $\mathcal{M}$ be a holonomic $\mathcal{D}_W$-module.*

i) *We have an isomorphism $f_{D!*}\big(\mathbf{D}_W(\mathcal{M})\big) \simeq \mathbf{D}_X\big(f_{D!*}(\mathcal{M})\big)$.*
ii) *We have canonical isomorphisms*
$$f_D^!\big(f_{D!*}(\mathcal{M})\big) \simeq \mathcal{M} \simeq f_D^*\big(f_{D!*}(\mathcal{M})\big).$$
iii) *$f_{D!*}$ preserves injections and surjections.*
iv) *If f is an open immersion and $\mathcal{N}$ is a holonomic $\mathcal{D}_X$-module such that $\mathcal{N} \subseteq \mathcal{H}^0\big(f_{D*}(\mathcal{N}|_U)\big)$ (in other words, if $\Gamma_{X\smallsetminus W}(\mathcal{N}) = 0$), then $f_{D!*}(\mathcal{N}|_U) \subseteq \mathcal{N}$. In particular, for every holonomic $\mathcal{D}_W$-module $\mathcal{M}$, the intermediate extension $f_{D!*}(\mathcal{M})$ is the smallest $\mathcal{D}_X$-submodule of $\mathcal{H}^0\big(f_{D*}(\mathcal{M})\big)$ whose restriction to W is $\mathcal{M}$; dually, $f_{D!*}(\mathcal{M})$ is the smallest $\mathcal{D}_X$-module quotient of $\mathcal{H}^0\big(f_{D!}(\mathcal{M})\big)$ whose restriction to W is $\mathcal{M}$.*
v) *If $\mathcal{M}$ is a simple[2] $\mathcal{D}_W$-module, then $f_{D!*}(\mathcal{M})$ is a simple $\mathcal{D}_X$-module. Moreover, $f_{D!*}(\mathcal{M})$ is the unique simple $\mathcal{D}_X$-submodule of $\mathcal{H}^0\big(f_{D*}(\mathcal{M})\big)$ and it is the unique simple quotient $\mathcal{D}_X$-module of $\mathcal{H}^0\big(f_{D!}(\mathcal{M})\big)$.*
vi) *If f is an open immersion, then $f_{D!*}(\mathcal{M})$ is the unique quasi-coherent $\mathcal{D}_X$-module (up to isomorphism) whose restriction to W is isomorphic to $\mathcal{M}$ and that has no submodule or quotient module supported on $X \smallsetminus W$.*
vii) *$f_{D!*}$ is a fully faithful functor $\mathcal{M}od_{\mathrm{hol}}(\mathcal{D}_W) \to \mathcal{M}od_{\mathrm{hol}}(\mathcal{D}_X)$.*

PROOF. For the assertion in i), note that it follows from Remark 6.75 that the morphism $\beta_f\big(\mathbf{D}_W(\mathcal{M})\big)$ is identified with $\mathbf{D}_X\big(\beta_f(\mathcal{M})\big)$. Using the exactness of $\mathbf{D}_X$, we can then identify $\gamma_f\big(\mathbf{D}_W(\mathcal{M})\big)$ to $\mathbf{D}_X\big(\gamma_f(\mathcal{M})\big)$ and obtain the assertion in i).

By Remark 6.78, in order to prove ii), it is enough to treat separately the cases when f is an open immersion or a closed immersion. If $f\colon W \to X$ is an open immersion, then it follows from the definition of $\beta_f(\mathcal{M})$ that the restriction of $\gamma_f(\mathcal{M})$ to W is identified with the identity map on $\mathcal{M}$, which gives the assertion in

[2]Recall that an object in an Abelian category is *simple* if it is nonzero and has no nonzero proper subobjects.

ii). On the other hand, if f is a closed immersion, it is enough to use the fact that we have canonical isomorphisms $f_D^! f_{D!}(\mathcal{M}) \simeq \mathcal{M} \simeq f_D^* f_{D*}(\mathcal{M})$ by Proposition 6.73v).

The assertions in iii) follow from the fact that $\mathcal{H}^0\big(f_{D*}(-)\big)$ is a left exact functor (since f_{D*} is left t-exact, see Proposition 6.73 and Remark 6.70) and, dually, $\mathcal{H}^0\big(f_{D!}(-)\big)$ is a right exact functor.

In order to prove iv), note that the canonical morphism $\beta_f(\mathcal{N}|_W)\colon f_{D!}(\mathcal{N}|_W) \to f_{D*}(\mathcal{N}|_W)$ factors as $f_{D!}(\mathcal{N}|_W) \to \mathcal{N} \to f_{D*}(\mathcal{N}|_W)$ and after taking $\mathcal{H}^0$, we see that $\gamma_f(\mathcal{N}|_W)$ factors as

$$\mathcal{H}^0\big(f_{D!}(\mathcal{N}|_W)\big) \to \mathcal{N} \hookrightarrow \mathcal{H}^0\big(f_{D*}(\mathcal{N}|_W),$$

where the second map is injective by our assumption. The first assertion in iv) then follows.

In order to prove the second assertion, note first that $j_{D!*}(\mathcal{M})|_W = \mathcal{M}$ by ii). Suppose now that $\mathcal{N} \subseteq \mathcal{H}^0\big(f_{D*}(\mathcal{M})\big)$ is such that $\mathcal{N}|_U = \mathcal{M}$. In this case, it follows from the first assertion that

$$f_{D!*}(\mathcal{M}) = f_{D!*}(\mathcal{N}|_U) \subseteq \mathcal{N}.$$

Finally, the last assertion in iv) follows by duality.

Let us prove v). Note first that the direct image of a simple holonomic $\mathcal{D}$-module by a closed immersion is again simple by Kashiwara's Equivalence Theorem, hence using Remark 6.78 we see that it is enough to prove the assertion in v) when $f\colon W \to X$ is an open immersion. Note that $\mathcal{H}^0\big(f_{D*}(\mathcal{M})\big)$ is holonomic and nonzero (its restriction to W is $\mathcal{M}$), and since every holonomic $\mathcal{D}_X$-module has finite length (see Proposition 6.37), we conclude that $\mathcal{H}^0\big(f_{D*}(\mathcal{M})\big)$ contains a simple $\mathcal{D}_X$-submodule $\mathcal{N}$. We will show that in this case $i_{D!*}(\mathcal{M}) \subseteq \mathcal{N}$, and since $i_{D!*}(\mathcal{M})$ is nonzero (its restriction to W being $\mathcal{M}$) and $\mathcal{N}$ is simple, this will imply that $\mathcal{N} = i_{D!*}(\mathcal{M})$. By restricting to W the inclusion map $\mathcal{N} \hookrightarrow \mathcal{H}^0\big(f_{D*}(\mathcal{M})\big)$, we obtain an injective map $\mathcal{N}|_W \hookrightarrow \mathcal{M}$. Since $\Gamma_{X\smallsetminus W} f_*(\mathcal{M}) = 0$, it follows that $\mathcal{N}|_W \neq 0$, and since $\mathcal{M}$ is simple, we conclude that $\mathcal{N}|_W = \mathcal{M}$. In this case the inclusion $i_{D!*}(\mathcal{M}) \subseteq \mathcal{N}$ follows from iv). The last assertion in v) follows from the second one by duality.

We now prove vi). The restriction of $f_{D!*}(\mathcal{M})$ to W is isomorphic to $\mathcal{M}$ by ii). Since $f_{D!*}(\mathcal{M})$ is a $\mathcal{D}_X$-submodule of $\mathcal{H}^0\big(f_{D*}(\mathcal{M})\big)$, whose underlying $\mathcal{O}_X$-module is $f_*(\mathcal{M})$, it follows that $f_{D!*}(\mathcal{M})$ contains no $\mathcal{D}_X$-module supported on Z. By duality, it follows that it has no quotient $\mathcal{D}_X$-module supported on Z.

Suppose now that $\mathcal{N}$ is a quasi-coherent $\mathcal{D}_X$-module that has no subobjects and quotient objects supported on Z and such that $\mathcal{N}|_W \simeq \mathcal{M}$. The fact that $\mathcal{N}$ has no submodules supported on Z implies that the canonical morphism $\mathcal{N} \to \mathcal{H}^0\big(f_{D*}(\mathcal{N}|_W)\big)$ is injective; in particular, $\mathcal{N}$ is holonomic. By duality, the fact that $\mathcal{N}$ has no quotient module supported on Z implies that the canonical morphism $\mathcal{H}^0\big(f_{D!}(\mathcal{N}|_W) \to \mathcal{N}$ is surjective. Since the composition

$$\mathcal{H}^0\big(f_{D!}(\mathcal{N}|_W) \to \mathcal{N} \to \mathcal{H}^0\big(f_{D*}(\mathcal{N}|_W)\big)$$

is the morphism $\gamma_f(\mathcal{N}|_W)$, we conclude that $\mathcal{N} \simeq f_{D!*}(\mathcal{N}|_W)$.

In order to prove the assertion in vii), we may assume that f is an open immersion: this follows from Remark 6.78 and Kashiwara's Equivalence Theorem. In this case, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are holonomic $\mathcal{D}_W$-modules, the composition

$$\mathrm{Hom}_{\mathcal{D}_W}(\mathcal{M}_1, \mathcal{M}_2) \to \mathrm{Hom}_{\mathcal{D}_X}\big(f_{D!*}(\mathcal{M}_1), f_{D!*}(\mathcal{M}_2)\big) \to \mathrm{Hom}_{\mathcal{D}_W}\big(f_{D!*}(\mathcal{M}_1)|_W, f_{D!*}(\mathcal{M}_2)|_W\big)$$

is an isomorphism by ii), hence the first map is injective. We conclude that $f_{D!*}$ is a faithful functor. Moreover, in order to prove that $f_{D!*}$ is fully faithful, it is enough to show that for every $\mathcal{M}_1$ and $\mathcal{M}_2$, if $\varphi\colon f_{D!*}(\mathcal{M}_1) \to f_{D!*}(\mathcal{M}_2)$ is such that $\varphi_W = 0$, then $\varphi = 0$. In this case $\operatorname{Im}(\varphi)$ has support inside $X \smallsetminus W$, and it follows from vi) that $\operatorname{Im}(\varphi) = 0$, hence $\varphi = 0$. This completes the proof of the theorem. □

EXAMPLE 6.80. If $j\colon W \hookrightarrow X$ is an open immersion, with X smooth and irreducible, and $\mathcal{E}$ is a $\mathcal{D}_X$-module on X that is coherent as an $\mathcal{O}_X$-module, then $j_{D!*}(\mathcal{E}|_W) \simeq \mathcal{E}$. Indeed, by assertion iv) in Theorem 6.79, $j_{D!*}(\mathcal{E}|_W)$ is the smallest $\mathcal{D}_X$-submodule of $\mathcal{H}^0\big(j_{D*}(\mathcal{E}|_W)\big) = j_*(\mathcal{E}|_W)$ whose restriction to W is $\mathcal{E}|_W$. Therefore we have $j_{D!*}(\mathcal{E}|_W) \subseteq \mathcal{E}$. The quotient $\mathcal{F} := \mathcal{E}/j_{D!*}(\mathcal{E}|_W)$ is a $\mathcal{D}_X$-module that's coherent as a $\mathcal{O}_X$-module, hence locally free by Proposition 3.18. Since $\operatorname{Supp}(\mathcal{F}) \subseteq X \smallsetminus W$, it follows that $\mathcal{F} = 0$.

EXERCISE 6.81. For every $\lambda \in \mathbf{C} \smallsetminus \mathbf{Z}$, let $\mathcal{M}_\lambda = \mathcal{D}_{\mathbf{A}^1}/\mathcal{D}_{\mathbf{A}^1} \cdot (x\partial_x - \lambda)$.

i) Show that $\mathbf{D}_{\mathbf{A}^1}(\mathcal{M}_\lambda) \simeq \mathcal{M}_{-\lambda-1}$.
ii) Show that $\mathcal{M}_\lambda$ has no $\mathcal{D}_{\mathbf{A}^1}$-submodules and quotient modules supported at 0.
iii) Deduce that if $\mathcal{L}_\lambda$ is the local system $\mathcal{M}_\lambda|_{\mathbf{A}^1 \smallsetminus \{0\}}$ and $j\colon \mathbf{A}^1 \smallsetminus \{0\} \hookrightarrow \mathbf{A}^1$ is the inclusion map, then $\mathcal{M}_\lambda \simeq j_{D!*}(\mathcal{L}_\lambda)$.

In the next proposition we treat the behavior of intermediate extension with respect to composition of morphisms.

PROPOSITION 6.82. If $W \xrightarrow{g} V \xrightarrow{f} X$ are locally closed immersions of smooth, irreducible varieties, then for every holonomic $\mathcal{D}_W$-module $\mathcal{M}$, we have a functorial isomorphism

$$(f \circ g)_{D!*}(\mathcal{M}) \simeq f_{D!*}\big(g_{D!*}(\mathcal{M})\big).$$

PROOF. We begin by treating two special cases.
Step 1. We consider the case when both f and g are open immersions. Note that we have canonical surjections

$$\mathcal{H}^0 f_{D!}\big(\mathcal{H}^0 g_{D!}(\mathcal{M})\big) \xrightarrow{u} f_{D!*}\big(\mathcal{H}^0 g_{D!}(\mathcal{M})\big) \xrightarrow{v} f_{D!*}\big(g_{D!*}(\mathcal{M})\big)$$

(the fact that v is surjective follows from the fact that $f_{D!*}$ preserves surjections by assertion iii) in Theorem 6.79). Since it is clear that the restriction of $f_{D!*}\big(g_{D!*}(\mathcal{M})\big)$ to W is $\mathcal{M}$, it follows from assertion iv) in Theorem 6.79 that there is an induced morphism

$$\alpha\colon f_{D!*}\big(g_{D!*}(\mathcal{M})\big) \to (f \circ g)_{D!*}(\mathcal{M})$$

such that the canonical morphism

$$\mathcal{H}^0 f_{D!}\big(\mathcal{H}^0 g_{D!}(\mathcal{M})\big) \simeq \mathcal{H}^0 (f \circ g)_{D!*}(\mathcal{M}) \to (f \circ g)_{D!*}(\mathcal{M}) \tag{6.10}$$

factors as $\alpha \circ v \circ u$ (note that the isomorphism in (6.10) is a consequence of the fact that for every morphism h, the functor $h_{D!}$ is right t-exact). Of course, α is surjective. Dually, we get an injective morphism

$$\beta\colon (f \circ g)_{D!*}(\mathcal{M}) \to f_{D!*}\big(g_{D!*}(\mathcal{M})\big).$$

It is easy to check that the composition $\beta \circ \alpha$ is the identity and using the fact that α is surjective, we conclude that α and β are inverse isomorphisms.

Step 2. The case when either f or g is a closed immersion is easy: this follows directly from the definition of intermediate extension, using the fact that if i is a closed immersion, then $i_{D!} \simeq i_{D*}$ is a t-exact functor.
Step 3. Finally, we treat the general case. In order to simplify the notation, we may and will assume that f and g are inclusion maps. We factor g as $W \overset{\alpha}{\hookrightarrow} U_2 \cap V \overset{\beta}{\hookrightarrow} V$, where U_2 is an open subset of X and α is a closed immersion, and similarly, we factor f as $V \overset{\gamma}{\hookrightarrow} U_1 \overset{\delta}{\hookrightarrow} X$, where U_1 is an open subset of X and γ is a closed immersion. Note that by Step 2 (in fact, by Remark 6.78), we have
(6.11)
$$g_{D!*}(\mathcal{M}) \simeq \beta_{D!*}\big(\alpha_{D!*}(\mathcal{M})\big) \quad \text{and} \quad f_{D!*}\big(g_{D!*}(\mathcal{M})\big) \simeq \delta_{D!*}\big(\gamma_{D!*}(g_{D!*}(\mathcal{M}))\big).$$
On the other hand, we can write $\gamma \circ \beta = \gamma' \circ \beta'$, where $\beta' \colon U_2 \cap V \hookrightarrow U_1 \cap U_2$ is a closed immersion and $\gamma' \colon U \cap V \hookrightarrow V$ is an open immersion.

By Step 2, we thus have
$$\gamma_{D!*}\big(\beta_{D!*}(\mathcal{M})\big) \simeq \gamma'_{D!*}\big(\beta'_{D!*}(\mathcal{M})\big). \tag{6.12}$$
By combining (6.11) and (6.12), we thus conclude that
$$f_{D!*}\big(g_{D!*}(\mathcal{M})\big) \simeq \delta_{D!*}\big(\gamma'_{D!*}\big(\beta'_{D!*}(\alpha_{D!*}(\mathcal{M}))\big)\big)$$
$$\simeq (\delta \circ \gamma')_{D!*}\big((\beta' \circ \alpha)_{D!*}(\mathcal{M})\big) \simeq (f \circ g)_{D!*}(\mathcal{M}),$$
where the second isomorphism follows using the assertions in Step 1 and Step 2 and the last isomorphism follows using Step 2 (or Remark 6.78). This completes the proof of the proposition. $\square$

One reason the intermediate extension construction is important is due to the fact that it provides a description of simple holonomic $\mathcal{D}$-modules, as follows.

THEOREM 6.83. *If X is a smooth, irreducible variety, then a $\mathcal{D}_X$-module $\mathcal{M}$ on X is holonomic and simple if and only if there is a locally closed immersion $f \colon W \hookrightarrow X$, with W smooth and irreducible, and a simple $\mathcal{D}_W$-module $\mathcal{E}$ that is coherent as an $\mathcal{O}_W$-module, such that $\mathcal{M} \simeq f_{D!*}(\mathcal{E})$.*

PROOF. We already know that if there is such a pair $(f, \mathcal{E})$, then $\mathcal{M}$ is simple by Theorem 6.79v). Suppose now that $\mathcal{M}$ is a simple holonomic $\mathcal{D}_X$-module.

We begin by showing that if $U \subseteq X$ is an open subset such that $\mathcal{M}|_U \neq 0$, and $j \colon U \hookrightarrow X$ is the inclusion, then $\mathcal{M} \simeq j_{D!*}(\mathcal{M}|_U)$. Since $\mathcal{M}$ is simple and $\mathcal{M}|_U \neq 0$, it follows that $\Gamma_Z(\mathcal{M}) = 0$, hence $\mathcal{M} \hookrightarrow \mathcal{H}^0\big(j_{D*}(\mathcal{M})\big)$. By Theorem 6.79iv), we thus have $j_{D!*}(\mathcal{M}|_U) \subseteq \mathcal{M}$, and since $\mathcal{M}$ is simple and $j_{D!*}(\mathcal{M}|_U) \neq 0$, we have $\mathcal{M} = j_{D!*}(\mathcal{M}|_U)$.

In particular, we see that for such U, the restriction $\mathcal{M}|_U$ is simple. Indeed, if $\mathcal{N} \subseteq \mathcal{M}|_U$ is a nonzero $\mathcal{D}_U$-submodule, it follows from Theorem 6.79iii) that $j_{D!*}(\mathcal{N}) \subseteq j_{D!*}(\mathcal{M}|_U) \simeq \mathcal{M}$. Since $\mathcal{M}$ is simple, we have $j_{D!*}(\mathcal{N}) = \mathcal{M}$, and after restriction to U we get $\mathcal{N} = \mathcal{M}|_U$. Furthermore, note that if we have a locally closed immersion $f \colon W \to U$ and a simple $\mathcal{D}_W$-module $\mathcal{E}$ that is coherent as an $\mathcal{O}_W$-module such that $\mathcal{M}|_U \simeq f_{D!*}(\mathcal{E})$, then $\mathcal{M} \simeq (j \circ f)_{D!*}(\mathcal{E})$ by what we have already shown and Proposition 6.82. Therefore we may always replace X by an open subset U such that $\mathcal{M}|_U \neq 0$. In particular, we may assume that $Z := \mathrm{Supp}(\mathcal{M})$ is smooth and irreducible. By Kashiwara's Equivalence Theorem, we can write $\mathcal{M} = i_+(\mathcal{N})$, where $i \colon Z \hookrightarrow X$ is the inclusion, and $\mathcal{N}$ is a simple $\mathcal{D}_Z$-module. It is clear that if we can find a locally closed immersion $f \colon W \to Z$ and a simple $\mathcal{D}_W$-module $\mathcal{E}$

that is coherent as an $\mathcal{O}_W$-module such that $\mathcal{N} \simeq f_{D!*}(\mathcal{E})$, then $\mathcal{M} \simeq (i \circ f)_{D!*}(\mathcal{E})$ by Proposition 6.82. Therefore we may replace $(X, \mathcal{M})$ by $(Z, \mathcal{N})$ to assume that $\mathrm{Supp}(\mathcal{M}) = X$. In this case it follows from Remark 6.30 that there is an open subset $U \subseteq X$ such that $\mathcal{M}|_U$ is $\mathcal{O}_U$-coherent (and nonzero). As we have seen, we may replace X and $\mathcal{M}$ by U and $\mathcal{M}|_U$, in which case the assertion in the theorem is trivial. $\square$

The intermediate extension is also important since it leads to the intersection cohomology of singular varieties.

Definition 6.84. Let X be a smooth variety and Z an irreducible closed subvariety of X. If Z_{sm} is the smooth locus of Z and $f\colon Z_{\mathrm{sm}} \hookrightarrow X$ is the inclusion, then the *intersection cohomology $\mathcal{D}$-module IC_Z* is given by

$$IC_Z := f_{D!*}(\mathcal{O}_{Z_{\mathrm{sm}}}).$$

More generally, if Z is a closed subvariety of X, with irreducible components $Z_1, \ldots, Z_r$, then we put

$$IC_Z := IC_{Z_1} \oplus \ldots \oplus IC_{Z_r}.$$

Remark 6.85. It is a consequence of assertions i) and v) in Theorem 6.79 that if Z is an irreducible, closed subvariety of the smooth variety X, then IC_Z is a simple holonomic $\mathcal{D}_X$-module, such that $\mathbf{D}_X(IC_Z) \simeq IC_Z$. It is also clear from the definition that $\mathrm{Supp}(IC_Z) = Z$ and that if $U = X \smallsetminus Z_{\mathrm{sing}}$, then $IC_Z|_U \simeq \mathcal{H}^r_{U\cap Z}(\mathcal{O}_U)$, where $r = \mathrm{codim}_X(Z)$.

Remark 6.86. If Z is an irreducible closed subvariety of the smooth variety X, V is any nonempty smooth open subset of Z, and $g\colon V \to X$ is the inclusion map, then $IC_Z \simeq g_{D!*}(\mathcal{O}_V)$. Indeed, g factors as $V \overset{j}{\hookrightarrow} Z_{\mathrm{sm}} \overset{f}{\hookrightarrow} X$, and we have

$$g_{D!*}(\mathcal{O}_V) \simeq f_{D!*}\big(j_{D!*}(\mathcal{O}_V)\big) \simeq f_{D!*}(\mathcal{O}_{X_{\mathrm{sm}}}) = IC_Z,$$

where the first isomorphism follows from Proposition 6.82 and the second one follows from Example 6.80.

Remark 6.87. If Z is an irreducible closed subvariety of the smooth variety X and if $j\colon U \hookrightarrow X$ is the inclusion of an open subset such that $U \cap Z \neq \emptyset$, then it follows from the previous remark and Proposition 6.82 that $IC_Z = j_{D!*}(IC_{Z\cap U})$. In particular, we have a canonical isomorphism $IC_Z|_U \simeq IC_{Z\cap U}$.

Remark 6.88. Suppose that Z is an irreducible closed subvariety of the smooth, irreducible variety X, and let $r = \mathrm{codim}_X(Z)$. The inclusion of Z_{sm} in X factors as $Z_{\mathrm{sm}} \overset{i}{\hookrightarrow} V = X \smallsetminus Z_{\mathrm{sing}} \overset{j}{\hookrightarrow} X$, hence

$$IC_Z = j_{D!*}\big(i_+(\mathcal{O}_{Z_{\mathrm{sm}}})\big) \simeq j_{D!*}\big(\mathcal{H}^r_{Z_{\mathrm{sm}}}(\mathcal{O}_V)\big).$$

It is not hard to see that we have an isomorphism $\mathcal{H}^0\big(j_{D*}(\mathcal{H}^r_{Z_{\mathrm{sm}}}(\mathcal{O}_V))\big) \simeq \mathcal{H}^r_Z(\mathcal{O}_X)$ (we leave this as an exercise). We thus deduce from assertions iv) and v) in Theorem 6.79 that IC_Z is isomorphic to the unique simple $\mathcal{D}_X$-submodule of $\mathcal{H}^r_Z(\mathcal{O}_X)$, which is also the smallest $\mathcal{D}_X$-submodule of $\mathcal{H}^r_Z(\mathcal{O}_X)$ whose restriction to V is $\mathcal{H}^r_{Z_{\mathrm{sm}}}(\mathcal{O}_V)$.

Definition 6.89. If X is a smooth projective variety and Z is a closed subvariety of X, then the *intersection cohomology* of Z is

$$IH(Z) := p_{D*}(IC_Z),$$

where $p\colon X \to \operatorname{Spec}(k)$ is the structure morphism. We put

$$IH^i(Z) := \mathcal{H}^i\big(IH(Z)\big) \quad \text{for} \quad i \in \mathbf{Z}.$$

Note that these are finite-dimensional k-vector spaces.

Remark 6.90. It follows from Example 6.16 that $IH(Z) \simeq \mathbf{R}\Gamma\big(\mathrm{DR}_X(IC_Z)\big)$.

Example 6.91. Suppose that Z is a smooth, irreducible, d-dimensional closed subvariety of X. If $i\colon Z \to X$ is the inclusion and $q = p \circ i$, where $p\colon X \to \operatorname{Spec}(k)$ is the structure morphism, then $IC_Z \simeq i_{D*}(\mathcal{O}_Z)$, hence

$$IH^i(Z) = \mathcal{H}^i\big(p_{D*}\big(i_{D*}(\mathcal{O}_Z)\big)\big) \simeq \mathcal{H}^i\big(q_{D*}(\mathcal{O}_Z)\big) \simeq H^{i+d}_{\mathrm{dR}}(Z).$$

Remark 6.92. Note that intersection cohomology satisfies (shifted) Poincaré duality, in the sense that if Z is a closed subvariety of the smooth, irreducible, projective variety X, then

$$IH^i(Z) \simeq IH^{-i}(Z)^\vee \quad \text{for all} \quad i \in \mathbf{Z}.$$

Indeed, as we have mentioned in Remark 6.85, we have $IC_Z \simeq \mathbf{D}_X(IC_Z)$, and since X is a projective variety, it follows from Theorem 6.59 that if $p\colon X \to \operatorname{Spec}(k)$ is the structure map, then

$$IH(Z) = p_{D*}(IC_Z) \simeq p_{D*}\big(\mathbf{D}_X(IC_Z)\big) \simeq \mathbf{D}_{\operatorname{Spec}(k)}\big(IH(Z)\big)$$

and we obtain our assertion by taking that i-th cohomology.

In fact, the intersection cohomology of Z is independent of the embedding in a smooth projective variety X since the IC_Z is independent of such an embedding, in a suitable sense. In order to make sense of this, we need to discuss the category of quasi-coherent $\mathcal{D}$-modules on a singular variety Z. The existence of this category is a consequence of Kashiwara's Equivalence Theorem.

Suppose that Z is a variety over k that admits a closed embedding $i\colon Z \hookrightarrow X$ in a smooth, irreducible variety X (for example, any quasi-projective variety satisfies this property). We define the category $\mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_Z)$ to be the full subcategory $\mathcal{M}od_{qc}(\mathcal{D}_{X,Z})$ of $\mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_X)$ consisting of those quasi-coherent $\mathcal{D}_X$-modules supported on Z. The fact that this is well-defined is a consequence of the following

Proposition 6.93. If $i\colon Z \hookrightarrow X$ and $i'\colon Z \hookrightarrow X'$ are two closed immersions in smooth, irreducible varieties X and X', then we have a natural equivalence of categories

$$\Phi_{i,i'}\colon \mathcal{M}od_{qc}(\mathcal{D}_{X,Z}) \simeq \mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_{X',Z}). \tag{6.13}$$

Moreover, if $X' = X \times W$ and if $i = p \circ i'$, where $p\colon X \times W \to X$ is the projection onto the first component, then p_+ induces an equivalence of categories $\mathcal{M}od_{qc}(\mathcal{D}_{X',Z}) \simeq \mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_{X,Z})$, with inverse induced by $p^\dagger$.

Proof. We first prove the second assertion in the proposition. Assume, to begin with, that X and W are both affine varieties. We choose a closed immersion $\varphi\colon W \hookrightarrow \mathbf{A}^N$.

We have $i' = (i, g)$, for a morphism $g\colon Z \to W$. Since i is a closed immersion, there is a morphism $\psi\colon X \to \mathbf{A}^N$ such that $\psi \circ i = \varphi \circ g$. We thus have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "i'"'] & X \arrow[d, "\beta"] \\ X \times W \arrow[r, "\alpha"'] & X \times \mathbf{A}^N, \end{tikzcd}$$

with $\alpha = (p, \varphi)$ and $\beta = (1_X, \psi)$, in which all morphisms are closed immersions. It is a consequence of Kashiwara's Equivalence Theorem that we have the following equivalences of categories

$$\mathcal{M}od_{qc}(\mathcal{D}_{X',Z}) \xrightarrow{\alpha_+} \mathcal{M}od_{qc}(\mathcal{D}_{X\times\mathbf{A}^N,Z}) \xleftarrow{\beta_+} \mathcal{M}od_{qc}(\mathcal{D}_{X,Z}). \tag{6.14}$$

Moreover, it follows from Proposition 6.24 that the inverse of the first equivalence is given by $\alpha^\dagger$ and the inverse of the second one is given by $\beta^\dagger$. If $q\colon X \times \mathbf{A}^N \to X$ is the projection onto the first component, then $q \circ \beta = \mathrm{Id}_X$, hence $q_+ \circ \beta_+$ is the identity on $\mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X)$ and thus $q_+ = \beta^\dagger$ on $\mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_{X\times\mathbf{A}^N,Z})$. Since $q \circ \alpha = p$, it follows that

$$p_+ = \beta_+^{-1} \circ \alpha_+ \colon \mathcal{M}od_{qc}(\mathcal{D}_{X',Z}) \to \mathcal{M}od_{qc}(\mathcal{D}_{X,Z}).$$

Furthermore, its inverse $\alpha_+^{-1} \circ \beta_+$ is given by $p^\dagger$. Indeed, this follows from the fact that $\beta^\dagger \circ q^\dagger$ is the identity on $\mathcal{D}^b_{\mathrm{qc}}(\mathcal{D}_X)$, hence $\beta_+ = q^\dagger$ on $\mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_{X,Z})$, and thus $\alpha_+^{-1} \circ \beta_+ = \alpha^\dagger \circ q^\dagger = p^\dagger$ on $\mathcal{M}od_{qc}(\mathcal{D}_{X,Z})$. This proves the last assertion in the theorem when both X and W are affine.

In the general case, we consider finite affine open covers $X = \bigcup_i U_i$ and $W = \bigcup_i V_i$ such that $g\big(i^{-1}(U_i)\big) \subseteq V_i$ for all i. We put $U_i' = U_i \times V_i$ and $Z_i = i^{-1}(U_i) = {i'}^{-1}(V_i)$. Using the fact that p_+ induces equivalences of categories

$$\mathcal{M}od_{qc}(\mathcal{D}_{U_i',Z_i}) \simeq \mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_{U_i,Z_i}) \quad \text{and}$$

$$\mathcal{M}od_{qc}(\mathcal{D}_{U_i'\cap U_j',Z_i\cap Z_j}) \simeq \mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_{U_i\cap U_j,Z_i\cap Z_j}),$$

with inverse induced by $p^\dagger$, we conclude that p_+ induces an equivalence of categories (6.13), whose inverse is induced by $p^\dagger$. We use the fact that

$$i'(Z) \subseteq X_0' := \bigcup_i (U_i \times V_i),$$

hence $\mathcal{M}od_{qc}(\mathcal{D}_{X',Z}) \simeq \mathcal{M}od_{qc}(\mathcal{D}_{X_0',Z})$ via restriction to X_0'.

It is easy to obtain now the first assertion in the proposition: given two closed immersions $i_1\colon Z \to X_1$ and $i_2\colon Z \to X_2$, with X_1 and X_2 smooth and irreducible, we consider also $i' = (i_1, i_2)\colon Z \to X' = X_1 \times X_2$. If $p\colon X' \to X_1$ and $q\colon X' \to X_2$ are the two projections, it follows from what we have already proved that we get an equivalence of categories as the composition

$$\Phi_{i_1 i_2}\colon \mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_{X_1,Z}) \xrightarrow{p^\dagger} \mathcal{M}od_{qc}(\mathcal{D}_{X',Z}) \xrightarrow{q_+} \mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_{X_2,Z}).$$

It is easy to check, using the base-change formula in Theorem 6.64, that if $i_3\colon Z \hookrightarrow X_3$ is another closed embedding in a smooth, irreducible variety, then we have an isomorphism of functors $\Phi_{i_2,i_3} \circ \Phi_{i_1,i_2} \simeq \Phi_{i_1,i_3}$; we leave this as an exercise for the reader. This completes the proof of the proposition. □

Corollary 6.94. If we have two closed embeddings $i\colon Z \hookrightarrow X$ and $i'\colon Z \hookrightarrow X'$, with X and X' smooth, irreducible varieties, and if IC_Z and IC_Z' are the corresponding $\mathcal{D}$-modules in $\mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_{X,Z})$ and $\mathcal{M}od_{\mathrm{qc}}(\mathcal{D}_{X',Z})$, then we have an isomorphism $IC_Z \simeq \Phi_{i,i'}(IC'_Z)$. In particular, the intersection cohomology of a projective variety Z is independent of the embedding in an ambient smooth, irreducible, projective variety X, up to isomorphism.

PROOF. Since X is smooth, using Nagata's theorem and resolution of singularities, we can find an open immersion $X \hookrightarrow \overline{X}$, where $\overline{X}$ is a complete, smooth, irreducible variety. If $\overline{Z}$ is the closure of Z in $\overline{X}$, then it follows from Remark 6.87 that we have a canonical isomorphism $IC_Z \simeq IC_{\overline{Z}}|_X$. Since it is easy to see that the equivalence $\Phi_{i,i'}$ commutes with restriction to open subsets, we deduce that also for the first assertion in the corollary we may assume that X and X' are complete varieties.

By the definition of the equivalence $\Phi_{i,i'}$, we see that for the first assertion in the corollary, it is enough to show that if $X' = X \times W$, with W a smooth, irreducible, complete variety, such that $i = p \circ i'$, where $p\colon X' \to X$ is the projection onto the first component, then $p_+(IC'_Z) \simeq IC_Z$. Note that since p is proper, the canonical functorial transformation $p_{D!} \to p_{D*}$ is an isomorphism. If $j\colon Z_{\rm sm} \hookrightarrow Z$ is the embedding of the smooth locus of Z and $f = i \circ j$ and $f' = i' \circ j$, since $p_{D!} \simeq p_{D*}$ is exact on $\mathcal{M}od_{\rm qc}(\mathcal{D}_{X',Z})$, it maps the image of

$$\gamma_{f'}\colon \mathcal{H}^0\big(f'_{D!}(\mathcal{O}_{Z_{\rm sm}})\big) \to \mathcal{H}^0\big(f'_{D*}(\mathcal{O}_{Z_{\rm sm}})\big)$$

to the image of

$$\gamma_f\colon \mathcal{H}^0\big(f_{D!}(\mathcal{O}_{Z_{\rm sm}})\big) \to \mathcal{H}^0\big(f_{D*}(\mathcal{O}_{Z_{\rm sm}})\big).$$

We thus get $p_+(IC'_Z) \simeq IC_Z$. The last assertion now follows from this and the fact that if $q\colon X \to \mathrm{Spec}(k)$ and $q'\colon X' \to \mathrm{Spec}(k)$ are the structure morphisms, then $q \circ p = q'$, hence we have an isomorphism of functors $q_+ \circ p_+ \simeq q'_+$. □

CHAPTER 7

The Riemann-Hilbert correspondence

In this chapter we work over the field $\mathbf{C}$ of complex numbers. Our goal is to describe one of the central results of the theory of $\mathcal{D}$-modules, the Riemann-Hilbert correspondence, relating $\mathcal{D}$-modules with objects of a topological nature. Since this result will not play a big role in what follows, we will be very brief: we will only prove the classical version of the result and Kashiwara's constructibility theorem. The other results will mostly be stated without proofs.

We begin by reviewing a few basic facts concerning complex algebraic varieties and the corresponding analytic spaces. Recall that if X is a complex algebraic variety, then we have a corresponding analytic space X^{an}, with sheaf of holomorphic functions $\mathcal{O}_{X^{\mathrm{an}}}$. Note that X is smooth if and only if X^{an} is a complex manifold and X is connected if and only if X^{an} has the same property. In general, we have a morphism of locally ringed spaces $i\colon (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$, which at the level of sets is given by the identity map. For every $\mathcal{O}_X$-module $\mathcal{F}$, we have a corresponding $\mathcal{O}_{X^{\mathrm{an}}}$-module $\mathcal{F}^{\mathrm{an}} := i^*(\mathcal{F}) = i^{-1}(\mathcal{F}) \otimes_{i^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{\mathrm{an}}}$. For every $x \in X$, the ring homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathrm{an}},x}$ is flat (in fact, it induces an isomorphism between the corresponding completions). Therefore the functor $\mathcal{F} \to \mathcal{F}^{\mathrm{an}}$ is an exact functor.

Suppose now that X is a smooth, irreducible complex algebraic variety. As we have mentioned in Remark 2.18, we have a sheaf $\mathcal{D}_{X^{\mathrm{an}}}$ of differential operators on X^{an}. This can be defined as the subsheaf of $\mathcal{E}nd_{\mathbf{C}}(\mathcal{O}_{X^{\mathrm{an}}})$ generated by $\mathcal{O}_{X^{\mathrm{an}}}$ and $\mathrm{Der}_{\mathbf{C}}(\mathcal{O}_{X^{\mathrm{an}}}) \simeq \mathcal{T}_X^{\mathrm{an}}$. It is then clear that we have a canonical isomorphism $\mathcal{D}_{X^{\mathrm{an}}} \simeq \mathcal{D}_X^{\mathrm{an}}$. Moreover, we see that if $\mathcal{M}$ is a left (or right) $\mathcal{D}_X$-module, then $\mathcal{M}^{\mathrm{an}}$ has a natural structure of left (respectively, right) $\mathcal{D}_X^{\mathrm{an}}$-module. In this way we get an exact functor

$$\mathcal{M}od(\mathcal{D}_X) \to \mathcal{M}od(\mathcal{D}_X^{\mathrm{an}}), \quad \mathcal{M} \mapsto \mathcal{M}^{\mathrm{an}},$$

where we denote by $\mathcal{M}od(\mathcal{D}_X^{\mathrm{an}})$ the category of left $\mathcal{D}_X^{\mathrm{an}}$-modules. We get an induced exact functor $D(\mathcal{D}_X) \to D(\mathcal{D}_X^{\mathrm{an}})$ between the corresponding derived categories.

For every commutative ring R and every topological space Y, we denote by $D(\underline{R}_Y)$ the derived category corresponding to the Abelian category of sheaves of R-modules on Y. We will almost exclusively consider the case when R is a field.

7.1. The classical Riemann-Hilbert correspondence

In this section we describe the classical correspondence between (holomorphic) vector bundles with integrable connection and local systems. Let M be a connected n-dimensional complex manifold[1], with sheaf of holomorphic functions $\mathcal{O}_M$. The notion of integrable connection on a vector bundle[2] $\mathcal{E}$ on M is defined as in the algebraic setting, see Chapter 3.2. Furthermore, given a vector bundle with integrable

[1] All manifolds are assumed to be separated, with a countable basis of open subsets.

[2] As usual, we identify vector bundles and locally free $\mathcal{O}_M$-modules.

connection $(\mathcal{E},\nabla)$, we have a corresponding *de Rham* complex $\mathrm{DR}(\mathcal{E})$:

$$0\to\mathcal{E}\xrightarrow{\nabla}\Omega_M^1\otimes_{\mathcal{O}_M}\mathcal{E}\xrightarrow{\nabla}\ldots\xrightarrow{\nabla}\Omega_M^n\otimes_{\mathcal{O}_M}\mathcal{E}\to 0,$$

placed in cohomological degrees $-n,\ldots,0$.

DEFINITION 7.1. Let K be a field. A *K-local system* on M is a sheaf $\mathcal{L}$ of K-vector spaces on M such that for every $x\in X$, there is an open neighborhood U_x of x such that $\mathcal{L}|_{U_x}\simeq\underline{W}_{U_x}$ for some finite dimensional K-vector space W (we denote by $\underline{W}_Y$ the constant sheaf associated to W on a topological space Y). A morphism of K-local systems is simply a morphism of sheaves of K-vector spaces. For simplicity, we often call a $\mathbf{C}$-local system just a *local system*.

REMARK 7.2. It follows from the definition that if $\mathcal{L}$ is a K-local system on M, then the function $M\ni x\to\dim_K\mathcal{L}_x$ is locally constant, and thus constant since M is connected. This is the *rank* of the K-local system $\mathcal{L}$.

REMARK 7.3. We note that the set of isomorphism classes K-local systems of rank r is in bijection with the set of representations $\pi_1(M)\to\mathrm{GL}_r(K)$, up to conjugation (this is the *monodromy representation* associated to the local system).

EXAMPLE 7.4. If $f\colon X\to S$ is a smooth, projective morphism of smooth complex algebraic varieties, with S connected, then for every $q\in\mathbf{Z}_{\geq 0}$, we have a K-local system on $R^qf_*\underline{K}_X$ on S^{an}. The fact that this is indeed a K-local system is a consequence of Ehresman's theorem, which says that for every $x\in S$, there is an open neighborhood U of x (in the analytic topology) and a $\mathcal{C}^\infty$-diffeomorphism $f^{-1}(U_x)\simeq U_x\times f^{-1}(x)$ over U_x.

The following result is known as the (classical) Riemann-Hilbert correspondence.

THEOREM 7.5. *For every manifold M, the functor $\mathcal{L}\mapsto(\mathcal{O}_M\otimes_{\mathbf{C}}\mathcal{L},d\otimes\mathrm{Id})$ from the category of local systems on M to the category of vector bundles on M with integrable connection is an equivalence of categories, whose inverse is given by $(\mathcal{E},\nabla)\mapsto\mathcal{E}^\nabla:=\ker(\nabla)\subseteq\mathcal{E}$.*

PROOF. It is clear that if $\mathcal{L}$ is a rank r local system on M, then $\mathcal{O}_M\otimes_{\mathbf{C}}\mathcal{L}$ is a vector bundle on M of rank r. Moreover, if $d\colon\mathcal{O}_M\to\Omega_M^1$ is the standard connection on $\mathcal{O}_M$ (given by $\nabla(f)=df$), then we get an integrable connection

$$\nabla\colon\mathcal{O}_M\otimes_{\mathbf{C}}\mathcal{L}\to\Omega_M^1\otimes_{\mathcal{O}_M}(\mathcal{O}_M\otimes_{\mathbf{C}}\mathcal{L})\simeq\Omega_M^1\otimes_{\mathbf{C}}\mathcal{L}$$

given by $\nabla(f\otimes u)=df\otimes u$. Furthermore, it is clear that $(\mathcal{O}_M\otimes_{\mathbf{C}}\mathcal{L})^\nabla\simeq\mathcal{L}$.

In order to conclude the proof, it is enough to show that if $\mathcal{E}$ is a rank r holomorphic vector bundle on M, with an integrable connection ∇, then $\mathcal{E}^\nabla$ is a rank r local system and the morphism $\mathcal{O}_M\otimes_{\mathbf{C}}\mathcal{E}^\nabla\to\mathcal{E}$ induced by the inclusion $\mathcal{E}^\nabla\subseteq\mathcal{E}$ (which is automatically a morphism of vector bundles with connection) is an isomorphism. This is an application of a classical result regarding the existence and uniqueness of a system of linear PDEs, with initial conditions. In what follows we give a geometric version of the argument.

It is enough to check the assertion locally, hence from now on we assume that M is an open subset of $\mathbf{C}^n$, with coordinates $x_1,\ldots,x_n$, and $\mathcal{E}=\mathcal{O}_M^{\oplus r}$, with the standard basis $e_1,\ldots,e_r$. In this case the connection ∇ on $\mathcal{E}$ is described by the functions $\Gamma_{ij}^k \in \mathcal{O}(M)$, for $1 \leq i \leq n$ and $1 \leq j, k \leq r$ such that

$$\nabla(e_j) = \sum_{i=1}^{n} \sum_{k=1}^{r} \Gamma_{ij}^k dx_i \otimes e_k$$

for $1 \leq j \leq n$ (the Γ_{ij}^k are the *Christoffel symbols* of the connection). The integrability condition $\nabla\big(\nabla(e_j)\big) = 0$ for all j translates as

$$\frac{\partial \Gamma_{ij}^q}{\partial x_p} - \frac{\partial \Gamma_{pj}^q}{\partial x_i} + \sum_{k=1}^{r} (\Gamma_{ij}^k \Gamma_{pk}^q - \Gamma_{pj}^k \Gamma_{ik}^q) = 0 \quad \text{for} \quad 1 \leq i, p \leq n,\ 1 \leq j, q \leq r. \tag{7.1}$$

Note also that if $U \subseteq M$ is an open subset and $s = \sum_{j=1}^{r} s_j e_j$, with $s_j \in \mathcal{O}_M(U)$ for all j, then $s \in \Gamma(U, \mathcal{E}^{\nabla})$ if and only if $s_1, \ldots, s_n$ satisfy the following system of linear PDEs:

$$\frac{\partial s_k}{\partial x_i} + \sum_{j=1}^{r} \Gamma_{ij}^k s_j = 0 \quad \text{for} \quad 1 \leq i \leq n,\ 1 \leq k \leq r. \tag{7.2}$$

Consider now $E = M \times \mathbf{C}^r$, with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_r$, the total space of $\mathcal{E}$, and let $\pi \colon E \to M$ be the projection. We consider on E the following vector fields:

$$v_i = \partial_{x_i} - \sum_{k=1}^{r} \left(\sum_{j=1}^{r} \Gamma_{ij}^k y_j \right) \partial_{y_k} \quad \text{for} \quad 1 \leq i \leq n. \tag{7.3}$$

It is clear that $v_1, \ldots, v_n$ span a rank n subbundle $\mathcal{F}$ of $\mathcal{T}_Y$. A straightforward computation shows that the integrability condition in (7.1) is equivalent to the fact that $[v_i, v_j] = 0$ for $1 \leq i, j \leq n$. This immediately implies that we have $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$, that is, $\mathcal{F}$ satisfies the Frobenius integrability condition. The Frobenius theorem (see [**War83**, Theorem 1.60]) implies that for every point $p \in E$, there is an integral submanifold Y_p of $\mathcal{F}$ at p: this is a closed submanifold of an open neighborhood of p, containing p, such that $\mathcal{T}_{Y_p} = \mathcal{F}|_{Y_p}$; moreover, any two such submanifolds are equal in a suitable neighborhood of p.

Note now that if we view $s \in \Gamma(U, \mathcal{E})$ as a section of π, and if we write $s(x) = \big(x, s_1(x), \ldots, s_r(x)\big)$, then $s(U)$ is a closed submanifold of $\pi^{-1}(U)$, with the tangent bundle trivialized by

$$\partial_{x_i} + \sum_{k=1}^{r} \tfrac{\partial s_k}{\partial x_i} \partial_{y_k} \quad \text{for} \quad 1 \leq i \leq n.$$

We thus see that $s(U) \subseteq \pi^{-1}(U)$ is an integral submanifold of $\mathcal{F}$ if and only if s satisfies the equations (7.2), so this is the case if and only if $s \in \Gamma\big(U, \mathcal{E}^{\nabla}\big)$. Moreover, if Y_p is an integral submanifold of $\mathcal{F}$ at $p \in E$, it follows from (7.3) that the linear map $T_pE \to T_{\pi(p)}M$ induced by π is an isomorphism, hence there is an open neighborhood U of $\pi(p)$ and $s \in \Gamma(U, \mathcal{E})$ such that $Y_p = s(U)$ in a neighborhood of p.

The existence and uniqueness of the integral submanifold of $\mathcal{F}$ at any point of E implies that for every $x \in M$, the composition

$$\mathcal{E}_x^{\nabla} \hookrightarrow \mathcal{E}_x \to \pi^{-1}(x) \simeq \mathbf{C}^r$$

is an isomorphism. For every such x, we can thus find an open neighborhood U_x of x and a $\mathbf{C}$-vector subspace $V \subseteq \Gamma(U_x, \mathcal{E}^{\nabla})$ such that the induced map $V \to \pi^{-1}(x)$

is an isomorphism. This implies that after possibly shrinking U_x, we may assume that the induced morphism $\mathcal{O}_{U_x} \otimes_{\mathbf{C}} V \to \mathcal{E}|_{U_x}$ is an isomorphism. It is then clear that $\mathcal{E}^{\nabla}|_{U_x} \simeq V \otimes_{\mathbf{C}} \underline{\mathbf{C}}_{U_x}$ is a trivial local system and the canonical morphism $\mathcal{O}_M \otimes_{\mathbf{C}} \mathcal{E}^{\nabla} \to \mathcal{E}$ is an isomorphism (of vector bundles with connection) on U_x. This completes the proof of the theorem.

□

Corollary 7.6. If M is an n-dimensional connected complex manifold and $\mathcal{E}$ is a vector bundle on M with an integrable connection, then in $\mathcal{D}(\underline{\mathbf{C}}_M)$ we have an isomorphism

$$\mathrm{DR}_M(\mathcal{E}) \simeq \mathcal{E}^{\nabla}[n].$$

Proof. It follows from Theorem 7.5 that we have an isomorphism $(\mathcal{E}, \nabla) \simeq (\mathcal{O}_M \otimes_{\mathbf{C}} \mathcal{E}^{\nabla}, d \otimes \mathrm{Id})$, which implies that $\mathrm{DR}_M(\mathcal{E})$ is isomorphic to $\mathrm{DR}_M(\mathcal{O}_M) \otimes_{\mathbf{C}} \mathcal{E}^{\nabla}$. Therefore it is enough to prove the assertion when $\mathcal{E} = \mathcal{O}_X$, with the standard connection, in which case the assertion is the well-known holomorphic Poincaré Lemma. □

Remark 7.7. If X is a smooth, irreducible, projective (or, more generally, proper) complex algebraic variety, then the functor $\mathcal{E} \mapsto \mathcal{E}^{\mathrm{an}}$ gives an equivalence of categories between the category of (algebraic) vector bundles on X, with integrable connection, and that of (holomorphic) vector bundles on X^{an}, with integrable connection. This is a consequence of Serre's GAGA theorem (due, in the proper case, to Deligne) once we describe integrable connections in terms of $\mathcal{O}$-linear information.

Consider on $X \times X$ the ideal $\mathcal{I}$ defining the diagonal and let $\mathcal{P} = \mathcal{O}_{X \times X}/\mathcal{I}^2$. Note that we have an exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{O}_X \to 0,$$

and we have an isomorphism $\Omega_X^1 \simeq \mathcal{I}/\mathcal{I}^2$, such that df corresponds to $f \otimes 1 - 1 \otimes f$ for every section f of $\mathcal{O}_X$. We have two $\mathcal{O}_X$-module structures on $\mathcal{P}$, one via the first projection $p_1 \colon X \times X \to X$ (that we consider on the left) and another one via the second projection $p_2 \colon X \times X \to X$ (that we consider on the right). Both these structures coincide on $\mathcal{I}/\mathcal{I}^2$, inducing the usual $\mathcal{O}_X$-module structure on Ω_X^1. Using this framework, giving a connection $\nabla \colon \mathcal{E} \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$ is equivalent to giving an $\mathcal{O}_X$-linear morphism $\varphi \colon \mathcal{E} \to \mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{E}$ such that $(\pi \otimes \mathrm{Id}_{\mathcal{E}}) \circ \varphi = \mathrm{Id}_{\mathcal{E}}$. Indeed, such a map has the form $u \mapsto 1 \otimes u + \nabla(u)$, where $\nabla(u) \in \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$ and it is straightforward to check that φ is $\mathcal{O}_X$-linear if and only if ∇ is a connection. The same considerations work in the analytic setting. Moreover, note that if ∇ is a connection on $\mathcal{E}$, then $\nabla \circ \nabla \colon \mathcal{E} \to \Omega_X^2 \otimes_{\mathcal{O}_X} \mathcal{E}$ is an $\mathcal{O}_X$-linear map, hence checking integrability of the connection comes down to the fact that a certain morphism between coherent $\mathcal{O}_X$-module is 0. We can thus apply GAGA to conclude that if X is a proper algebraic variety, then the categories of algebraic (holomorphic) vector bundles with integrable connection on X (respectively, on X^{an}) are equivalent.

7.2. The functorial formalism in the topological setting

In this section we describe very briefly the functors associated to a map in the topological setting. For a detailed introduction, see for example [Ive86].

We assume that all topological spaces here are separated and locally compact (note that this is the case for X^{an}, when X is a complex algebraic variety). We will soon impose one additional finite-dimensionality condition.

Let K be a field and X a topological space. Recall that we denote by $\mathcal{D}(\underline{K}_X)$ the derived category corresponding to the Abelian category of sheaves of K-vector spaces on X. In particular, when X is a point, we have the derived category $\mathcal{D}(K)$ of K-vector spaces. We want to describe several exact functors associated to a continuous map $f\colon X \to Y$ at the level of derived categories.

First, we have the (derived) *push-forward* functor $f_*\colon \mathcal{D}^+(\underline{K}_X) \to \mathcal{D}^+(\underline{K}_Y)$, the right derived functor of the usual push-forward functor for sheaves. When f is the map to a point, we also write $\mathbf{R}\Gamma$ for f_* and $H^i(X,-)$ for its i-th cohomology.

We also have the *pull-back* functor $f^*\colon \mathcal{D}(\underline{K}_Y) \to \mathcal{D}(\underline{K}_X)$, induced by an exact functor at the level of Abelian categories (this is the functor that is often denoted by f^{-1}). It restricts to a functor $\mathcal{D}^+(\underline{K}_Y) \to \mathcal{D}^+(\underline{K}_X)$ which is the left adjoint of f_*. If X is the topological subspace of Y and f is the inclusion map, then for $u \in \mathcal{D}^b(\underline{K}_Y)$, we often write $u|_X$ instead of $f^*(u)$.

We also have the (derived) *push-forward with compact supports* functor

$$f_!\colon \mathcal{D}^+(\underline{K}_X) \to \mathcal{D}^+(\underline{K}_Y),$$

defined as follows. If $\mathcal{F}$ is a sheaf of K-vector spaces on X, then we get a sheaf on Y, whose sections on $V \subseteq Y$ are given by those $s \in \Gamma\big(f^{-1}(V),\mathcal{F}\big)$ with the induced map $\mathrm{Supp}(s) \to V$ proper[3]. This gives a left exact functor, whose right derived functor is $f_!$. If f is the map to a point, we also write $\mathbf{R}\Gamma_c$ for $f_!$ and $H^i_c(X,-)$ for its i-th cohomology.

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two continuous maps, then we have isomorphisms of functors

$$(g\circ f)_* \simeq g_*\circ f_*, \quad (g\circ f)_! \simeq g_!\circ f_!, \quad \text{and} \quad (g\circ f)^* \simeq f^*\circ g^*.$$

EXAMPLE 7.8. It follows from definition that for every f, we have a morphism of functors $f_! \to f_*$, which is an isomorphism when f is proper. On the other hand, if $j\colon U \hookrightarrow X$ is the inclusion map of an open subset, then $j_!$ is induced at the level of derived categories by the exact functor that maps a sheaf $\mathcal{F}$ on U to its *extension by 0* on X.

In this topological setting we have the following general base-change theorem: given a Cartesian diagram of topological spaces

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y, \end{tikzcd}$$

we have an isomorphism of functors $u^*\circ f_! \simeq g_!\circ v^*$.

The more subtle functor is the functor $f^!$, that's only defined at the level of derived categories. Its existence requires one extra assumption.

DEFINITION 7.9. We say that a topological space X has *finite cohomological dimension* if there is N such that $H^i_c(X,\mathcal{F}) = 0$ for all sheaves $\mathcal{F}$ on X and all

[3]This means that the inverse image of a compact set is compact.

$i > N$. The smallest N with this property is the *cohomological dimension* $\mathrm{cdim}(X)$ of X.

One can show that if X has finite cohomological dimension, then every locally closed subspace Y of X has finite cohomological dimension (in fact, we have $\mathrm{cdim}(Y) \leq \mathrm{cdim}(X)$).

REMARK 7.10. If $f\colon X \to Y$ is a continuous map and $\mathrm{cdim}(X) = N$, then $\mathrm{cdim}\big(f^{-1}(y)\big) \leq N$ for all $y \in Y$, hence we deduce from the base-change formula that $\mathcal{H}^i\big(f_!(\mathcal{F})\big) = 0$ for all $i > N$. In particular, we see that $f_!$ induces a functor $f_!\colon \mathcal{D}^b(\underline{K}_X) \to \mathcal{D}^b(\underline{K}_Y)$.

From now on, we assume that all topological spaces we consider have finite cohomological dimension (for example, this is the case for X^{an}, whenever X is a complex algebraic variety). The following is a nontrivial result:

THEOREM 7.11. *If $f\colon X \to Y$ is a continuous map of topological spaces (assumed to be separated, locally compact, and of finite cohomological dimension), then the functor $f_!\colon \mathcal{D}^b(\underline{K}_X) \to \mathcal{D}^b(\underline{K}_Y)$ has a right adjoint $f^!\colon \mathcal{D}^b(\underline{K}_Y) \to \mathcal{D}^b(\underline{K}_X)$. Moreover, for every $u \in \mathcal{D}^b(\underline{K}_X)$ and $v \in \mathcal{D}^b(\underline{K}_Y)$, we have a functorial isomorphism*

$$\mathbf{R}\mathcal{H}om\big(f_!(u), v\big) \simeq f_*\mathbf{R}\mathcal{H}om\big(u, f^!(v)\big). \tag{7.4}$$

Note that by taking $\mathbf{R}\Gamma$ on both sides in (7.4), we obtain an isomorphism

$$\mathbf{R}\mathrm{Hom}\big(f_!(u), v\big) \simeq \mathbf{R}\mathrm{Hom}\big(u, f^!(v)\big)$$

in $\mathcal{D}^b(K)$. This is known as *global Verdier duality*, while the isomorphism in (7.4) is known as *local Verdier duality*.

EXAMPLE 7.12. If $j\colon U \hookrightarrow Y$ is the inclusion of an open subset, then we have an isomorphim of functors $j^! \simeq j^*$. On the other hand, if $i\colon Z \hookrightarrow Y$ is the inclusion of $Z = X \smallsetminus U$, then we have an isomorphism of functors $i_*i^! \simeq \mathbf{R}\Gamma_Z$, where Γ_Z is the left exact functor that associates to a sheaf its subsheaf of sections supported on Z and $\mathbf{R}\Gamma_Z$ is the corresponding right derived functor. For every $u \in \mathcal{D}(\underline{K}_Y)$, we thus have an exact triangle

$$i_!i^!(u) \to u \to j_*j^*(u) \xrightarrow{+1} . \tag{7.5}$$

We note that we have another exact triangle

$$j_!j^!(u) \to u \to i_*i^*(u) \xrightarrow{+1}, \tag{7.6}$$

as can be easily checked using the definitions.

DEFINITION 7.13. If $f\colon X \to \{*\}$ is the morphism to a point, then the *dualizing complex* of X is $\omega_X := f^!(K) \in \mathcal{D}^b(\underline{K}_X)$. The *Verdier duality functor* is the contravariant functor $\mathbf{D}_X = \mathbf{R}\mathcal{H}om(-, \omega_X)\colon \mathcal{D}^b(\underline{K}_X) \to \mathcal{D}^b(\underline{K}_X)$.

REMARK 7.14. It is an immediate consequence of the definition of $f^!$ as an adjoint functor that if $f\colon X \to Y$ and $g\colon Y \to Z$ are two continuous maps, then we have an isomorphism of functors $(g \circ f)^! \simeq f^! \circ g^!$. In particular, by taking Z to be a point, we obtain an isomorphism $f^!(\omega_Y) \simeq \omega_X$. By taking $v = \omega_Y$ in (7.4), we see that we have an isomorphism of functors

$$\mathbf{D}_Y \circ f_! \simeq f_* \circ \mathbf{D}_X. \tag{7.7}$$

EXAMPLE 7.15. If X is an orientable real manifold of dimension n, then $\omega_X \simeq \underline{K}_X[n]$. Note that, in this case, Verdier duality for the morphism to a point gives Poincaré duality: for every sheaf of K-vector spaces on X, we have an isomorphism

$$\mathrm{Ext}^j(\mathcal{F}, \underline{K}_X) \simeq H_c^{n-j}(X, \mathcal{F})^\vee.$$

In particular, by taking $\mathcal{F} = \underline{K}_X$, we get $H^j(X, K) \simeq H_c^{n-j}(X, K)^\vee$.

We will be interested in the analytic space associated to an algebraic variety, when we will focus on the following subcategory of the derived category of sheaves of K-vector spaces.

DEFINITION 7.16. Let Z be an arbitrary complex algebraic variety. A sheaf of K-vector spaces $\mathcal{F}$ on Z^{an} is *constructible* if there is a partition $Z = Z_1 \sqcup \ldots \sqcup Z_N$, with each Z_i a smooth, irreducible, locally closed subvariety of Z such that $\mathcal{F}|_{Z_i}$ is a K-local system for every i. We denote by $\mathcal{D}_c^b(\underline{K}_Z)$ the subcategory of $\mathcal{D}^b(\underline{K}_{Z^{\mathrm{an}}})$ consisting of all u such that $\mathcal{H}^i(u)$ is a constructible sheaf for every $i \in \mathbf{Z}$: this is the *constructible derived category of sheaves (of K-vector spaces) on* Z^{an} (we use the simpler notation $\mathcal{D}_c^b(\underline{K}_Z)$ instead of $\mathcal{D}_c^b(\underline{K}_{Z^{\mathrm{an}}})$ since the constructible category only makes sense in the classical topology).

REMARK 7.17. It is easy to check that $\mathcal{D}_c^b(\underline{K}_Z)$ is a triangulated subcategory of $\mathcal{D}^b(\underline{K}_{Z^{\mathrm{an}}})$. It is a highly nontrivial result that if $f\colon X \to Y$ is a morphism of complex algebraic varieties, then the functors f_*, $f_!$, f^*, $f^!$, and $\mathbf{D}_{Z^{\mathrm{an}}}$ preserve the constructible derived category of sheaves.

REMARK 7.18. One can show that if Z is a complex algebraic variety, then there is an isomorphism of functors $\mathbf{D}_{Z^{\mathrm{an}}} \circ \mathbf{D}_{Z^{\mathrm{an}}} \simeq \mathrm{Id}$ on $\mathcal{D}_c^b(\underline{K}_Z)$. Using this and the isomorphism (7.7), it is easy to see that if $f\colon X \to Y$ is a morphism of complex algebraic varieties, then $\mathbf{D}_{X^{\mathrm{an}}} \circ f^! \circ \mathbf{D}_{Y^{\mathrm{an}}}$ is a left adjoint of f_*, hence we have an isomorphism of functors $\mathcal{D}_c^b(\underline{K}_{Y^{\mathrm{an}}}) \to \mathcal{D}_c^b(\underline{K}_X)$:

$$\mathbf{D}_{X^{\mathrm{an}}} \circ f^* \simeq f^! \circ \mathbf{D}_{Y^{\mathrm{an}}}.$$

7.3. Kashiwara's constructibility theorem

Let X be a smooth, irreducible, n-dimensional complex algebraic variety. Recall that for every $\mathcal{D}_X$-module $\mathcal{M}$, we defined the (algebraic) de Rham complex $\mathrm{DR}_X(\mathcal{M})$. Note that this represents $\Omega_X^n \otimes_{\mathcal{D}_X}^L \mathcal{M}$ in the bounded derived category $\mathcal{D}^b(\underline{\mathbf{C}}_X)$ of sheaves of $\mathbf{C}$-vector spaces on X (this follows using the resolution of Ω_X^n given in Example 3.65). With a slight abuse of notation, we will denote by DR_X also the functor

$$\mathcal{D}^b(\mathcal{D}_X) \to \mathcal{D}^b(\underline{\mathbf{C}}_X), \quad u \mapsto \Omega_X^n \otimes_{\mathcal{D}_X}^L u.$$

From now on, we will mostly be concerned with the following analytic version of the above functor:

$$\mathrm{DR}_X^{\mathrm{an}}\colon \mathcal{D}^b(\mathcal{D}_X) \to \mathcal{D}^b(\underline{\mathbf{C}}_{X^{\mathrm{an}}}), \quad u \mapsto \Omega_{X^{\mathrm{an}}}^n \otimes_{\mathcal{D}_X^{\mathrm{an}}}^L u^{\mathrm{an}}.$$

This is the composition

$$\mathcal{D}^b(\mathcal{D}_X) \to \mathcal{D}^b(\mathcal{D}_X) \to \mathcal{D}^b(\underline{\mathbf{C}}_{X^{\mathrm{an}}}),$$

where the first functor is the analytification functor and the second one maps $v \in \mathcal{D}^b(\mathcal{D}_X^{\mathrm{an}})$ to $\Omega_{X^{\mathrm{an}}}^n \otimes_{\mathcal{D}_X^{\mathrm{an}}}^L v$.

Note that the resolution of Ω_X^n in Example 3.65 induces a corresponding resolution of $\Omega^n_{X^{\mathrm{an}}}$, hence for a $\mathcal{D}_X$-module $\mathcal{M}$, $\mathrm{DR}_X^{\mathrm{an}}(\mathcal{M})$ is represented indeed by the *analytic de Rham complex* of $\mathcal{M}$:

$$0 \to \mathcal{M}^{\mathrm{an}} \to \Omega^1_{X^{\mathrm{an}}} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathcal{M}^{\mathrm{an}} \to \ldots \to \Omega^n_{X^{\mathrm{an}}} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathcal{M}^{\mathrm{an}} \to 0.$$

Example 7.19. It follows from Corollary 7.6 that if $\mathcal{E}$ is a vector bundle with integrable connection on X, then $\mathrm{DR}_X^{\mathrm{an}}(\mathcal{E}) \simeq (\mathcal{E}^{\mathrm{an}})^{\nabla}[n]$ is a (cohomological shift of a) local system on X^{an}.

Example 7.20. For general $\mathcal{D}_X$-modules (even holonomic ones), the behavior of the two functors DR_X and $\mathrm{DR}_X^{\mathrm{an}}$ can be quite different. For example, consider $X = \mathbf{A}^1_{\mathbf{C}}$ and let $\mathcal{M}$ be the holonomic $\mathcal{D}_X$-module $\mathcal{D}_X/\mathcal{D}_X(\partial_x - \lambda)$, for some $\lambda \neq 0$. Using the isomorphism of $\mathcal{O}_X$-modules $\mathcal{M} \simeq \mathcal{O}_X$ that maps $\overline{\partial_x}$ to λ, we see that $\mathrm{DR}_X(\mathcal{M})$ is isomorphic to the complex

$$0 \to \mathcal{O}_X \to \Omega^1_X \to 0, \quad f \mapsto \left(\tfrac{\partial f}{\partial x} + \lambda f\right) dx,$$

which is easily seen to be quasi-isomorphic to 0. On the other hand, the corresponding analytic version is quasi-isomorphic to $\mathbf{C}_{X^{\mathrm{an}}}[1]$, with $\mathcal{H}^{-1}\big(\mathrm{DR}_X^{\mathrm{an}}(\mathcal{M})\big)$ generated by $\varphi_x = \exp(-\lambda x)$. As we will see in the next section, this pathology disappears if we work with holonomic $\mathcal{D}$-modules with regular singularities.

The following result shows that this construction commutes with proper push-forward of $\mathcal{D}$-modules.

Proposition 7.21. For every morphism $f\colon X \to Y$ of smooth, irreducible, complex algebraic varieties, and every $u \in \mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_X)$, there is a functorial map

$$\mathrm{DR}_Y^{\mathrm{an}}\big(f_+(u)\big) \to f_*\big(\mathrm{DR}_X^{\mathrm{an}}(u)\big).$$

This is an isomorphism if f is proper.

Proof. We do not give the proof, but only mention that the corresponding assertion for $u \in \mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}_X^{\mathrm{an}})$ follows easily from the definition of push-forward and holds for arbitrary f (see [**HTT08**, Theorem 4.2.5]. Therefore the key point is the fact that analytification commutes with push-forward of $\mathcal{D}$-modules. If X is proper and Y is a point, the assertion follows using GAGA to compare the cohomology of the de Rham complex of $\mathcal{M}$ and that of $\mathcal{M}^{\mathrm{an}}$, when $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module. For a detailed proof in the general case, see [**HTT08**, Proposition 4.7.2]. ☐

The following is Kashiwara's constructibility theorem. We give a simplified proof in the algebraic case, due to Bernstein.

Theorem 7.22. *If X is a smooth, irreducible, complex algebraic variety and $w \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$, then $\mathrm{DR}_X^{\mathrm{an}}(w) \in \mathcal{D}^b_c(\underline{\mathbf{C}}_X)$.*

Proof. We break the proof in a few steps, following [**Bo+87**, Chapter VIII, Section 20].

Step 1. The key ingredient is that if $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, then it follows from Remark 6.30 that there is a nonempty open subset $U \subseteq X$ such that $\mathcal{M}|_U$ is $\mathcal{O}_U$-coherent. In this case, using Corollary 7.6 we deduce that

$$\mathrm{DR}_X^{\mathrm{an}}(\mathcal{M})|_U \simeq \mathrm{DR}_U^{\mathrm{an}}(\mathcal{M}|_U) \simeq \mathcal{L}[n],$$

where $\mathcal{L}$ is a local system on U and $n = \dim(X)$. Applying this for each nonzero $\mathcal{H}^i(w)$, we conclude that there is a nonempty open subset U of X such that each

$\mathrm{DR}_X^{\mathrm{an}}\big(\mathcal{H}^i(w)\big)|_U \in \mathcal{D}_c^b(\underline{\mathbf{C}}_U)$. In this case, an argument using truncation functors (see Remark A.20) implies that if $v = \mathrm{DR}_X^{\mathrm{an}}(w)$, then $v|_U \in \mathcal{D}_c^b(\underline{\mathbf{C}}_U)$. Since X is a Noetherian topological space, it follows that it is enough to prove the following

Claim: If $w \in \mathcal{D}_{\mathrm{hol}}^b(\mathcal{D}_X)$ and $v = \mathrm{DR}_X^{\mathrm{an}}(w)$ and we have an open subset $U \subseteq X$ such that $v|_U \in \mathcal{D}_c^b(\underline{\mathbf{C}}_U)$ then for every nonempty open subset W of $X \smallsetminus U$, there is a nonempty open subset $W_0 \subseteq W$ (so $U \cup W_0$ is open in X) such that $v|_{U\cup W_0} \in \mathcal{D}_c^b(\underline{\mathbf{C}}_{U\cup W_0})$.

In fact, it is enough to prove the assertion in the claim when $W = X \smallsetminus U$, but it will be convenient to treat the more general statement.

Step 2. We first show that the claim holds if there is a proper morphism $p\colon X \to W$ whose restriction to W is the identity. Note first that in this case W is (Zariski) closed in X: indeed, we have $W = \{x \in X \mid p(x) = 1_X(x)\}$, and this is closed since X is separated. In particular, W is both open and closed in $Z = X \smallsetminus U$, hence there is W' open in Z such that $Z = W \sqcup W'$.

If $j\colon U \hookrightarrow X$, $i\colon W \hookrightarrow X$, and $i'\colon W' \to X$ are the inclusions, then the exact triangle (7.6) becomes

$$j_!j^!(v) \to v \to i_*i^*(v) \oplus i'_*i'^*(v) \xrightarrow{+1} .$$

Applying p_*, we obtain the exact triangle

$$p_*j_!(v|_U) \to p_*(v) \to i^*(v) \oplus p_*i'_*i'^*(v) \xrightarrow{+1} . \tag{7.8}$$

By assumption, we know that $v|_U \in \mathcal{D}_c^b(\underline{\mathbf{C}}_U)$, hence $p_*j_!(v|_U) \in \mathcal{D}_c^b(\underline{\mathbf{C}}_W)$. On the other hand, since p is proper, it follows from Proposition 7.21 that $p_*(v) \simeq \mathrm{DR}_W^{\mathrm{an}}\big(p_+(u)\big)$. Since $p_+(u) \in \mathcal{D}_{\mathrm{hol}}^b(W)$, it follows from Step 1 that there is a nonempty open subset W_0 of W such that $p_*(v)|_{W_0} \in \mathcal{D}_c^b(\underline{\mathbf{C}}_{W_0})$. By restricting the exact triangle to W_0, we thus conclude[4] that $i^*(v)|_{W_0} \simeq v|_{W_0} \in \mathcal{D}_c^b(\underline{\mathbf{C}}_{W_0})$. We thus conclude that $v|_{U\cup W_0} \in \mathcal{D}_c^b(\underline{\mathbf{C}}_{U\cup W_0})$, completing the proof of the claim in this case.

Step 3. We now proceed to proving the general case. In this step we reduce the claim to the case when $X = \mathbf{A}^d \times \mathbf{P}^N$ for some d and N, such that the projection on the first component induces an étale morphism $W \to \mathbf{A}^d$. Note first that in order to prove the claim, we may always replace X by an open subset whose intersection with W is nonempty. We may thus assume that X is affine and we have algebraic coordinates $x_1, \ldots, x_n$ on X such that W is defined by $(x_{d+1}, \ldots, x_n)$. We thus get an étale morphism

$$f = (x_1, \ldots, x_n)\colon X \to \mathbf{A}^n = \mathbf{A}^d \times \mathbf{A}^{n-d}$$

such that the projection $\pi_1\colon \mathbf{A}^d \times \mathbf{A}^{n-d} \to \mathbf{A}^d$ induces an étale morphism $W \to \mathbf{A}^d$. We also choose a locally closed immersion $g\colon X \to \mathbf{P}^N$ and let $h = (\pi_1 \circ f, g)\colon X \to X' := \mathbf{A}^d \times \mathbf{P}^N$, which is a locally closed immersion as well. Let $\overline{X}$ be the closure of $h(X)$ in X' and $U' = h(U) \cup (X' \smallsetminus \overline{X})$, which is an open subset of X'. We also put $W' = h(W)$, which is an open subset of $X' \smallsetminus U'$. Let $w' := h_+(w) \in \mathcal{D}_{\mathrm{hol}}^b(\mathcal{D}_{X'})$ and $v' := \mathrm{DR}_{X'}^{\mathrm{an}}(w')$.

[4] We are using the fact that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are sheaves of $\mathbf{C}$-vector spaces on Y, such that $\mathcal{M}_1 \oplus \mathcal{M}_2$ is constructible, then $\mathcal{M}_1$ and $\mathcal{M}_2$ are constructible. Indeed, it is enough to prove this when $\mathcal{M}_1 \oplus \mathcal{M}_2$ is a local system and use the fact that a sheaf $\mathcal{F}$ of $\mathbf{C}$-vector spaces is a local system if and only if for every $y \in Y$, there is an open neighborhood U_y of Y such that $\Gamma(U_y, \mathcal{F})$ is a finite-dimensional vector space and the canonical map $\Gamma(U_y, \mathcal{F}) \to \mathcal{F}_z$ is an isomorphism for all $z \in U_y$.

Note now that for every locally closed immersion $\nu\colon Z \to X'$, if $u \in \mathcal{D}^b_{\mathrm{hol}}(Z)$, then we have a canonical isomorphism

$$\nu^*\big(\mathrm{DR}^{\mathrm{an}}_{X'}(\nu_+(u))\big) \simeq \mathrm{DR}^{\mathrm{an}}_{Z}(u).$$

Indeed, is enough to check this separately for open and closed immersions. The assertion is clear for open immersions, while for closed immersions it follows from Proposition 7.21 and the fact that if ν is a closed immersion, then $\nu^*\nu_* \simeq \mathrm{Id}$.

In particular, we have an isomorphism $h^*(v') \simeq v$. Moreover, we see that $v'|_{U'} \in \mathcal{D}^b_c(\underline{\mathbf{C}}_{U'})$ and if we have a nonempty open subset $W_0' \subseteq W'$ such that $v'|_{U' \cup W_0'} \in \mathcal{D}^b_c(\underline{\mathbf{C}}_{U'\cup W_0'})$, then $v|_{U\cup W_0} \in \mathcal{D}^b_c(\underline{\mathbf{C}}_{U\cup W_0})$, where $W_0 = h^{-1}(W_0')$.

Step 4. We thus may and will assume that $X = \mathbf{A}^d \times \mathbf{P}^N$ and W is such that the projection onto the first component induced an étale morphism $\varphi\colon W \to \mathbf{A}^d$. Consider the étale morphism $\psi = (\varphi, \mathrm{Id})\colon W \times \mathbf{P}^N \to \mathbf{A}^d \times \mathbf{P}^N$ and let

$$\widetilde{W} := \psi^{-1}(W) = \big\{(x, y_1, y_2) \in \mathbf{A}^d \times \mathbf{P}^N \times \mathbf{P}^N \mid (x, y_1) \in W, (x, y_2) \in W\big\}.$$

Note that we have a closed immersion $\alpha\colon W \hookrightarrow \widetilde{W}$ given by $\alpha(x, y) = (x, y, y)$, so that the composition $W \xrightarrow{\alpha} \widetilde{W} \to W$ is the identity and the second morphism is étale. Therefore α maps W isomorphically onto a connected component of $\widetilde{W}$.

Since ψ is étale, it is easy to see that $\mathrm{DR}^{\mathrm{an}}_{W\times\mathbf{P}^N}\big(\psi^\dagger(w)\big) \simeq \psi^*(v)$. By assumption, we have $\psi^*(v)|_{\psi^{-1}(U)} \in \mathcal{D}^b_c(\underline{\mathbf{C}}_{\psi^{-1}(U)})$ and $\alpha(W)$ is an open subset of $W \times \mathbf{P}^N \smallsetminus \psi^{-1}(U)$. If we can find a nonempty open subset W_0 of W such that the restriction of $\psi^*(v)$ to $\psi^{-1}(U) \cup \alpha(W_0)$ lies in the constructible derived category, then $v|_{U\cup W_0}$ lies in the constructible derived category, proving the claim. We may thus replace (X, U, W, u) by $\big(W \times \mathbf{P}^N, \psi^{-1}(U), \alpha(W), \psi^\dagger(u)\big)$. Since the composition $W \times \mathbf{P}^N \to W \xrightarrow{\alpha} \alpha(W)$ is a proper morphism whose restriction to $\alpha(W)$ is the identity, we can apply the case covered in Step 2. This completes the proof of the theorem. $\square$

7.4. The Riemann-Hilbert correspondence

Our goal in this section is to discuss the general form of the Riemann-Hilbert correspondence for $\mathcal{D}$-modules in the algebraic setting. This requires an extra condition on holonomic $\mathcal{D}$-modules: regular singularities. In order to motivate this definition, by begin with a brief overview of Deligne's version of the Riemann-Hilbert correspondence, see [**Del70**].

7.4.1. Differentials with log poles. Let us recall the definition of the sheaves of differential forms with log poles along an SNC divisor. Suppose that X is a smooth, irreducible algebraic variety over an algebraically closed field k and D is a reduced divisor on X with *simple normal crossings* (SNC, for short). Recall that this means that for every point $P \in X$, there are algebraic coordinates $x_1, \ldots, x_n$ in an open neighborhood U_P of P, which are centered at P, and such that in a neighborhood the divisor $D|_{U_P}$ is defined by $x_1 \cdots x_r$. The sheaf $\Omega^1_X(\log D)$ is the $\mathcal{O}_X$-submodule of the sheaf $\Omega_X \otimes_{\mathcal{O}_X} k(X)$ of rational 1-forms that in the neighborhood U_P is generated by $\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}, dx_{r+1}, \ldots, dx_n$. It is straightforward to see that this does not depend on the choice of local coordinates and therefore the subsheaves defined in such open subsets glue to a subsheaf of $\Omega_X \otimes_{\mathcal{O}_X} k(X)$. It is clear from the definition that $\Omega_X^1(\log D)$ is a locally free sheaf of rank n such that $\Omega_X^1 \subseteq \Omega_X^1(\log D)$. The higher sheaves of differentials with log poles are defined by

$$\Omega_X^p(\log D) := \wedge^p \Omega_X^1(\log D).$$

If D_1 is a prime divisor that appears in D and $D' = D - D_1$, then there is a short exact sequence

$$0 \to \Omega_X^1(\log D') \overset{f}{\hookrightarrow} \Omega_X^1(\log D) \xrightarrow{\mathrm{Res}_{D_1}} \mathcal{O}_{D_1} \to 0, \tag{7.9}$$

where f is the inclusion and the *residue* map $\mathrm{Res}_{D_1} \colon \Omega_X^1(\log D) \to \mathcal{O}_{D_1}$ is defined as follows: in an open subset U, with algebraic coordinates $x_1, \ldots, x_n$ such that D is defined by $x_1 \cdots x_r$ and D_1 is defined by x_1, we put

$$\mathrm{Res}_{D_1}\big(f_1 \tfrac{dx_1}{x_1} + \ldots + f_r \tfrac{dx_r}{x_r} + f_{r+1}dx_{r+1} + \ldots + f_n dx_n\big) = f_1|_{D_1}.$$

The exactness of (7.9) is straightforward to check.

DEFINITION 7.23. Let X and D be as above. If $\mathcal{E}$ is a vector bundle on X, a *connection on $\mathcal{E}$ with log poles along D* is an additive map

$$\nabla \colon \mathcal{E} \to \Omega_X^1(\log D) \otimes_{\mathcal{O}_X} \mathcal{E},$$

which satisfies the Leibniz rule: $\nabla(fu) = f\nabla(u) + df \otimes u$ for every $u \in \mathcal{E}$ and $f \in \mathcal{O}_X$. As in the case of usual connections, we say that ∇ is integrable if $\nabla^2 \colon \mathcal{E} \to \Omega_X^2(\log D) \otimes_{\mathcal{O}_X} \mathcal{E}$ vanishes.

Given a connection ∇ on $\mathcal{E}$ with log poles along D, if we write $D = \sum_{i=1}^N D_i$, then for every i we get an induced *residue endomorphism* $\mathrm{Res}_{D_i}^{\nabla} \colon \mathcal{E}|_{D_i} \to \mathcal{E}|_{D_i}$, as induced by the composition

$$\mathcal{E} \xrightarrow{\nabla} \Omega_X^1(\log D) \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\mathrm{Res}_{D_i} \otimes 1_{\mathcal{E}}} \mathcal{O}_{D_i} \otimes_{\mathcal{O}_X} \mathcal{E}$$

(it is easy to check that this is $\mathcal{O}_X$-linear and vanishes on $\mathcal{E}(-D_i)$). We will be especially interested in the case when X is proper over k, in which case we can talk about the eigenvalues of $\mathrm{Res}_{D_i}^{\nabla}$, which are elements of k (note that in this case the coefficients of the characteristic polynomial of $\mathrm{Res}_{D_i}^{\nabla}$ are global sections of $\mathcal{O}_X$, hence are constant on X).

If X is a smooth, irreducible, complex algebraic variety, we also have analytic versions of the above definitions: in particular, if D is an SNC divisor on X and $\mathcal{E}$ is a holomorphic vector bundle on X^{an}, we may consider connections on $\mathcal{E}$ with log poles along D, using $\Omega_{X^{\mathrm{an}}}^1(\log D) := \big(\Omega_X^1(\log D)\big)^{\mathrm{an}}$.

7.4.2. Deligne's Riemann-Hilbert correspondence. Recall that if X is a smooth, irreducible, proper complex algebraic variety, then the classical Riemann-Hilbert correspondence induces an equivalence of categories between local systems on X^{an} and algebraic vector bundles on X with integrable connection (see Remark 7.7). The goal of Deligne's theory is to give a similar algebraic description for the category of local systems on U^{an}, where U is an arbitrary smooth, irreducible, complex algebraic variety.

By Nagata's theorem, we have an open immersion $U \hookrightarrow X$, where X is a proper irreducible algebraic variety, By Hironaka's theorem on resolution of singularities, after replacing X by a suitable resolution, we may assume that X is smooth and $D = X \smallsetminus U$ is a (reduced) SNC divisor on X.

The following is Deligne's fundamental theorem.

THEOREM 7.24. *Let X be a smooth, irreducible, proper complex algebraic variety, $D=\sum_{i=1}^N D_i$ a reduced SNC divisor on X, and $U=X\smallsetminus D$. Given a section $\sigma\colon \mathbf{C}/\mathbf{Z}\to\mathbf{C}$ of the canonical projection, we have an equivalence of categories, that maps $(\mathcal{E},\nabla)\to(\mathcal{E}^{\rm an}|_U,\nabla|_U)$, between the category of (algebraic) vector bundles $\mathcal{E}$ on X with an integrable connection ∇ with log poles along D and such that for all i, the eigenvalues of $\mathrm{Res}^{\nabla}_{D_i}$ lie in $\sigma(\mathbf{C}/\mathbf{Z})$, and the category of holomorphic vector bundles on $U^{\rm an}$, with an integrable connection.*

A nice sketch of the proof is given in [**Kat76**]. Here we only discuss briefly the proof of essential surjectivity. Note first that by GAGA, the categories of algebraic, respectively holomorphic vector bundles on X are equivalent and an argument similar to the one in Remark 7.7 shows that the same holds for the corresponding categories of vector bundles with integrable connections with log poles along D.

Suppose now that we have a holomorphic vector bundle $(\mathcal{E},\nabla)$ on U, with an integrable connection. Note that, a priori, it is not clear that there is an extension of $\mathcal{E}$ to a vector bundle on X. We will construct such an extension locally. Let $P\in X$ and consider algebraic coordinates $x_1,\ldots,x_n$ in a neighborhood of P, centered at P, such that D is defined by $x_1\cdots x_r$. Let V be an open neighborhood of P, in the classical topology, that we identify via $(x_1,\ldots,x_n)$ to the polydisc

$$\{x\in\mathbf{C}^n\mid |x_j|<\epsilon \text{ for } 1\leq j\leq n\},$$

hence $U\cap V$ gets identified to $(\Delta_\epsilon^*)^r\times\Delta_\epsilon^{n-r}$, where $\Delta_\epsilon=\{x\in\mathbf{C}\mid |x|<\epsilon\}$ and $\Delta_\epsilon^*=\Delta_\epsilon\smallsetminus\{0\}$. By Theorem 7.5, $(\mathcal{E}|_{U\cap V},\nabla|_{U\cap V})$ corresponds to a local system on $U\cap V$, hence to a representation ρ of $\pi_1(U\cap V)$ (see Remark 7.3). Note that $\pi_1(U\cap V)\simeq\mathbf{Z}^r$, with generators giving by the loops $\gamma_1,\ldots,\gamma_r$, with γ_i going once counterclockwise around the divisor defined by z_i. We thus have a finite-dimensional vector space W and commuting invertible endomorphisms $A_1,\ldots,A_r\in\mathrm{GL}(W)$, with $A_j=\rho(\gamma_j)$.

In this case we have unique commuting endomorphisms $B_1,\ldots,B_r\in\mathrm{End}(W)$, such that for every j, with $1\leq j\leq r$, we have $\exp(2\pi iB_j)=A_j$ and all eigenvalues of B_j lie in $\sigma(\mathbf{C}/\mathbf{Z})$ (this follows easily by considering the Jordan canonical form). In this case we consider on V the holomorphic vector bundle $\widetilde{E}_V:=\mathcal{O}_V\otimes_{\mathbf{C}}W$, with the connection with log poles along $D|_V$ given by

$$\nabla\colon\mathcal{O}_V\otimes_{\mathbf{C}}W\to\Omega^1_{X^{\rm an}}|_V\otimes_{\mathcal{O}_V}(\mathcal{O}_V\otimes_{\mathbf{C}}W)\simeq\Omega^1_{X^{\rm an}}|_V\otimes_{\mathbf{C}}W,$$

$$\nabla(f\otimes w)=df\otimes w-f\cdot\sum_{j=1}^r\tfrac{dx_j}{x_j}\otimes B_j(w).$$

It is easy to see that this is, indeed, an integrable connection and it follows from the formula that $\mathrm{Res}^{\nabla}_{D_j}$ is given by the action of B_j, hence by assumption all its eigenvalues are in $\sigma(\mathbf{C}/\mathbf{Z})$. Finally, we can easily describe the flat sections on $V\cap U$: at every point of $V\cap U$, a basis of flat sections is given by the columns of the matrix $\prod_{j=1}^r x_j^{B_j}:=\exp\bigl(\log(x_1)B_1+\ldots+\log(x_r)B_r\bigr)$ (this is a straightforward computation). As a consequence, the monodromy action maps $\gamma_j\in\pi_1(U\cap V)$ to $\exp(2\pi iB_j)=A_j$. We thus conclude that we have an isomorphism of vector bundles with connection $\widetilde{E}_W|_{U\cap V}\simeq\mathcal{E}|_{U\cap V}$. One can check that in fact the $(\widetilde{E}_W,\nabla)$ patch together to a vector bundle with integrable connection with log poles along D whose restriction to U is isomorphic to $(\mathcal{E},\nabla)$. This proves that the functor in the theorem is essentially surjective.

7.4.3. Regular holonomic $\mathcal{D}$-modules. Before stating the general version of the Riemann-Hilbert correspondence, we discuss briefly the notion of regular holonomic $\mathcal{D}$-modules. For details, proofs of the main results, and the treatment in the analytic setting, see [**HTT08**, Chapters IV, V]. This concept is closely related to the notion of system of ODEs with regular singularities, but we do not discuss this aspect.

We begin with the case of a vector bundle with an integrable connection on a curve. Suppose that C is a smooth curve and $(\mathcal{E},\nabla)$ is a vector bundle with an integrable connection on C. Let $\overline{C}$ be the smooth projective curve that has an open embedding $j\colon C\hookrightarrow \overline{C}$.

DEFINITION 7.25. We say that $(\mathcal{E},\nabla)$ has *regular singularities* at $P\in\overline{C}\smallsetminus C$ if there is a vector bundle $\mathcal{E}'$ on $C\cup\{P\}$ with an integrable connection with log poles at P such that $(\mathcal{E}',\nabla)|_C\simeq(\mathcal{E},\nabla)$. We say that $(\mathcal{E},\nabla)$ has *regular singularities* if it has regular singularities at every $P\in\overline{C}\smallsetminus C$.

REMARK 7.26. With the notation in the above definition, note that if $j\colon C\hookrightarrow U_P=C\cup\{P\}$ is the inclusion and $(\mathcal{E}',\nabla)$ is a vector bundle on U_P with an integrable connection with a log pole at P such that $(\mathcal{E}',\nabla)|_C\simeq(\mathcal{E},\nabla)$, then we have an injective morphism $\mathcal{E}'\hookrightarrow j_+(\mathcal{E})$ and the connection on $\mathcal{E}'$ is induced by the one on $j_+(\mathcal{E})$. We thus conclude that if t is a local coordinate centered at P, then $\mathcal{E}$ has regular singularities at P if and only if there is an $\mathcal{O}_{\overline{C},P}$-submodule of $j_+(\mathcal{E})_P$ that is preserved by the action of $t\partial_t$ and which is equal to $j_+(\mathcal{E})_P$ after inverting t.

EXAMPLE 7.27. Let $C=\mathbf{A}^1$ and let $\mathcal{E}=\mathcal{D}_{\mathbf{A}^1}/\mathcal{D}_{\mathbf{A}^1}\cdot(\partial_x-\lambda)$, for some $\lambda\in\mathbf{C}$. We have an isomorphism $\mathcal{E}\simeq\mathcal{O}_{\mathbf{A}^1}$ such that via this isomorphism, the connection is given by

$$\nabla\colon\mathcal{O}_{\mathbf{A}^1}\to\Omega^1_{\mathbf{A}^1},\ \nabla(f)=dx\otimes\big(\tfrac{\partial f}{\partial x}+\lambda f\big).$$

In order to determine whether $(\mathcal{O}_{\mathbf{A}^1},\nabla_\lambda)$ has regular singularities at $\infty\in\mathbf{P}^1\smallsetminus\mathbf{A}^1$, we consider the coordinate $y=\frac{1}{x}$ centered at ∞. Since $dy=-\frac{1}{x^2}dx$, we see that ∇ is given in a punctured neighborhood of ∞ by

$$\nabla(g)=dy\otimes\left(\frac{\partial g}{\partial y}-\frac{\lambda g}{y^2}\right).$$

Since $\mathcal{O}_{\overline{C},\infty}$ is a DVR, with uniformizer y, it follows that every $\mathcal{O}_{\overline{C},P}$-submodule of the function field of $\overline{C}$ is equal to (y^m), for some $m\in\mathbf{Z}$. By definition, $(\mathcal{O}_{\mathbf{A}^1},\nabla_\lambda)$ has regular singularities at ∞ if and only if there is such m such that

$$\big(my^m-\lambda y^{m-1}\big)=y\cdot\nabla_{\partial_y}(y^m)\in\mathcal{O}_{\overline{C},P}\cdot(y^m),$$

which is the case if and only if $\lambda=0$.

EXAMPLE 7.28. Suppose that $C=\mathbf{A}^1\smallsetminus\{0\}$ and let $\mathcal{E}$ be the restriction to C of the $\mathcal{D}_{\mathbf{A}^1}$-module $\mathcal{D}_{\mathbf{A}^1}/\mathcal{D}_{\mathbf{A}^1}\cdot(x\partial_x-\lambda)$, for some $\lambda\in\mathbf{C}$. Note that we have $\mathcal{E}\simeq\mathcal{O}_C$, with the connection given by

$$\nabla_\lambda\colon\mathcal{O}_C\to\Omega^1_C,\quad \nabla_\lambda(g)=dx\otimes\left(\tfrac{\partial g}{\partial x}+\tfrac{\lambda}{x}\right).$$

It is then clear that $\mathcal{E}$ has regular singularities at 0. Similarly, if we consider the coordinate $y=\frac{1}{x}$ centered at ∞ on $\overline{C}=\mathbf{P}^1$, then the connection ∇_λ is given in a punctured neighborhood of ∞ by

$$\nabla_\lambda(g) = dy \otimes \left(\tfrac{\partial g}{\partial y} - \tfrac{\lambda}{y}\right),$$

hence $\mathcal{E}$ has regular singularities also at ∞.

DEFINITION 7.29. More generally, if $\mathcal{M}$ is a holonomic $\mathcal{D}$-module on a smooth curve C, we say that $\mathcal{M}$ has *regular singularities* if for some nonempty open subset $U \subseteq C$ such that $\mathcal{M}|_U$ is $\mathcal{O}_U$-coherent, $\mathcal{M}|_U$ has regular singularities. It is clear that the definition is independent of the choice of U.

We next move to higher dimensions. Let X be a smooth, irreducible, complex algebraic variety.

DEFINITION 7.30. If $\mathcal{E}$ is an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module, then we say that $\mathcal{E}$ has *regular singularities* if for every locally closed immersion $f\colon C \to X$, where C is a smooth curve, $f^*(\mathcal{E})$ has regular singularities. Note that if $\dim(X) = 1$, this agrees with the notion we have already discussed. The following result, due to Deligne, shows that also in higher dimensions we can characterize $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules with regular singularities in terms of connections with log poles.

THEOREM 7.31. *Let $\mathcal{E}$ be an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module on a smooth, irreducible, complex algebraic variety X. Given an open immersion $j\colon X \hookrightarrow \overline{X}$, with $\overline{X}$ proper, smooth, and irreducible, such that $D = \overline{X} \smallsetminus X$ is an SNC divisor, then $\mathcal{E}$ has regular singularities if and only if there is a vector bundle $\overline{\mathcal{E}}$ on $\overline{X}$, with an integrable connection ∇ with log poles along D, such that $(\overline{\mathcal{E}}, \nabla)|_X \simeq (\mathcal{E}, \nabla)$.*

Finally, we treat arbitrary holonomic $\mathcal{D}_X$-modules. Recall that by Theorem 6.83, every simple holonomic $\mathcal{D}_X$-module is isomorphic to $f_{D!*}(\mathcal{L})$, for some locally closed immersion $f\colon W \to X$, with W smooth and irreducible, and some simple $\mathcal{O}_W$-coherent $\mathcal{D}_W$-module $\mathcal{L}$.

DEFINITION 7.32. We say that a *simple* holonomic $\mathcal{D}_X$-module has *regular singularities* if $\mathcal{M} \simeq f_{D!*}(\mathcal{L})$, for some locally closed immersion $f\colon W \to X$, with W smooth and irreducible, and for some $\mathcal{O}_W$-coherent $\mathcal{D}_W$-module $\mathcal{L}$ with regular singularities. We say that an arbitrary holonomic $\mathcal{D}_X$-module $\mathcal{M}$ has *regular singularities* (or, simply, that it is *regular holonomic*) if all its simple factors have regular singularities.

It is clear from this definition that the subquotients of a regular holonomic $\mathcal{D}_X$-module are again regular holonomic. In particular, the full subcategory $\mathcal{M}od_{rh}(\mathcal{D}_X)$ of $\mathcal{M}od_{\mathrm{hol}}(\mathcal{D}_X)$ consisting of regular holonomic $\mathcal{D}_X$-modules is an Abelian category. Furthermore, it follows from the definition that the category is closed under extensions. We may thus consider the full subcategory $\mathcal{D}^b_{rh}(\mathcal{D}_X)$ of $\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ consisting of those u with all $\mathcal{H}^i(u)$ regular holonomic. This is a triangulated subcategory.

The following result says that this subcategory is preserved by the functors we have defined. For a proof, see [**HTT08**, Chapter 6].

THEOREM 7.33. *Let X be a smooth, irreducible, complex algebraic variety.*

i) *The subcategory $\mathcal{D}^b_{rh}(\mathcal{D}_X)$ of $\mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ is preserved by $\mathbf{D}_X$.*
ii) *If Y is another smooth, irreducible, complex algebraic variety and $f\colon X \to Y$ is a morphism, then f_{D*} and $f_{D!}$ map $\mathcal{D}^b_{rh}(\mathcal{D}_X)$ to $\mathcal{D}^b_{rh}(\mathcal{D}_Y)$ and $f^!_D$ and f^*_D map $\mathcal{D}^b_{rh}(\mathcal{D}_Y)$ to $\mathcal{D}^b_{rh}(\mathcal{D}_X)$.*

7.4.4. The statement of the Riemann-Hilbert correspondence. The following result, known as the *Riemann-Hilbert correspondence* is one of the main results in the theory of $\mathcal{D}$-modules. For a proof, see [**HTT08**, Chapter VII].

THEOREM 7.34. *For every smooth, irreducible, complex algebraic variety X, the functor $\mathrm{DR}_X^{\mathrm{an}}\colon \mathcal{D}^b_{rh}(\mathcal{D}_X) \to \mathcal{D}^b_c(\underline{\mathbf{C}}_X)$ is an equivalence of categories. Furthermore, the functor commutes with duality, that is, we have an isomorphism of functors $\mathbf{D}_{X^{\mathrm{an}}} \circ \mathrm{DR}_X^{\mathrm{an}} \simeq \mathrm{DR}_X^{\mathrm{an}} \circ \mathbf{D}_X$, and for every morphism of smooth, irreducible, complex algebraic varieties $f\colon X \to Y$, we have isomorphisms of functors:*

i) $\mathrm{DR}_Y^{\mathrm{an}} \circ f_{D*} \simeq f_* \circ \mathrm{DR}_X^{\mathrm{an}}$.
ii) $\mathrm{DR}_Y^{\mathrm{an}} \circ f_{D!} \simeq f_! \circ \mathrm{DR}_X^{\mathrm{an}}$.
iii) $\mathrm{DR}_X^{\mathrm{an}} \circ f_D^* \simeq f^* \circ \mathrm{DR}_Y^{\mathrm{an}}$.
iv) $\mathrm{DR}_X^{\mathrm{an}} \circ f_D^! \simeq f^! \circ \mathrm{DR}_Y^{\mathrm{an}}$.

REMARK 7.35. In the algebraic setting, for every smooth, irreducible, n-dimensional variety X and every $u, v \in \mathcal{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$, it follows from Lemma 6.61 that we have an isomorphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(u, v) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(u, \mathcal{D}_X) \otimes^L_{\mathcal{D}_X} v \simeq \mathbf{D}_X(u)^r \otimes^L_{\mathcal{D}_X} v[-n],$$

where we denote by $w \mapsto w^r$ the functor induced at the level of derived categories by the equivalence between left and right $\mathcal{D}_X$-modules in Chapter 3.6. Using also Exercise 3.60, we see that

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}\big(\mathbf{D}_X(u), v\big) \simeq u^r \otimes^L_{\mathcal{D}_X} v[-n] \simeq v^r \otimes^L_{\mathcal{D}_X} u[-n] \simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}\big(\mathbf{D}_X(v), u\big).$$

By taking $u = \mathcal{O}_X$, we thus conclude that

$$\mathrm{DR}_X(v) = \Omega^n_X \otimes^L_{\mathcal{D}_X} v \simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, v)[n] \simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}\big(\mathbf{D}_X(v), \mathcal{O}_X\big)[n]$$

and thus

$$\mathrm{DR}_X\big(\mathbf{D}_X(v)\big) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}\big(v, \mathcal{O}_X\big)[n].$$

Similar results hold in the analytic setting, and we see that

$$\mathrm{DR}_X^{\mathrm{an}}\big(\mathbf{D}_X(v)\big)[-n] \simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_X^{\mathrm{an}}}(v^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) =: \mathrm{Sol}_X^{\mathrm{an}}(v).$$

The functor $\mathrm{Sol}_X^{\mathrm{an}}$ is called the *solution functor* and it follows from Theorem 7.34 that it gives an anti-equivalence of categories between $\mathcal{D}^b_{rh}(\mathcal{D}_X)$ and $\mathcal{D}^b_c(\underline{\mathbf{C}}_X)$. The name is motivated by the fact that if X is affine and $v = \mathcal{D}_X/I$, then

$$\mathrm{Hom}_{\mathcal{D}_X^{\mathrm{an}}}(v, \mathcal{O}_{X^{\mathrm{an}}}) = \{\varphi \in \mathcal{O}(X^{\mathrm{an}}) \mid P\varphi = 0 \text{ for all } P \in I\}.$$

It follows from Theorem 7.34 that the category of regular holonomic $\mathcal{D}_X$-modules, that sits as a subcategory in $\mathcal{D}^b_{rh}(\mathcal{D}_X)$, is equivalent to an Abelian subcategory of $\mathcal{D}^b_c(\underline{\mathbf{C}}_X)$. This is the famous category of perverse sheaves.

DEFINITION 7.36. Let Z be a (possibly singular) complex algebraic variety and let K be a field. The category $\mathcal{P}erv_K(Z^{\mathrm{an}})$ of *K-perverse sheaves* on Z^{an} is the full subcategory of $\mathcal{D}^b_c(\underline{K}_Z)$ consisting of those u that satisfy the following two properties:

i) For every j in $\mathbf{Z}$, we have $\dim\big(\mathrm{Supp}(\mathcal{H}^j(u))\big) \leq -j$.
ii) The dual $\mathbf{D}_{Z^{\mathrm{an}}}(u)$ also satisfies the condition in i).

If $K = \mathbf{C}$, then we simply call these perverse sheaves.

For an introduction to the theory of perverse sheaves, see for example [**BBD82**] or [**Ach21**]. It is an important result that $\mathcal{P}erv_K(Z^{\mathrm{an}})$ is always an Abelian category in which every object has finite length.

THEOREM 7.37. *For every smooth, irreducible, complex algebraic variety X, the equivalence $\mathrm{DR}_X^{\mathrm{an}}\colon \mathcal{D}^b_{\mathrm{rh}}(\mathcal{D}_X) \to \mathcal{D}^b_c(\underline{\mathbf{C}}_{X^{\mathrm{an}}})$ induces an equivalence of categories*

$$\mathcal{M}od_{rh}(\mathcal{D}_X) \to \mathcal{P}erv_{\mathbf{C}}(X^{\mathrm{an}}).$$

PROOF. We only show that if $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module, then $\mathrm{DR}_X^{\mathrm{an}}(\mathcal{M})$ is a perverse sheaf. For the full proof of the assertion in the theorem, see [**HTT08**, Theorem 7.2.5]. Since

$$\mathbf{D}_{X^{\mathrm{an}}}\big(\mathrm{DR}_X^{\mathrm{an}}(\mathcal{M})\big) \simeq \mathrm{DR}_X^{\mathrm{an}}\big(\mathbf{D}_X(\mathcal{M})\big),$$

it follows that it is enough to show that $\mathrm{DR}_X^{\mathrm{an}}(\mathcal{M})$ satisfies condition i) in Definition 7.36. Given $j \in \mathbf{Z}$, let W be an irreducible component of the support of $\mathcal{H}^j\big(\mathrm{DR}_X^{\mathrm{an}}(\mathcal{M})\big)$. We need to show that if $d = \dim(W)$, then $d \leq -j$. Let W_0 be a nonempty smooth open subset of W and let $f\colon W_0 \to X$ be the inclusion map. By assumption, we have

$$0 \neq f^*\mathcal{H}^j\big(\mathrm{DR}_X^{\mathrm{an}}(\mathcal{M})\big) \simeq \mathcal{H}^j\big(f^*(\mathrm{DR}_X^{\mathrm{an}}(\mathcal{M}))\big) \simeq \mathcal{H}^j\big(\mathrm{DR}_{W_0}^{\mathrm{an}}(f_D^*(\mathcal{M})\big),$$

where the last isomorphism follows from Theorem 7.34. Recall that $f_D^*(\mathcal{M}) \simeq \mathbf{D}_{W_0}\big(f^\dagger(\mathbf{D}_X(\mathcal{M}))\big)$. If $\dim(X) = n$, then W_0 is locally cut out in a suitable open subset of X by $n-d$ equations, hence

$$\mathcal{H}^i\big(f^\dagger(\mathbf{D}_X(\mathcal{M}))\big) \simeq L^{i+d-n}f^*\big(\mathbf{D}_X(\mathcal{M})\big) = 0 \quad \text{for all} \quad i < 0,$$

and thus $\mathcal{H}^i\big(f_D^*(\mathcal{M})\big) = 0$ for all $i > 0$ (see Remark 6.71). After possibly shrinking W_0, we may assume that $\mathcal{H}^i\big(f_D^*(\mathcal{M})\big)$ is $\mathcal{O}_{W_0}$-coherent for all i, in which case it follows from Lemma 7.38 below that $\mathcal{H}^i\big(\mathrm{DR}_{W_0}^{\mathrm{an}}(f_D^*(\mathcal{M})\big) = 0$ for all $i > -d$, hence $j \leq -d$. $\square$

LEMMA 7.38. *Let W be a d-dimensional smooth, irreducible, complex algebraic variety. If $u \in \mathcal{D}^b(\mathcal{D}_W)$ is such that $\mathcal{H}^i(u)$ is $\mathcal{O}_W$-coherent for every i and $m \in \mathbf{Z}$ is such that $\mathcal{H}^i(u) = 0$ for all $i > m$, then $\mathcal{H}^i\big(\mathrm{DR}_X^{\mathrm{an}}(u)\big) = 0$ for all $i > m - d$.*

PROOF. Let q be such that $\mathcal{H}^i(u) = 0$ for $i \notin [m-q, m]$ and we argue by induction on q. If $q = 0$, then $u \simeq \mathcal{M}[-m]$ for an $\mathcal{O}_W$-coherent $\mathcal{D}_W$-module $\mathcal{M}$, in which case $\mathrm{DR}_X^{\mathrm{an}}(u) \simeq \mathcal{L}[d-m]$, for a local system $\mathcal{L}$, and the assertion is clear. For the induction step, we use the exact triangle

$$u' \to u \to u'' \to u'[1],$$

where $u' = \tau^{\leq m-1}(u)$ and $u'' = \tau^{\geq m}(u) \simeq \mathcal{H}^m(u)[-m]$ (see Example A.20). We can apply the induction hypothesis for u' to conclude that $\mathcal{H}^i\big(\mathrm{DR}_X^{\mathrm{an}}(u')\big) = 0$ for all $i > m-1-d$, while $\mathrm{DR}_X^{\mathrm{an}}(u'') \simeq \mathcal{L}[d-m]$ for a local system $\mathcal{L}$. We conclude that $\mathcal{H}^i\big(\mathrm{DR}_X^{\mathrm{an}}(u)\big) = 0$ for all $i > m-d$ using the exact sequence

$$0 = \mathcal{H}^i\big(\mathrm{DR}_X^{\mathrm{an}}(u')\big) \to \mathcal{H}^i\big(\mathrm{DR}_X^{\mathrm{an}}(u)\big) \to \mathcal{H}^i\big(\mathrm{DR}_X^{\mathrm{an}}(u'')\big) = 0.$$

$\square$

CHAPTER 8

V-filtrations and Bernstein-Sato polynomials

In this chapter we discuss the notion of V-filtration, due to Malgrange and Kashiwara, and its connection with b-functions. We also discuss in more detail the Bernstein-Sato polynomial of a regular function and prove, in particular, the rationality of its roots, following [**Kas77**] and [**Lic89**].

We go back to the assumption that the ground field k is an arbitrary algebraically closed field of characteristic 0.

8.1. The V-filtration with respect to a smooth hypersurface

We begin by discussing the V-filtration with respect to a *smooth* hypersurface, following [**Sai88**, Chapter 3.1]. Let X be a smooth, irreducible, algebraic variety over k and let $t \in \mathcal{O}_X(X)$ be nonzero, defining a smooth, irreducible hypersurface H. We put $\dim(X) = n+1$, so $\dim(H) = n$.

For every $m \in \mathbf{Z}$, we put

$$V^m\mathcal{D}_X = \{P \in \mathcal{D}_X \mid P \cdot (t)^q \subseteq (t)^{q+m} \text{ for all } q \in \mathbf{Z}\}$$

(with the convention that $(t)^i = \mathcal{O}_X$ if $i \leq 0$). It follows from the definition that we have $V^{m_1}\mathcal{D}_X \cdot V^{m_2}\mathcal{D}_X \subseteq V^{m_1+m_2}\mathcal{D}_X$ for all $m_1, m_2 \in \mathbf{Z}$.

Remark 8.1. Note that the decreasing filtration $V^\bullet\mathcal{D}_X$ only depends on H, but not on the function t defining H.

It is easy to describe $V^m\mathcal{D}_X$ locally. Note first that if $U \subseteq X$ is an open subset such that $H \cap U = \emptyset$, then $V^m\mathcal{D}_X|_U = \mathcal{D}_U$ for all $m \in \mathbf{Z}$. On the other hand, if U is an affine open subset with algebraic coordinates $x_1, \ldots, x_n, t$, then given $P \in \Gamma(U, \mathcal{D}_X)$, we write $P = \sum_{\alpha,j} P_{\alpha,j}\partial_x^\alpha\partial_t^j$, where the sum is over $\alpha \in \mathbf{Z}_{\geq 0}^n$ and $j \in \mathbf{Z}_{\geq 0}$, and all $P_{\alpha,j}$ lie in $\mathcal{O}_X(U)$. In this case, it is straightforward to see that $P \in \Gamma(U, V^m\mathcal{D}_X)$ if and only if $P_{\alpha,j} \in (t^{m+j})$ for all α and j.

Given coordinates, as above, it is convenient to also describe $V^m\mathcal{D}_X$ in terms of the Euler operator $t\partial_t$. In fact, for reasons that will become clearer later, we will use instead $s := -\partial_t t = -t\partial_t - 1$. For future reference, we collect in the next lemma some easy identities involving s, t, and ∂_t.

Lemma 8.2. *The operator $s = -\partial_t t$ satisfies the following properties:*

i) *For every $P \in k[s]$ and every $m \geq 0$, we have $P(s)t^m = t^m P(s-m)$ and $P(s)\partial_t^m = \partial_t^m P(s+m)$.*

ii) *For every $m \in \mathbf{Z}_{>0}$, we have*

$$t^m\partial_t^m = (-1)^m \prod_{j=1}^{m}(s+j) \quad \textit{and} \quad \partial_t^m t^m = (-1)^m \prod_{j=0}^{m-1}(s-j).$$

PROOF. In order to prove the first formula in i), we may and will assume that $P = s^j$. Arguing by induction on j, we see that it is enough to prove the case $j = 1$. In order to prove this, we compute

$$[s, t^m] = -\partial_t t^{m+1} + t^m \partial_t t = -[\partial_t, t^m]t = -mt^m,$$

where the last equality follows from Lemma 2.1. This proves the first assertion in i) and the proof of the second one is similar.

We prove the first assertion in ii) by induction on m, the case $m = 1$ being clear. Let us assume that we know the assertion for m. Using this and the first formula in i), we obtain

$$\begin{aligned} t^{m+1}\partial_t^{m+1} &= (-1)^m t(s+1)\cdots(s+m)\partial_t = (-1)^m (s+2)\cdots(s+m+1)t\partial_t \\ &= (-1)^{m+1}\prod_{j=1}^{m+1}(s+j). \end{aligned}$$

The second formula in ii) follows from the first one by applying the automorphism of $A_1(k)$ that maps t to ∂_t and ∂_t to $-t$, hence maps s to $t\partial_t = -s-1$. ☐

Using the above lemma, we see that $t^i\partial_t^j$ lies in $\mathbf{C}[s]\cdot t^{i-j}$ if $i \geq j$ and it lies in $\mathbf{C}[s]\cdot\partial_t^{j-i}$ if $i \leq j$. We thus conclude that, given coordinates $x_1, \ldots, x_n, t$ as above,

$$V^0\mathcal{D}_U = \mathcal{O}_X\langle \partial_{x_1}, \ldots, \partial_{x_n}, s\rangle. \tag{8.1}$$

We also have

$$V^m\mathcal{D}_U = V^0\mathcal{D}_U\cdot t^m = t^m\cdot V^0\mathcal{D}_U \quad \text{for all} \quad m \geq 0 \quad \text{and} \tag{8.2}$$

$$V^{-m}\mathcal{D}_U = \sum_{i=0}^m V^0\mathcal{D}_U\cdot\partial_t^i = \sum_{i=0}^m \partial_t^i\cdot V^0\mathcal{D}_U \quad \text{for all} \quad m \geq 0. \tag{8.3}$$

REMARK 8.3. Note that all $V^i\mathcal{D}_X$ are quasi-coherent sheaves of $\mathcal{O}_X$-modules and for every affine open subset $U \subseteq X$, the ring $\Gamma(U, V^0\mathcal{D}_X)$ is both left and right Noetherian. In fact, if we consider the Rees-type sheaf of rings

$$\mathcal{R}_+(V^\bullet\mathcal{D}_X) := \bigoplus_{i\geq 0} V^i\mathcal{D}_X z^i,$$

then $\Gamma\big(U, \mathcal{R}_+(V^\bullet\mathcal{D}_X)\big)$ is both left and right Noetherian. Of course, it is enough to prove the assertion for the latter ring, the former ring being a quotient of this one. The assertion is straightforward to see if $U \cap H = \emptyset$. Otherwise, we may assume that we have coordinates $x_1, \ldots, x_n, t$ on U. In this case, if $F_pV^i\mathcal{D}_X := F_p\mathcal{D}_X \cap V^i\mathcal{D}_X$ for every $p \geq 0$ and if we put $F_p\mathcal{R}_+(V^\bullet\mathcal{D}_X) = \bigoplus_{i\geq 0} F_pV^i\mathcal{D}_X z^i$, then we get a filtration on $\mathcal{R}_+(V^\bullet\mathcal{D}_X)$ such that $\mathrm{Gr}_\bullet^F\big(\mathcal{R}_+(V^\bullet\mathcal{D}_X)\big)|_U$ is isomorphic to a polynomial algebra over $\mathcal{O}_U$ in $y_1, \ldots, y_n, s$, and tz; this follows from the description of $V^i\mathcal{D}_X$ in (8.1) and the fact that $st = t(s-1)$. The fact that $\Gamma\big(U, \mathcal{R}_+(V^\bullet\mathcal{D}_X)\big)$ is both left and right Noetherian now follows by arguing as in the proof of Corollary 2.22.

REMARK 8.4. Note that the operators ∂_t and $s = -\partial_t t$ depend on the choice of coordinates. More precisely, if we choose a different set of local coordinates $x_1', \ldots, x_n', t$ and write ∂_t' for the corresponding derivation with respect to this system of coordinates, we have $\partial_t' = \partial_t + \sum_{i=1}^n \partial_t'(x_i)\partial_{x_i}$, hence

$$\partial_t' - \partial_t \in V^0\mathcal{D}_X \quad \text{and} - \partial_t' t + \partial_t t \in V^1\mathcal{D}_X.$$

We also note that if we replace t by $t' = ut$, for some invertible function u, then a similar computation shows that, with respect to the same coordinates $x_1, \ldots, x_n$, we have $\partial_{t'} - \partial_t u^{-1} \in V^0\mathcal{D}_X$ and thus $\partial_{t'}t' - \partial_t t \in V^1\mathcal{D}_X$.

We now come to the definition of the V-filtration with respect to t. The role of the V-filtration is, roughly speaking, to put the operator $\partial_t t$ on a given $\mathcal{D}$-module in upper triangular form.

DEFINITION 8.5. A *weak*[1] *V-filtration* on a coherent $\mathcal{D}_X$-module $\mathcal{M}$ (with respect to $t \in \mathcal{O}_X(X)$) is a decreasing filtration $V^\bullet\mathcal{M} = (V^\alpha\mathcal{M})_{\alpha\in\mathbf{Q}}$ by quasi-coherent $\mathcal{O}_X$-modules, indexed by rational numbers, which is exhaustive[2], discrete and left-continuous[3], and satisfies the following properties:

i) We have $V^i\mathcal{D}_X \cdot V^\alpha\mathcal{M} \subseteq V^{\alpha+i}\mathcal{M}$ for every $\alpha \in \mathbf{Q}$ and $i \in \mathbf{Z}$.
ii) For every $\alpha \in \mathbf{Q}$, the operator $(s + \alpha)$ is nilpotent on $\mathrm{Gr}_V^\alpha(\mathcal{M})$.

We say that $V^\bullet\mathcal{M}$ is a *V-filtration* if it satisfies, in addition, the following two conditions:

iii) Each $V^\alpha\mathcal{M}$ is locally finitely generated over $V^0\mathcal{D}_X$.
iv) We have $t \cdot V^\alpha\mathcal{M} = V^{\alpha+1}\mathcal{M}$ for all $\alpha > 0$.

Note that in the above definition, we put $\mathrm{Gr}_V^\alpha(\mathcal{M}) := V^\alpha\mathcal{M}/V^{>\alpha}\mathcal{M}$, where

$$V^{>\alpha}\mathcal{M} = \bigcup_{\beta>\alpha} V^\beta\mathcal{M} = V^{\alpha+\epsilon}\mathcal{M} \quad \text{for} \quad 0 < \epsilon \ll 1.$$

By assumption, there is a positive integer ℓ such that $\mathrm{Gr}_V^\alpha(\mathcal{M}) = 0$ unless $\ell\alpha \in \mathbf{Z}$ (we will refer to this property by saying that the filtration is *discrete*).

REMARK 8.6. It is clear from the definition that if $V^\bullet\mathcal{M}$ is a (weak) V-filtration on $\mathcal{M}$ with respect to t, then for every open subset $U \subseteq X$, the restriction $V^\bullet\mathcal{M}|_U$ is a (weak) V-filtration on $\mathcal{M}|_U$ with respect to $t|_U$.

REMARK 8.7. If $V^\bullet\mathcal{M}$ is a weak V-filtration on $\mathcal{M}$, then it follows from condition i) in Definition 8.5 that for every $\alpha \in \mathbf{Q}$, the quotient $V^\alpha\mathcal{M}/V^{\alpha+1}\mathcal{M}$ is annihilated by (t); in fact, it is naturally a $\mathcal{D}_H$-module. In particular, each $\mathrm{Gr}_V^\alpha(\mathcal{M})$ is naturally a $\mathcal{D}_H$-module. Since the filtration is exhaustive and discrete, and $\mathrm{Gr}_V^\alpha(\mathcal{M})|_{X\smallsetminus H} = 0$ for all α, we see that $V^\alpha\mathcal{M}|_{X\smallsetminus H} = \mathcal{M}|_{X\smallsetminus H}$ for all $\alpha \in \mathbf{Q}$.

REMARK 8.8. Since $\mathrm{Gr}_V^\alpha(\mathcal{M})$ is supported on H, we only put the condition in ii) in Definition 8.5 at the points in H; note that the operator s is defined around these points. While this operator is only defined locally and depends on the choice of local coordinates, its action on $\mathrm{Gr}_V^\alpha(\mathcal{M})$ is well-defined: indeed, as we have noticed in Remark 8.4, if s' is the same operator corresponding to a different choice of coordinates, then $s - s' \in V^1\mathcal{D}_X$, hence the actions of s and s' on $\mathrm{Gr}_V^\alpha(\mathcal{M})$ agree by condition i).

Furthermore, it follows from the same remark that if we replace t by ut, for some invertible function u, and if we denote by s' the new operator, then the actions of s and s' on $\mathrm{Gr}_V^\alpha(\mathcal{M})$ again coincide. Therefore the notion of V-filtration on $\mathcal{M}$ only depends on H and it does not depend on the choice of function t.

[1]This is not standard terminology. We will only use it in a few remarks in this section, in order to emphasize the different roles of the conditions in the definition of a V-filtration.

[2]This means that $\cup_{\alpha\in\mathbf{Q}} V^\alpha\mathcal{M} = \mathcal{M}$.

[3]These conditions mean that there is a positive integer ℓ such that $V^\alpha\mathcal{M}$ takes a constant value for α in an interval of the form $\big(i/\ell, (i+1)/\ell\big]$, where i is any integer.

REMARK 8.9. In the presence of local coordinates $x_1, \ldots, x_n, t$, it follows from the formulas (8.1), (8.2) and (8.3) that the condition in i) in Definition 8.5 is equivalent to the fact that each $V^\alpha\mathcal{M}$ is preserved by the action of $\mathcal{O}_X\langle\partial_{x_1}, \ldots, \partial_{x_n}\rangle$ and

$$t \cdot V^\alpha\mathcal{M} \subseteq V^{\alpha+1}\mathcal{M} \quad \text{and} \quad \partial_t \cdot V^\alpha\mathcal{M} \subseteq V^{\alpha-1}\mathcal{M} \quad \text{for all} \quad \alpha \in \mathbf{Q}.$$

REMARK 8.10. If $V^\bullet\mathcal{M}$ is a V-filtration on $\mathcal{M}$, then it follows from condition iii) in the definition that each $\mathrm{Gr}_V^\alpha(\mathcal{M})$ is locally finitely generated over $\mathcal{D}_H[s]$. Since $(s+\alpha)$ is nilpotent by condition ii) in the definition, we conclude that $\mathrm{Gr}_V^\alpha(\mathcal{M})$ is a coherent $\mathcal{D}_H$-module.

PROPOSITION 8.11. If $V^\bullet\mathcal{M}$ is a weak V-filtration on the coherent $\mathcal{D}_X$-module $\mathcal{M}$, then for every $\alpha \neq 0$, the maps

$$\mathrm{Gr}_V^\alpha(\mathcal{M}) \xrightarrow{t\cdot} \mathrm{Gr}_V^{\alpha+1}(\mathcal{M}) \xrightarrow{\partial_t\cdot} \mathrm{Gr}_V^\alpha(\mathcal{M}) \tag{8.4}$$

are isomorphisms of $\mathcal{D}_H$-modules.

PROOF. Let ν_t and ν_{∂_t} be the first, respectively the second, map in (8.4). It is clear that they are morphisms of $\mathcal{D}_H$-modules, hence we only need to show that they are bijective. It follows from condition ii) in Definition 8.5 that since $\alpha \neq 0$, the composition $\nu_{\partial_t} \circ \nu_t$ is invertible, hence ν_t is injective and ν_{∂_t} is surjective. On the other hand, since $t\partial_t = \partial_t t - 1$ and $\alpha + 1 \neq 1$, it follows from condition ii) in Definition 8.5 that $\nu_t \circ \nu_{\partial_t}$ is invertible, hence ν_t is surjective and ν_{∂_t} is injective. Therefore both ν_t and ν_{∂_t} are bijective. $\square$

It follows from the above proposition that the interesting maps are the following morphisms of $\mathcal{D}_H$-modules:

$$\mathrm{can}\colon \mathrm{Gr}_V^1(\mathcal{M}) \xrightarrow{\partial_t\cdot} \mathrm{Gr}_V^0(\mathcal{M}) \quad \text{and}$$
$$\mathrm{Var}\colon \mathrm{Gr}_V^0(\mathcal{M}) \xrightarrow{t\cdot} \mathrm{Gr}_V^1(\mathcal{M})$$

(the notation for these two maps is justified by the connection to topological vanishing and nearby cycles, See Section 8.2.1 below).

COROLLARY 8.12. If $V^\bullet\mathcal{M}$ is a weak V-filtration on $\mathcal{M}$, then the following hold:

i) If $\alpha \leq 0$ and $u \in \mathcal{M}$ is such that $tu \in V^{\alpha+1}\mathcal{M}$, then $u \in V^\alpha\mathcal{M}$. In particular, we have
$$\{u \in \mathcal{M} \mid tu = 0\} \subseteq V^0\mathcal{M}.$$
ii) If $\alpha \leq 1$ and $v \in \mathcal{M}$ is such that $\partial_t v \in V^{\alpha-1}\mathcal{M}$, then $v \in V^\alpha\mathcal{M}$.

iii) For every positive integer m and every $\alpha \in \mathbf{Q}$ such that $-m \leq \alpha < -m+1$, we have

$$V^\alpha\mathcal{M} = \partial_t^m \cdot V^{\alpha+m}\mathcal{M} + \sum_{j=0}^{m-1} \partial_t^j \cdot V^0\mathcal{M} \quad \text{if} \quad -m \leq \alpha < -m+1. \tag{8.5}$$

PROOF. In the setting of i), since the V-filtration is exhaustive, it follows that there is $\beta \in \mathbf{Q}$ such that $u \in V^\beta\mathcal{M}$. If $\beta \geq \alpha$, then we are done. Otherwise, since $tu \in V^{>\beta+1}$ and $\beta + 1 < \alpha + 1 \leq 1$, it follows from Proposition 8.11 that $u \in V^{>\beta}\mathcal{M}$. Since the V-filtration is discrete, after repeating this finitely many steps, we conclude that $u \in V^\alpha\mathcal{M}$. The second assertion in i) follows by taking $\alpha = 0$.

The argument for ii) is similar: let γ be such that $v \in V^{\gamma}\mathcal{M}$. If $\gamma \geq \alpha$, then we are done. Otherwise, since $\partial_t v \in V^{>\gamma-1}\mathcal{M}$ and $\gamma - 1 < \alpha - 1 \leq 0$, it follows from Proposition 8.11 that $v \in V^{>\gamma}\mathcal{M}$. After repeating this finitely many steps, we conclude that $v \in V^{\alpha}\mathcal{M}$.

For the assertion in iii), we only need to prove the inclusion "$\subseteq$". Since $\alpha < 0$, for every $w \in V^{\alpha}\mathcal{M}$, it follows from Proposition 8.11 that we can write $w = \partial_t w' + w''$, where $w' \in V^{\alpha+1}\mathcal{M}$ and $w'' \in V^{\beta}\mathcal{M}$, for some $\beta > \alpha$. Since the filtration is discrete, we can iterate this argument to see that $w \in \partial_t \cdot V^{\alpha+1}\mathcal{M} + V^{\alpha+1}\mathcal{M}$ if $\alpha \leq -1$ and $w \in \partial_t \cdot V^{\alpha+1}\mathcal{M} + V^{0}\mathcal{M}$ if $-1 < \alpha < 0$. The formula in (8.5) then follows by an easy induction on m. $\square$

Note that the formula in (8.5) gives an explicit description of $V^{\alpha}\mathcal{M}$ for $\alpha < 0$ in terms of $V^{\beta}\mathcal{M}$, for $\beta \in [0, 1)$. In the next proposition we discuss the difference between V-filtrations and their weak version.

PROPOSITION 8.13. If $V^{\bullet}\mathcal{M}$ is a weak filtration on the coherent $\mathcal{D}_X$-module $\mathcal{M}$, then the following are equivalent:

a) $V^{\bullet}\mathcal{M}$ is a V-filtration.
b) The $V^0\mathcal{D}_X$-module $V^0\mathcal{M}$ is locally finitely generated and $t \cdot V^i\mathcal{M} = V^{i+1}\mathcal{M}$ for $i \in \mathbf{Z}$, $i \gg 0$.
c) The $\mathcal{R}_+(V^{\bullet}\mathcal{D}_X)$-module $\mathcal{R}_+(V^{\bullet}\mathcal{M}) := \bigoplus_{i \geq 0} V^i\mathcal{M}z^i$ is locally finitely generated.

PROOF. Since $\mathcal{R}_+(V^{\bullet}\mathcal{D}_X) = \bigoplus_{i \geq 0} V^0\mathcal{D}_X t^i z^i$, it is easy to see that the condition in c) is equivalent to the fact that $V^i\mathcal{M}$ is locally finitely generated over $V^0\mathcal{D}_X$ for all $i \in \mathbf{Z}_{\geq 0}$ and $V^{i+1}\mathcal{M} = t \cdot V^i\mathcal{M}$ for all $i \in \mathbf{Z}$, with $i \gg 0$. In order to prove that the 3 assertions in the statement are equivalent, it is thus enough to show the following two facts:

i) If $V^0\mathcal{M}$ is locally finitely generated over $V^0\mathcal{D}_X$, then all $V^{\alpha}\mathcal{M}$ are finitely generated over $V^0\mathcal{D}_X$.
ii) If $V^{\alpha+1}\mathcal{M} = t \cdot V^{\alpha}\mathcal{M}$ for all $\alpha \in \mathbf{Z}$, with $\alpha \gg 0$, then the same assertion holds for all $\alpha > 0$.

The assertion in i) follows from the fact that $V^0\mathcal{D}_X$ has Noetherian section rings over affine open subsets (see Remark 8.3) and the fact that $V^{\alpha}\mathcal{M} \subseteq V^0\mathcal{M}$ for $\alpha \geq 0$, while $V^{\alpha}\mathcal{M}$ for $\alpha < 0$ is given by (8.5). For the assertion in ii), note that it follows from Proposition 8.11 that if $u \in V^{\alpha+1}\mathcal{M}$, with $\alpha > 0$, then $u \in t \cdot V^{\alpha}\mathcal{M} + V^{>\alpha+1}\mathcal{M}$. Since the filtration is discrete, we can repeat this to conclude that for every $\beta > \alpha$, we have $u \in t \cdot V^{\alpha}\mathcal{M} + V^{\beta}\mathcal{M}$. If we take $\beta \geq \alpha + 1$ to be an integer such that $V^{\beta}\mathcal{M} = t \cdot V^{\beta-1}\mathcal{M}$, we conclude that $u \in t \cdot V^{\alpha}\mathcal{M} + t \cdot V^{\beta-1}\mathcal{M} \subseteq t \cdot V^{\alpha}\mathcal{M}$. $\square$

REMARK 8.14. For future reference, we note that if $V^{\bullet}\mathcal{M}$ is a V-filtration on $\mathcal{M}$ and $\mathcal{M}_0 \subseteq \mathcal{M}$ is a coherent $\mathcal{O}_X$-submodule such that $\mathcal{D}_X \cdot \mathcal{M}_0 = \mathcal{M}$, then there is $q \in \mathbf{Z}_{>0}$ such that

$$V^{i+q}\mathcal{D}_X \cdot \mathcal{M}_0 \subseteq V^i\mathcal{M} \subseteq V^{i-q}\mathcal{D}_X \cdot \mathcal{M}_0 \quad \text{for} \quad i \in \mathbf{Z}, i > 1.$$

Indeed, since $\mathcal{M}_0$ is $\mathcal{O}_X$-coherent and $V^{\bullet}\mathcal{M}$ is exhaustive, there is $q \in \mathbf{Z}_{>0}$ such that $\mathcal{M}_0 \subseteq V^{-q}\mathcal{M}$, and thus

$$V^{i+q}\mathcal{D}_X \cdot \mathcal{M}_0 \subseteq V^{i+q}\mathcal{D}_X \cdot V^{-q}\mathcal{M} \subseteq V^i\mathcal{M} \quad \text{for all} \quad i \in \mathbf{Z}.$$

Since $\mathcal{D}_X \cdot \mathcal{M}_0 = \mathcal{M}$ and $V^1\mathcal{M}$ is locally finitely generated over $V^0\mathcal{D}_X$, it follows that after possibly increasing q, we may assume that $V^1\mathcal{M} \subseteq V^{1-q}\mathcal{D}_X \cdot \mathcal{M}_0$, in which case for every $i \in \mathbf{Z}$, with $i > 1$, we have

$$V^i\mathcal{M} = t^{i-1} \cdot V^1\mathcal{M} \subseteq t^{i-1} \cdot V^{1-q}\mathcal{D}_X \cdot \mathcal{M}_0 \subseteq V^{i-q}\mathcal{D}_X \cdot \mathcal{M}_0.$$

PROPOSITION 8.15. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ admits at most one V-filtration with respect to t.

PROOF. Suppose that $V^\bullet\mathcal{M}$ and $\widetilde{V}^\bullet\mathcal{M}$ are two V-filtrations on $\mathcal{M}$. By symmetry, it is enough to show that for every $\alpha \in \mathbf{Q}$, we have

$$V^\alpha\mathcal{M} \subseteq \widetilde{V}^\alpha\mathcal{M} \quad \text{for all} \quad \alpha \in \mathbf{Q}. \tag{8.6}$$

This is clear on $X \smallsetminus H$, hence we only need to check it around the points in H. Note first that

$$\frac{V^\beta\mathcal{M} \cap \widetilde{V}^\gamma\mathcal{M}}{(V^{>\beta}\mathcal{M} \cap \widetilde{V}^\gamma\mathcal{M}) + (V^\beta\mathcal{M} \cap \widetilde{V}^{>\gamma}\mathcal{M})} \tag{8.7}$$

is a subquotient of both $\mathrm{Gr}_V^\beta(\mathcal{M})$ and $\mathrm{Gr}_{\widetilde{V}}^\gamma(\mathcal{M})$. Property iv) in Definition 8.5 thus implies that both $s+\beta$ and $s+\gamma$ are nilpotent on (8.7), and thus (8.7) is 0 for all $\beta, \gamma \in \mathbf{Q}$, with $\beta \neq \gamma$.

Let us fix now $\alpha \in \mathbf{Q}$. Note first that since the filtration $\widetilde{V}^\bullet\mathcal{M}$ is an exhaustive filtration by $V^0\mathcal{D}_X$-submodules and $V^\alpha\mathcal{M}$ is locally finitely generated over $V^0\mathcal{D}_X$, there is $\gamma \in \mathbf{Q}$ such that

$$V^\alpha\mathcal{M} \subseteq \widetilde{V}^\gamma\mathcal{M}. \tag{8.8}$$

If $\gamma \geq \alpha$, then (8.6) holds and we are done, hence we may and will assume that $\gamma < \alpha$ and that γ is maximal such that (8.8) holds, aiming for a contradiction (we may assume this since the filtration $\widetilde{V}^\bullet\mathcal{M}$ is discrete and left continuous). On the other hand, since $V^\bullet\mathcal{M}$ satisfies property iii), it follows that there is $m_0 \geq 0$ such that for every $m \in \mathbf{Z}_{>0}$, we have

$$V^{\alpha+m_0+m}\mathcal{M} \subseteq t^m \cdot V^{\alpha+m_0}\mathcal{M} \subseteq t^m \cdot \widetilde{V}^\gamma\mathcal{M} \subseteq \widetilde{V}^{>\gamma}\mathcal{M}.$$

Since $V^\alpha\mathcal{M} \not\subseteq \widetilde{V}^{>\gamma}\mathcal{M}$ and the filtration $V^\bullet\mathcal{M}$ is discrete and left continuous, it follows that there is $\beta \geq \alpha$ such that

$$V^\beta\mathcal{M} \not\subseteq \widetilde{V}^{>\gamma}\mathcal{M} \quad \text{and} \tag{8.9}$$

$$V^{>\beta}\mathcal{M} \subseteq \widetilde{V}^{>\gamma}\mathcal{M}. \tag{8.10}$$

Note that $\gamma < \beta$, hence the quotient (8.7) for β and γ is 0, and therefore

$$V^\beta\mathcal{M} \subseteq V^\beta\mathcal{M} \cap \widetilde{V}^\gamma\mathcal{M} \subseteq (V^{>\beta}\mathcal{M} \cap \widetilde{V}^\gamma\mathcal{M}) + (V^\beta\mathcal{M} \cap V^{>\gamma}\mathcal{M}) \subseteq V^{>\gamma}\mathcal{M},$$

where the first inclusion follows from (8.8), since $\beta \geq \alpha$, and the last inclusion follows from (8.10). However, this contradicts (8.9), and thus completes the proof of the proposition. $\square$

REMARK 8.16. The proof of Proposition 8.15 shows that if $V^\bullet\mathcal{M}$ is a V-filtration on $\mathcal{M}$ and $\widetilde{V}^\bullet\mathcal{M}$ is a weak V-filtration on $\mathcal{M}$, then $V^\alpha\mathcal{M} \subseteq \widetilde{V}^\alpha\mathcal{M}$ for all $\alpha \in \mathbf{Q}$.

COROLLARY 8.17. Given a coherent $\mathcal{D}_X$-module $\mathcal{M}$ and an open cover $X = \bigcup_i U_i$, then $\mathcal{M}$ has a V-filtration with respect to t if and only if the same holds for each $\mathcal{M}|_{U_i}$ with respect to $t|_{U_i}$.

PROOF. Given a filtration $V^\bullet\mathcal{M}$ on $\mathcal{M}$ it is clear that this is a V-filtration if and only if it induces a V-filtration on each U_i. The assertion in the corollary now follows from the fact that by the proposition, given V-filtrations for each $\mathcal{M}|_{U_i}$, these glue to a filtration on $\mathcal{M}$. $\square$

EXAMPLE 8.18. If $\mathcal{M}$ is a $\mathcal{D}_X$-module that is coherent as an $\mathcal{O}_X$-module, then $\mathcal{M}$ has a V-filtration given by $V^\alpha\mathcal{M} = (t)^{\lceil\alpha\rceil-1}\cdot\mathcal{M}$ for all $\alpha \in \mathbf{Q}$, with the convention that $(t)^m = \mathcal{O}_X$ if $m \leq 0$. It is straightforward to check that the conditions in Definition 8.5 are satisfied. We only note that each $V^\alpha\mathcal{M}$ is clearly locally finitely generated over $V^0\mathcal{D}_X$ since it is coherent as an $\mathcal{O}_X$-module. Also, condition ii) in the definition holds since if $m \in \mathbf{Z}_{>0}$, then $(\partial_t t - m)t^{m-1}u = t^m\partial_t u \in V^{m+1}\mathcal{M}$, hence $(s+m)\cdot \mathrm{Gr}_V^m(\mathcal{M}) = 0$.

EXAMPLE 8.19. If $\mathcal{N}$ is a coherent $\mathcal{D}_X$-module supported on H, then $\mathcal{N}$ has a V-filtration such that $V^\alpha\mathcal{N} = 0$ for all $\alpha > 0$ and $\mathrm{Gr}_V^\alpha(\mathcal{N}) = 0$ for all $\alpha \in \mathbf{Q}_{<0} \smallsetminus \mathbf{Z}$. Indeed, note first that by Corollary 8.17, in order to check this, we may and will assume that X is affine and we have coordinates $x_1, \ldots, x_n, t$ on X such that H is defined by (t). Recall that we have seen in the proof of Theorem 6.20 that if $\mathcal{N}_i = \{u \in \mathcal{N} \mid (s+i)u = 0\}$, then the following hold:

i) $t\cdot\mathcal{N}_i \subseteq \mathcal{N}_{i+1}$ and $\partial_t\cdot\mathcal{N}_i \subseteq \mathcal{N}_{i-1}$ for all $i \in \mathbf{Z}$.
ii) $\mathcal{N}_i = 0$ for $i > 0$ and $\mathcal{N} = \bigoplus_{i\leq 0}\mathcal{N}_i$.
iii) $\mathcal{N}_{-m} = \partial_t^m\cdot\mathcal{N}_0$ for all $m \geq 0$.

Furthermore, since $\mathcal{N}$ is a coherent $\mathcal{D}_X$-module, it follows that $\mathcal{N}_0$ is a coherent $\mathcal{D}_H$-module, and thus by iii) above, we see that every $\bigoplus_{i=0}^m\mathcal{N}_{-i}$ is a finitely generated $V^0\mathcal{D}_X$-module for every $m \geq 0$. It is then clear that we get a V-filtration on $\mathcal{N}$ by putting

$$V^\alpha\mathcal{N} = \bigoplus_{i=0}^{-\lceil\alpha\rceil}\mathcal{N}_{-i} \quad \text{for} \quad \alpha \leq 0$$

and $V^\alpha\mathcal{N} = 0$ for $\alpha > 0$.

PROPOSITION 8.20. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module that has a V-filtration $V^\bullet\mathcal{M}$ with respect to t.

i) For every $\alpha > 0$, the map $V^\alpha\mathcal{M} \xrightarrow{t\cdot} V^{\alpha+1}\mathcal{M}$ is bijective.
ii) If $\mathcal{M}'$ is a $\mathcal{D}_X$-submodule of $\mathcal{M}$ and $p\colon \mathcal{M} \to \mathcal{M}'' = \mathcal{M}/\mathcal{M}'$ is the canonical projection, then $\mathcal{M}'$ and $\mathcal{M}''$ have V-filtrations with respect to t given by

$$V^\alpha\mathcal{M}' = V^\alpha\mathcal{M}\cap\mathcal{M}' \quad \text{and} \quad V^\alpha\mathcal{M}'' = p(V^\alpha\mathcal{M}) \quad \text{for all} \quad \alpha \in \mathbf{Q}. \tag{8.11}$$

PROOF. Suppose first that we are in the setting of ii) and we define the filtrations $V^\bullet\mathcal{M}'$ and $V^\bullet\mathcal{M}''$ as in (8.11). Note that for every $\alpha \in \mathbf{Q}$, we have a short exact sequence

$$0 \to \mathrm{Gr}_V^\alpha(\mathcal{M}') \to \mathrm{Gr}_V^\alpha(\mathcal{M}) \to \mathrm{Gr}_V^\alpha(\mathcal{M}'') \to 0.$$

It is straightforward to check that $V^\bullet\mathcal{M}''$ is a V-filtration on $\mathcal{M}''$. It is also clear that $V^\bullet\mathcal{M}'$ satisfies all the conditions to make it a V-filtration on $\mathcal{M}'$ with the exception of condition iv) in Definition 8.5 (the fact that each $V^\alpha\mathcal{M}'$ is locally finitely generated over $V^0\mathcal{D}_X$ follows from the fact that $V^\alpha\mathcal{M}$ has this property and Remark 8.3). However, this condition also follows if we know that if $u\in V^\alpha\mathcal{M}$, with $\alpha>0$, is such that $tu\subseteq\mathcal{M}'$, then $u\in\mathcal{M}'$; in other words, it is enough to know that $\mathcal{M}''$ satisfies assertion i) in the proposition.

We now prove that every $\mathcal{D}_X$-module $\mathcal{M}$ that has a V-filtration satisfies the assertion in i). We consider the short exact sequence

$$0\to\mathcal{M}_1\to\mathcal{M}\to\mathcal{M}_2\to 0,$$

such that $\mathcal{M}_1=\Gamma_H(\mathcal{M})$. Note that in this case it is clear that if $tu\in\mathcal{M}_1$, then $u\in\mathcal{M}_1$. The previous discussion thus implies that the V-filtration on $\mathcal{M}$ induces V-filtrations on $\mathcal{M}_1$ and $\mathcal{M}_2$. On the other hand, since $\mathcal{M}_1$ is supported on H, it follows from the uniqueness of the V-filtration and Example 8.19 that $V^\alpha\mathcal{M}\cap\mathcal{M}_1=0$ for all $\alpha>0$. Therefore $V^\alpha\mathcal{M}\simeq V^\alpha\mathcal{M}_2$ for all $\alpha>0$, and since multiplication by t is injective on $\mathcal{M}_2$, we see that multiplication by t is injective on $V^\alpha\mathcal{M}$ for all $\alpha>0$. This completes the proof of the proposition. $\square$

Corollary 8.21. It $\varphi\colon\mathcal{M}\to\mathcal{N}$ is a morphism of coherent $\mathcal{D}_X$-modules such that both $\mathcal{M}$ and $\mathcal{N}$ have V-filtrations with respect to t, then φ is a filtered morphism and it is strict[4]. In particular, the category of coherent $\mathcal{D}_X$-modules that carry a V-filtration with respect to t is an Abelian category.

Proof. It follows from assertion ii) in Proposition 8.20 that if we put

$$V^\alpha\varphi(\mathcal{M})=\varphi(V^\alpha\mathcal{M})\quad\text{and}\quad\widetilde{V}^\alpha\varphi(\mathcal{M})=V^\alpha\mathcal{N}\cap\varphi(\mathcal{M}),$$

then both these give a V-filtration on $\varphi(\mathcal{M})$. Therefore they agree by uniqueness of the V-filtration, hence φ is a filtered morphism and it is strict. The last assertion in the corollary is an immediate consequence of this. $\square$

The following proposition describes the behavior of V-filtration with respect to $\mathcal{D}$-module push-forward via closed immersions.

Proposition 8.22. Let $i\colon Z\hookrightarrow X$ be the inclusion map of a smooth, irreducible, closed subvariety such that $t|_Z$ defines a smooth, irreducible hypersurface in Z. If $\mathcal{N}$ is a coherent $\mathcal{D}_Z$-module and $\mathcal{M}=i_+(\mathcal{N})$, then $\mathcal{N}$ has a V-filtration with respect to $t|_Z$ if and only if $\mathcal{M}$ has a V-filtration with respect to t. Moreover, in this case, if $x_1,\ldots,x_m,y_1,\ldots,y_r,t$ are local coordinates on X such that Z is defined by $(y_1,\ldots,y_r)$, so $\mathcal{M}\simeq\mathcal{N}\otimes_k k[\partial_{y_1},\ldots,\partial_{y_r}]$, then

$$V^\alpha\mathcal{M}\simeq V^\alpha\mathcal{N}\otimes_k k[\partial_{y_1},\ldots,\partial_{y_r}]\quad\text{for all}\quad\alpha\in\mathbf{Q}.$$

Proof. If $\mathcal{M}$ has a V-filtration with respect to t, then by assumption each $V^\alpha\mathcal{M}$ is preserved by the action of $y_i\partial_{y_i}$ for all i. This implies that if $\sum_\beta u_\beta\otimes\partial_y^\beta\in V^\alpha\mathcal{M}$, then $u_\beta\otimes\partial_y^\beta\in V^\alpha\mathcal{M}$ for all β. It is then straightforward to see that if we put $V^\alpha\mathcal{N}=\{u\mid u\otimes 1\in V^\alpha\mathcal{M}\}$ for all α, then $V^\bullet\mathcal{N}$ is a V-filtration on $\mathcal{N}$ with respect to $t|_Z$. Conversely, if $\mathcal{N}$ has a V-filtration with respect to $t|_Z$ and we put $V^\alpha\mathcal{M}=V^\alpha\mathcal{N}\otimes_k k[\partial_{y_1},\ldots,\partial_{y_r}]$, it is straightforward to see that this is a V-filtration of $\mathcal{M}$ with respect to t. $\square$

[4]The fact that φ is a filtered morphism means that $\varphi(V^\alpha\mathcal{M})\subseteq V^\alpha\mathcal{N}$ for all $\alpha\in\mathbf{Q}$. The fact that it is strict means that, in addition, the filtration on $\varphi(\mathcal{M})$ induced from $\mathcal{M}$ is the same as the one induced from $\mathcal{N}$, that is, $\varphi(V^\alpha\mathcal{M})=\varphi(\mathcal{M})\cap V^\alpha\mathcal{N}$ for all $\alpha\in\mathbf{Q}$.

We end this section by discussing in more detail the two maps Var and can.

PROPOSITION 8.23. If $\mathcal{M}$ has a V-filtration with respect to t and $i\colon H \hookrightarrow X$ is the inclusion, then $i^{\dagger}(\mathcal{M})$ is computed by the complex

$$\mathrm{Gr}_V^0(\mathcal{M}) \xrightarrow{t\cdot} \mathrm{Gr}_V^1(\mathcal{M}),$$

placed in cohomological degrees 0 and 1. In particular, we have

$$\mathcal{H}^0_H(\mathcal{M}) \simeq i_+\big(\mathrm{Ker}(\mathrm{Var})\big) \quad \text{and} \quad \mathcal{H}^1_H(\mathcal{M}) \simeq i_+\big(\mathrm{Coker}(\mathrm{Var})\big).$$

PROOF. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & V^{>0}\mathcal{M} \arrow[r] \arrow[d, "t\cdot"'] & V^0\mathcal{M} \arrow[r] \arrow[d, "t\cdot"'] & \mathrm{Gr}_V^0(\mathcal{M}) \arrow[r] \arrow[d, "t\cdot"'] & 0 \\ 0 \arrow[r] & V^{>1}\mathcal{M} \arrow[r] & V^1\mathcal{M} \arrow[r] & \mathrm{Gr}_V^1(\mathcal{M}) \arrow[r] & 0. \end{tikzcd}$$

Since the first vertical map is an isomorphism by Proposition 8.20i), it follows that the second and third columns are quasi-isomorphic. Therefore the first assertion in the proposition follows if we show that the inclusion of complexes

$$\begin{tikzcd} V^0\mathcal{M} \arrow[r] \arrow[d, "t\cdot"'] & \mathcal{M} \arrow[d, "t\cdot"] \\ V^1\mathcal{M} \arrow[r] & \mathcal{M} \end{tikzcd} \tag{8.12}$$

is an isomorphism.

We first show that the induced morphism $\sigma\colon V^1\mathcal{M}/t\cdot V^0\mathcal{M} \to \mathcal{M}/t\mathcal{M}$ is an isomorphism. It follows from Proposition 8.11 that if $u \in V^{\alpha}\mathcal{M}$, with $\alpha < 1$, then there is $u' \in V^{>\alpha}$ such that $u - u' \in t\mathcal{M}$. Since the V-filtration is discrete, after iterating this finitely many times, we see that $\overline{u} \in \mathcal{M}/t\mathcal{M}$ lies in the image of σ. In order to prove that σ is injective, consider $u \in V^1\mathcal{M} \cap t\mathcal{M}$. If we write $u = tv$, then it follows from Corollary 8.12i) that $v \in V^0\mathcal{M}$, hence $u \in t \cdot V^0\mathcal{M}$. We have thus proved that the induced morphism between the cokernels of the vertical maps in (8.12) is an isomorphism.

We next show that the induced map

$$\{u \in V^0\mathcal{M} \mid tu = 0\} \to \{u \in \mathcal{M} \mid tu = 0\} \tag{8.13}$$

is an isomorphism. This is clearly injective and surjectivity follows from Corollary 8.12i)

The last assertion in the proposition follows directly from the first one and the isomorphism $i_+i^{\dagger}(\mathcal{M}) \simeq \mathbf{R}\Gamma_H(\mathcal{M})$ (see Example 6.62). $\square$

COROLLARY 8.24. If $V^{\bullet}\mathcal{M}$ is a V-filtration on $\mathcal{M}$ and $\mathcal{M}$ has no t-torsion, then for every $u \in \mathcal{M}$ and every $\alpha \in \mathbf{Q}$, we have $u \in V^{\alpha}\mathcal{M}$ if and only if $tu \in V^{\alpha+1}\mathcal{M}$. In particular, if the action of t on $\mathcal{M}$ is invertible, then $V^{\alpha+1}\mathcal{M} = t \cdot V^{\alpha}\mathcal{M}$ for all $\alpha \in \mathbf{Q}$.

PROOF. For every $\beta \in \mathbf{Q}$, the map

$$\mathrm{Gr}_V^{\beta}(\mathcal{M}) \xrightarrow{t\cdot} \mathrm{Gr}_V^{\beta+1}(\mathcal{M})$$

is injective. Indeed, for $\beta \neq 0$ this follows from Proposition 8.11, while for $\beta = 0$ it follows from the fact that $\mathcal{M}$ has no t-torsion by Proposition 8.23. This implies that if $u \in V^\beta\mathcal{M}$ is such that $tu \in V^{\alpha+1}\mathcal{M}$ for some $\alpha > \beta$, then $u \in V^{>\beta}$. We thus get the first assertion in the corollary using the fact that the V-filtration is discrete. The last assertion in the statement is an immediate consequence. $\square$

PROPOSITION 8.25. If the coherent $\mathcal{D}_X$-module $\mathcal{M}$ has a V-filtration with respect to t and $i\colon H \hookrightarrow X$ is the inclusion, then the following hold:

i) The $\mathcal{D}_X$-submodule $\mathcal{M}' := \mathcal{D}_X \cdot V^{>0}\mathcal{M}$ is the smallest $\mathcal{D}_X$-submodule of $\mathcal{M}$ with the property that $\mathcal{M}/\mathcal{M}'$ is supported on H.

ii) We have an isomorphism of $\mathcal{D}_H$-modules

$$\mathcal{M}/\mathcal{M}' \simeq i_+\mathrm{Coker}\big(\mathrm{Gr}_V^1(\mathcal{M}) \xrightarrow{\partial_t\cdot} \mathrm{Gr}_V^0(\mathcal{M})\big).$$

PROOF. Given any $u \in \mathcal{M}$, if $u \in V^\alpha\mathcal{M}$ and $m + \alpha > 0$, then $t^m u \in V^{>0}\mathcal{M}$. Therefore $\mathcal{M}/\mathcal{M}'$ is supported on H. On the other hand, if $\mathcal{N}$ is any $\mathcal{D}_X$-submodule of $\mathcal{M}$ such that $\mathcal{M}/\mathcal{N}$ is supported on H, then it follows from Example 8.19 and Proposition 8.20 that $V^{>0}\mathcal{M} = V^{>0}\mathcal{N} \subseteq \mathcal{N}$, hence $\mathcal{M}' \subseteq \mathcal{N}$.

In order to prove the isomorphism in ii), note first that since $\mathcal{M}/\mathcal{M}'$ is supported on H, it follows from Propositions 6.24 and Example 8.19 that

$$\mathcal{M}/\mathcal{M}' \simeq i_+i^\dagger(\mathcal{M}/\mathcal{M}') \simeq i_+\mathrm{Gr}_V^0(\mathcal{M}/\mathcal{M}') \simeq i_+\big(V^0\mathcal{M}/(V^{>0}\mathcal{M} + (V^0\mathcal{M}\cap\mathcal{M}'))\big).$$

The assertion in ii) thus follows if we show that

$$V^{>0}\mathcal{M} + (V^0\mathcal{M}\cap\mathcal{M}') = V^{>0}\mathcal{M} + \partial_t \cdot V^1\mathcal{M} \tag{8.14}$$

(we may and will assume that we are in an open subset of X where we have coordinates $x_1, \ldots, x_n, t$ on X, so that we have the operator ∂_t acting on $\mathcal{M}$). The inclusion "$\supseteq$" in (8.14) is clear, hence we only need to prove the reverse inclusion. Note that since $V^{>0}\mathcal{M}$ is a $V^0\mathcal{D}_X$-submodule of $\mathcal{M}$, it follows that

$$\mathcal{M}' = \sum_{m\geq 0}\partial_t^m \cdot V^{>0}\mathcal{M}.$$

Suppose now that $u \in V^0\mathcal{M}\cap\mathcal{M}'$. We can write $u = \sum_{i=0}^N \partial_t^i u_i$, with all $u_i \in V^{>0}\mathcal{M}$. If $w = \sum_{i=1}^N \partial_t^{i-1}u_i$, then $\partial_t w = u - u_0 \in V^0\mathcal{M}$, hence $w \in V^1\mathcal{M}$ by Corollary 8.12ii). Therefore $u = u_0 + \partial_t w \in V^{>0}\mathcal{M} + \partial_t \cdot V^1\mathcal{M}$, which completes the proof of the proposition. $\square$

REMARK 8.26. If $j\colon U = X \smallsetminus H \hookrightarrow X$ is the inclusion, $\mathcal{N}$ is a holonomic $\mathcal{D}_U$-module, and $\mathcal{M} = j_{D*}(\mathcal{N})$ has a V-filtration (note that $\mathcal{H}^i\big(j_{D*}(\mathcal{N})\big) = 0$ for $i \neq 0$ since U is the complement of a hypersurface), then it follows from Theorem 6.79 that the $\mathcal{D}_X$-submodule $\mathcal{M}'$ of $\mathcal{M}$ in Proposition 8.25 is $j_{D!*}(\mathcal{N})$.

8.2. The V-filtration with respect to an arbitrary function

Suppose now that X is a smooth, irreducible, n-dimensional variety, and $f \in \mathcal{O}_X(X)$ is nonzero, defining the (possibly singular) hypersurface H in X. The idea, due to Kashiwara, is to construct the V-filtration on the push-forward of a given $\mathcal{D}_X$-module via the graph embedding of f.

More precisely, we work on the smooth, irreducible variety $X \times \mathbf{A}^1$, of dimension $n+1$. We denote by t the coordinate on $\mathbf{A}^1$, defining the smooth hypersurface $X \times \{0\}$. Consider the closed immersion $\iota = \iota_f\colon X \hookrightarrow X \times \mathbf{A}^1$ given by $\iota(x) =$

$\big(x, f(x)\big)$. Given a coherent $\mathcal{D}_X$-module $\mathcal{M}$, we consider the coherent $\mathcal{D}_{X\times\mathbf{A}^1}$-module $\iota_+\mathcal{M}$. With a slight abuse of terminology, we will say that $\mathcal{M}$ *has a V-filtration with respect to f* if $\iota_+(\mathcal{M})$ has a V-filtration with respect to t.

In fact, we prefer to work with sheaves on X, rather than on $X\times\mathbf{A}^1$. In other words, we will tacitly identify a quasi-coherent $\mathcal{D}_{X\times\mathbf{A}^1}$-module $\mathcal{F}$ with the quasi-coherent $\mathcal{D}_X\langle t,\partial_t\rangle$-module $p_*(\mathcal{F})$, where $p\colon X\times\mathbf{A}^1\to X$ is the projection onto the first component.

Note that the formulas (8.1), (8.2), and (8.3) become in our setting

$$V^0\mathcal{D}_{X\times\mathbf{A}^1}=\mathcal{D}_X\langle s,t\rangle\subseteq\mathcal{D}_X\langle t,\partial_t\rangle, \tag{8.15}$$

$$V^m\mathcal{D}_{X\times\mathbf{A}^1}=V^0\mathcal{D}_{X\times\mathbf{A}^1}\cdot t^m=t^m\cdot V^0\mathcal{D}_{X\times\mathbf{A}^1}\quad\text{for all}\quad m\geq 0\quad\text{and} \tag{8.16}$$

$$V^{-m}\mathcal{D}_{X\times\mathbf{A}^1}=\sum_{i=0}^{m}V^0\mathcal{D}_{X\times\mathbf{A}^1}\cdot\partial_t^i\quad\text{for all}\quad m\geq 0. \tag{8.17}$$

Let us describe explicitly $\iota_+(\mathcal{M})$. Let's consider first the case when $\mathcal{M}=\mathcal{O}_X$. It follows from Example 6.62 that we have an isomorphism $\iota_+(\mathcal{O}_X)\simeq\mathcal{H}_{\iota(X)}(\mathcal{O}_Y)$, hence

$$B_f:=\iota_+(\mathcal{O}_X)\simeq\mathcal{O}_X[t]_{f-t}/\mathcal{O}_X[t] \tag{8.18}$$

(recall that we view this as a sheaf of $\mathcal{D}_X\langle t,\partial_t\rangle$-modules on X).

We will denote by δ the class of $\frac{1}{f-t}$ in B_f. Since $\mathcal{O}_X[t]_{f-t}/\mathcal{O}_X[t]=\bigoplus_{j\geq 1}\mathcal{O}_X\frac{1}{(f-t)^j}$ and $\frac{(j-1)!}{(f-t)^j}=\partial_t^{j-1}\cdot\frac{1}{(f-t)}$, we conclude that

$$B_f=\bigoplus_{j\geq 0}\mathcal{O}_X\partial_t^j\delta.$$

The action of $\mathcal{O}_X$ and ∂_t with respect to this decomposition is clear, while the actions of $\mathcal{D}er_k(\mathcal{O}_X)$ and of t are given by

$$D\cdot h\partial_t^j\delta=D(h)\partial_t^j\delta-hD(f)\partial_t^{j+1}\delta\quad\text{and}\quad t\cdot h\partial_t^j\delta=hf\partial_t^j\delta-jh\partial_t^{j-1}\delta \tag{8.19}$$

for every $D\in\mathcal{D}er_k(\mathcal{O}_X)$ and $h\in\mathcal{O}_X$.

Note now that if $\mathcal{M}$ is an arbitrary coherent $\mathcal{D}_X$-module, then we have an isomorphism

$$\iota_+(\mathcal{M})\simeq\mathcal{M}\otimes_{\mathcal{O}_X}B_f=\bigoplus_{j\geq 0}\mathcal{M}\otimes\partial_t^j\delta,$$

with the actions of $\mathcal{D}er_k(\mathcal{O}_X)$ and of t being given by the analogues of the formulas in (8.19):

$$D\cdot(u\otimes\partial_t^j\delta)=Du\otimes\partial_t^j\delta-D(f)u\otimes\partial_t^{j+1}\delta\quad\text{and}\quad t\cdot(u\otimes\partial_t^j\delta)=fu\otimes\partial_t^j\delta-ju\otimes\partial_t^{j-1}\delta \tag{8.20}$$

for every $D\in\mathcal{D}er_k(\mathcal{O}_X)$ and $u\in\mathcal{M}$.

EXAMPLE 8.27. Besides B_f, one important example is that of

$$B'_f:=\iota_+\big(\mathcal{O}_X[1/f]\big)=\bigoplus_{j\geq 0}\mathcal{O}_X[1/f]\partial_t^j\delta.$$

REMARK 8.28. If U is an open subset of X, we consider the restriction $g=f|_U$ of f, and let $\iota_f\colon X\hookrightarrow X\times\mathbf{A}^1$ and $\iota_g\colon U\hookrightarrow U\times\mathbf{A}^1$ be the corresponding graph embeddings. It is then clear that for every coherent $\mathcal{D}_X$-module $\mathcal{M}$, we have a canonical isomorphism

$$(\iota_g)_+(\mathcal{M}|_U) \simeq (\iota_f)_+(\mathcal{M})|_U$$

(recall that we consider both sides as sheaves on U). In this case it follows from Remark 8.6 that if $V^\bullet(\iota_f)_+\mathcal{M}$ is a V-filtration on $(\iota_f)_+(\mathcal{M})$, then $V^\bullet(\iota_f)_+\mathcal{M}|_U$ gives a V-filtration on $(\iota_g)_+(\mathcal{M}|_U)$.

Remark 8.29. It follows from the definition that if $\mathcal{M}$ has a V-filtration with respect to f, then $t \cdot \mathrm{Gr}_V^\alpha(\iota_+\mathcal{M}) = 0$ for all $\alpha \in \mathbf{Q}$. A related fact is that as a $\mathcal{D}_X$-module, $\mathrm{Gr}_V^\alpha(\iota_+\mathcal{M}) = 0$ is supported on H for all $\alpha \in \mathbf{Q}$. Indeed, note that if $u \in V^\alpha\iota_+(\mathcal{M}) \cap \sum_{j\leq p} \mathcal{M} \otimes \partial_t^j\delta$, then

$$fu = tu + (f-t)u \in V^{>\alpha}\iota_+(\mathcal{M}) + \left(V^\alpha\iota_+(\mathcal{M}) \cap \sum_{j\leq p-1} \mathcal{M} \otimes \partial_t^j\delta\right).$$

We deduce arguing by induction on p that $f^{p+1}u \in V^{>\alpha}\iota_+(\mathcal{M})$.

Remark 8.30. It follows from the formula (8.20) that in $\iota_+(\mathcal{M})$ we have

$$t\cdot\sum_{i=0}^N u_i\otimes\partial_t^i\delta = fu_N\otimes\partial_t^N\delta+(fu_{N-1}-Nu_N)\otimes\partial_t^{N-1}\delta+\ldots+(fu_0-u_1)\otimes\delta. \tag{8.21}$$

We deduce that multiplication by t on $\iota_+(\mathcal{M})$ is injective (or bijective) if and only if multiplication by f on $\mathcal{M}$ is injective (respectively, bijective). We thus conclude using Corollary 8.24 that if $\mathcal{M}$ admits a V-filtration with respect to f and $\mathcal{M}$ has no f-torsion (or multiplication by f is invertible), then for every $u \in \iota_+(\mathcal{M})$ and $\alpha \in \mathbf{Q}$, we have $u \in V^\alpha\iota_+(\mathcal{M})$ if and only if $tu \in V^{\alpha+1}\iota_+(\mathcal{M})$ (respectively, for every $\alpha \in \mathbf{Q}$, we have $t \cdot V^\alpha\iota_+(\mathcal{M}) = V^{\alpha+1}\iota_+(\mathcal{M})$).

Remark 8.31. If $U = X \smallsetminus H$, then $\iota(U)$ is contained in the complement of the hypersurface defined by t, hence it follows from Remarks 8.7 and 8.28 that if $\mathcal{M}$ has a V-filtration with respect to f, then $V^\alpha\iota_+(\mathcal{M})|_U = \iota_+(\mathcal{M})|_U$ for all $\alpha \in \mathbf{Q}$. Moreover, if f is invertible (so $H = \emptyset$), then every $\mathcal{M}$ has a V-filtration with respect to f.

Example 8.32. Note that $\mathrm{Supp}(\mathcal{M}) \subseteq H$ if and only if $\mathrm{Supp}\big(\iota_+(\mathcal{M})\big) \subseteq X \times \{0\}$ and in this case it follows from Example 8.19 that $\mathcal{M}$ has a V-filtration with respect to f. Moreover, we have $V^\alpha\iota_+(\mathcal{M}) = 0$ for all $\alpha > 0$ and

$$\begin{aligned}\mathrm{Gr}_V^0\big(\iota_+(\mathcal{M})\big) &\simeq \big\{u \in \iota_+(\mathcal{M}) \mid t\cdot u = 0\big\} \\ &= \left\{\sum_{j\geq 0} \tfrac{f^j}{j!}u_0 \otimes \partial_t^j\delta \mid u_0 \in \mathcal{M}\right\},\end{aligned}$$

where the last equality follows easily from formula (8.21) (note that $f^j u = 0$ for $j \gg 0$ by the assumption on $\mathcal{M}$).

Remark 8.33. We note that if f defines a smooth hypersurface in X, then the two notions of V-filtrations determine each other. More precisely, we have a V-filtration on $\mathcal{M}$ with respect to f if and only if we have a V-filtration on $\iota_+(\mathcal{M})$ with respect to t. Indeed, this follows from Proposition 8.22, since $t\circ\iota = f$. Moreover, an easy computation using the proposition implies that the two V-filtrations determine each other as follows: if we choose local coordinates $x_1, \ldots, x_{n-1}, y$ on X, with $y = f$, then

$$V^\alpha \iota_+(\mathcal{M}) = \sum_{j\geq 0} \partial_y^j \cdot (V^\alpha \mathcal{M} \otimes \delta).$$

For example, if we take $\mathcal{M} = \mathcal{O}_X$, then it follows from Example 8.18 that if the hypersurface defined by f is smooth, then the V-filtration on B_f is given by

$$V^\alpha \iota_+(\mathcal{O}_X) = \mathcal{D}_X \cdot y^{\lceil \alpha \rceil - 1}\delta = \sum_{j\geq 0} \mathcal{O}_X \cdot \partial_y^j y^{\lceil \alpha \rceil - 1}\delta \quad \text{for all} \quad \alpha \in \mathbf{Q},$$

with the convention that $y^m = 1$ if $m \leq 0$. In particular, we see that $V^1\iota_+(\mathcal{O}_X) = \iota_+(\mathcal{O}_X)$.

Example 8.34. For a more interesting example, let us consider the case of a simple normal crossing divisor. Suppose that $x_1, \ldots, x_n$ are algebraic coordinates on X, and $f = \prod_{i=1}^r x_i^{a_i}$, where $a_1, \ldots, a_r$ are positive integers. For every $\lambda \in \mathbf{Q}$, we put

$$I(f^\lambda) = \big(x_1^{\lceil \lambda a_1 \rceil - 1} \cdots x_r^{\lceil \lambda a_r \rceil - 1}\big),$$

with the convention that this is $\mathcal{O}_X$ for $\lambda < 0$. It is clear from the formula that $I(f^{\lambda_1}) \subseteq I(f^{\lambda_2})$ if $\lambda_1 \geq \lambda_2$). Note also that we have $f \cdot I(f^\lambda) \subseteq I(f^{\lambda+1})$, with equality if $\lambda > 0$.

Let us show that if we put

$$V^\lambda \iota_+(\mathcal{O}_X) = \sum_{j\geq 0} \mathcal{D}_X \cdot I(f^{\lambda+j})\partial_t^j \delta \quad \text{for all} \quad \lambda \in \mathbf{Q}, \tag{8.22}$$

then $V^\bullet \iota_+(\mathcal{O}_X)$ is a V-filtration with respect to t. A more general result is proved in [**Sai90**, Theorem 3.4], with a rather involved proof. We here give a direct proof, by checking that the formula in (8.22) satisfies the properties of the V-filtration. It is clear that this is a decreasing, exhaustive filtration. Note also that if $N = \mathrm{lcm}(a_1, \ldots, a_r)$, then $I(f^\lambda)$ is constant for $\lambda \in (i/N, (i+1)/N]$ for every $i \in \mathbf{Z}$, hence the filtration we defined on $\iota_+(\mathcal{O}_X)$ has the same property, and thus it is discrete and left continuous.

Note first that the sum in the formula (8.22) can be replaced by a finite sum. Indeed, for every i, with $1 \leq i \leq r$, and every $b \in \mathbf{Z}_{\geq 0}^r$, we have

$$(x_i \partial_{x_i} - b_i) \cdot x^b \partial_t^j \delta = -a_i x^{a+b} \partial_t^{j+1},$$

which implies

$$I(f^{\lambda+1})\partial_t^{j+1}\delta \subseteq \mathcal{D}_X \cdot I(f^\lambda)\partial_t^j \delta \quad \text{for} \quad \lambda > 0.$$

This implies that for $\lambda > 0$ we have $V^\lambda \iota_+(\mathcal{O}_X) = \mathcal{D}_X \cdot I(f^\lambda)\delta$ and, more generally, we have

$$V^\lambda \iota_+(\mathcal{O}_X) = \sum_{j \leq \max\{1-\lambda, 0\}} \mathcal{D}_X \cdot I(f^{\lambda+j})\partial_t^j \quad \text{for all} \quad \lambda \in \mathbf{Q}.$$

This implies that every $V^\lambda \iota_+(\mathcal{O}_X)$ is finitely generated over $\mathcal{D}_X$ (hence also over $V^0\mathcal{D}_{X\times\mathbf{A}^1}$).

Note next that for every $j \geq 0$, we have

$$t \cdot I(f^{\lambda+j})\partial_t^j \delta \subseteq I(f^{\lambda+j+1})\partial_t^j \delta + I(f^\lambda)\partial_t^{j-1}\delta \subseteq V^{\lambda+1}\iota_+(\mathcal{O}_X),$$

so $t\cdot V^{\lambda}\iota_+(\mathcal{O}_X)\subseteq V^{\lambda+1}\iota_+(\mathcal{O}_X)$ for every $\lambda\in\mathbf{Q}$. Moreover, this is an equality for $\lambda>0$. Indeed, we have seen that in this case we have

$$V^{\lambda+1}\iota_+(\mathcal{O}_X)=\mathcal{D}_X\cdot I(f^{\lambda+1})\delta=f\cdot I(f^{\lambda})\delta=t\cdot I(f^{\lambda})\delta=t\cdot V^{\lambda}\iota_+(\mathcal{O}_X).$$

Note also that we have

$$\partial_t\cdot I(f^{\lambda+j})\partial_t^j=I(f^{\lambda+j})\partial_t^{j+1}\delta\subseteq V^{\lambda-1}\iota_+(\mathcal{O}_X),$$

hence $\partial_t\cdot V^{\lambda}\iota_+(\mathcal{O}_X)\subseteq V^{\lambda-1}\iota_+(\mathcal{O}_X)$ for all $\lambda\in\mathbf{Q}$.

In order to conclude, it is enough to show that if $b\in\mathbf{Z}^r_{\geq 0}$ is such that $x^b\in I(f^{\lambda+j})$, then

$$(\partial_t t-\lambda)^r x^b\partial_t^j\delta\in V^{>\alpha}\iota_+(\mathcal{O}_X).$$

Note that $x^b\partial_t^j\delta\in V^{>\lambda}\iota_+(\mathcal{O}_X)$, unless there is i such that $b_i=(\lambda+j)a_i-1$. In this case, if $e_1,\ldots,e_r$ is the standard basis of $\mathbf{Z}^r$, a simple computation gives

$$\partial_{x_i}\cdot x^{b+e_i}\partial_t^j\delta=(b_i+1)x^b\partial_t^j\delta-a_ix^{a+b}\partial_t^{j+1}\delta=-a_i(\partial_t t-\lambda)x^b\partial_t^j\delta.$$

We thus see that if $J=\big\{i\mid b_i=(\lambda+j)a_i-1\big\}$, then

$$(\partial_t t-\lambda)^{|J|}\cdot x^b\partial_t^j\delta\in\mathcal{D}_X\cdot x^{b'}\partial_t^j\delta,$$

for some $b'\in\mathbf{Z}^r_{\geq 0}$ such that $x^{b'}\in I(f^{\lambda'+j})$ for some $\lambda'>\lambda$. We thus conclude that $(\partial_t t-\lambda)^{|J|}\cdot x^b\partial_t^j\delta\in V^{>\lambda}\iota_+(\mathcal{O}_X)$.

We now discuss the connection between V-filtrations and b-functions. Suppose that $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module on which multiplication by f is bijective. Recall that in this case we have a $\mathcal{D}_X[s]$-module $\mathcal{M}[s]f^s$ on X (see Chapter 6.5). In fact, we have on $\mathcal{M}[s]f^s$ an action of $\mathcal{D}_X\langle t,s\rangle$, where t acts on $\mathcal{M}[s]f^s$ via the automorphism T. We note that since $ts=(s+1)t$, it follows from Lemma 8.2 that we have an injective homomorphism $\mathcal{D}_X\langle t,s\rangle\hookrightarrow\mathcal{D}_X\langle t,\partial_t\rangle$, with s mapping to $-\partial_t t$.

PROPOSITION 8.35. If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module on which multiplication by f is bijective, then we have an isomorphism of $\mathcal{D}_X\langle t,s\rangle$-modules

$$\tau\colon\mathcal{M}[s]f^s\xrightarrow{\sim}\iota_+(\mathcal{M}),\quad P(s)uf^s\mapsto P(-\partial_t t)(u\otimes\delta).\tag{8.23}$$

Moreover, the action of ∂_t on the right-hand side corresponds to the action of $(-s)T^{-1}$ on the left-hand side.

PROOF. Let's check the compatibility of τ with the $\mathcal{D}_X\langle t,s\rangle$-action. It is clear that τ is $\mathcal{O}_X[s]$-linear, hence we only need to check the compatibility with the action of $\mathcal{D}er_k(\mathcal{O}_X)$ and that of t. Note that

$$\begin{aligned}\tau\big(t\cdot P(s)uf^s\big)&=\tau\big(P(s+1)fuf^s\big)=P(-\partial_t t+1)fu\otimes\delta\\&=P(-\partial_t t+1)tu\otimes\delta=tP(-\partial_t t)u\otimes\delta=t\cdot\tau\big(P(s)uf^s\big),\end{aligned}$$

where the second to last equality follows from Lemma 8.2i).

Suppose now that $D\in\mathcal{D}er_k(\mathcal{O}_X)$. We have

$$\begin{aligned}\tau\big(D\cdot P(s)uf^s\big)=\tau\big(P(s)Duf^s+sP(s)\tfrac{D(f)}{f}uf^s\big)&=P(-\partial_t t)Du\otimes\delta-P(-\partial_t t)\partial_t t\tfrac{D(f)}{f}u\otimes\delta\\&=P(-\partial_t t)Du\otimes\delta-P(-\partial_t t)D(f)u\otimes\partial_t\delta,\quad\text{while}\end{aligned}$$

$$D\cdot\tau\big(P(s)uf^s\big)=D\cdot P(-\partial_t t)u\otimes\delta=P(-\partial_t t)\big(Du\otimes\delta-D(f)u\otimes\partial_t\delta\big),$$

hence $\tau\big(D\cdot P(s)uf^s\big)=D\cdot\tau\big(P(s)uf^s\big)$.

Note next that it follows from the second formula in Lemma 8.2ii) that

$$(-1)^m\tau\big(s(s-1)\cdots(s-m+1)uf^s\big) = f^m u\otimes\partial_t^m\delta \quad \text{for all} \quad m.$$

Since $k[s]$ has a basis given by $\prod_{j=0}^{m-1}(s-j)$, for $m\geq 0$, and since multiplication by f is invertible on M, it follows that τ is a bijective map. This completes the proof of the proposition. $\square$

Definition 8.36. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We say that a section $w\in\iota_+(\mathcal{M})$ *has a b-function* if there is a nonzero $b(s)\in k[s]$ such that $b(s)w\in V^1\mathcal{D}_{X\times\mathbf{A}^1}\cdot w$. In this case it follows that the set of such polynomials $b(s)$ is a nonzero ideal of $k[s]$. Its monic generator is the *b-function* $b_w(s)$.

Remark 8.37. With the notation in the above definition, we note that in fact $b_w(s)$ satisfies $b_w(s)\cdot\frac{V^0\mathcal{D}_{X\times\mathbf{A}^1}\cdot w}{V^1\mathcal{D}_{X\times\mathbf{A}^1}\cdot w} = 0$. This is due to the fact that $V^0\mathcal{D}_{X\times\mathbf{A}^1} = \mathcal{D}_X[s]+V^1\mathcal{D}_{X\times\mathbf{A}^1}$ and $b_w(s)$ commutes with the elements of $\mathcal{D}_X[s]$.

Moreover, if $\mathcal{N}=V^0\mathcal{D}_{X\times\mathbf{A}^1}\cdot w$, then b_w is the minimal polynomial of the action of s on $\mathcal{N}/t\mathcal{N}$. This follows from the fact that $V^1\mathcal{D}_{X\times\mathbf{A}^1}\cdot w = \big(t\cdot\mathcal{D}_X\langle s,t\rangle\big)\cdot w = t\mathcal{N}$.

Remark 8.38. Suppose that $X=U_1\cup\ldots\cup U_r$ is an open cover, $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, and $f\in\mathcal{O}_X(X)$ is nonzero. If $f_j=f|_{U_j}$ and $\iota_f\colon X\hookrightarrow X\times\mathbf{A}^1$ and $\iota_{f_j}\colon U_j\hookrightarrow U_j\times\mathbf{A}^1$ are the corresponding graph embeddings, then we have canonical isomorphisms

$$\iota_{f_j}(\mathcal{M}|_{U_j})\simeq\iota_f(\mathcal{M})|_{U_j}.$$

$w\in\Gamma\big(X,\iota_+(\mathcal{M})\big)$ and $w_j=w|_{U_j}$, then it follows from definition that

$$b_w=\mathrm{lcm}\big\{b_{w_j}\mid 1\leq j\leq r\big\}$$

(this means that b_w exists if and only if each b_{w_j} exists, and if this is the case, then we have the stated equality).

Remark 8.39. Note that if $w\in\iota_+(\mathcal{M})$ and $p\in\mathcal{O}_X(X)$ is an invertible function, then b_w exists if and only if b_{pw} exists and, if this is the case, then $b_w=b_{pw}$. Indeed, if $b(s)w=Q\cdot w$, for some $Q\in V^1\mathcal{D}_{X\times\mathbf{A}^1}$, then

$$b(s)pw=(pQ)\cdot w=(pQp^{-1})pw$$

and $pQp^{-1}\in V^1\mathcal{D}_{X\times\mathbf{A}^1}$. This implies that if b_w exists, then b_{pw} exists and b_{pw} divides b_w. The converse follows by symmetry since p is invertible.

Remark 8.40. Keeping the notation in Definition 8.36, note that if multiplication by f is invertible on $\mathcal{M}$ and $w=u\otimes\delta\in\iota_+\mathcal{M}$, then w corresponds to uf^s via the isomorphism τ in Proposition 8.35 and in this case $b_w(s)$ is the monic polynomial of minimal degree such that $b_w(s)uf^s\in\mathcal{D}_X[s]\cdot fuf^s$ (in particular, in this case we recover the definition in Chapter 6.5). Indeed, this follows from the fact that $V^1\mathcal{D}_{X\times\mathbf{A}^1}=\mathcal{D}_X\langle s,t\rangle\cdot t$, hence $V^1\mathcal{D}_{X\times\mathbf{A}^1}\cdot(u\otimes\delta)=\mathcal{D}_X[s]\cdot(fu\otimes\delta)$.

Remark 8.41. Given a coherent $\mathcal{D}_X$-module $\mathcal{M}$, it is easy to compare the V-filtrations of $\mathcal{M}$ with respect to f and $g=pf$, where $p\in\mathcal{O}_X(X)$ is an invertible function. Note that if ι_f and ι_g are the graph embeddings corresponding to f and g, respectively, then $\iota_g=\varphi\circ\iota_f$, where $\varphi\colon X\times\mathbf{A}^1\to X\times\mathbf{A}^1$ is the isomorphism given by $\varphi(x,t)=\big(x,p(x)t\big)$, so that $t\circ\varphi=pt$. We thus get an isomorphism of $\mathcal{D}_{X\times\mathbf{A}^1}$-modules

$$\varphi^*(\iota_g)_+(\mathcal{M})\simeq(\iota_f)_+(\mathcal{M}),$$

which in turn induces an isomorphism of $\mathcal{O}_X$-modules

$$\tau\colon (\iota_g)_+(\mathcal{M}) \to (\iota_f)_+(\mathcal{M}),\ \tau\Big(\sum_{j=1}^{N} \tfrac{u_j}{(g-t)^j}\Big) = \sum_{j=1}^{N} \tfrac{p^{-j}u_j}{(f-t)^j}.$$

This has the property that $\tau(Pw) = \tau_0(P)w$, where $\tau_0\colon \mathcal{D}_X\langle t,\partial_t\rangle \to \mathcal{D}_X\langle t,\partial_t\rangle$ is the isomorphism of sheaves of rings which is the identity on $\mathcal{O}_X$ and satisfies

$$\tau_0(t) = pt,\ \tau_0(\partial_t) = p^{-1}\partial_t,\ \text{and} \quad \tau_0(Q) = Q - Q(p)p^{-1}t\partial_t \tag{8.24}$$

for $Q \in \mathrm{Der}_k(\mathcal{O}_X)$ (note that, in particular, $\tau_0(s) = s$). By the last assertion in Remark 8.8, the V-filtrations on $(\iota_f)_+(\mathcal{M})$ with respect to t and pt coincide, hence $\mathcal{M}$ has a V-filtration with respect to f if and only if it has a V-filtration with respect to g, and in this case we have

$$\tau\big(V^\alpha(\iota_g)_+(\mathcal{M})\big) = V^\alpha(\iota_f)_+(\mathcal{M}) \quad \text{for all} \quad \alpha \in \mathbf{Q}.$$

Moreover, since τ preserves $V^\bullet\mathcal{D}_X\langle t,\partial_t\rangle$ (this follows from the definition of $V^\bullet\mathcal{D}_{X\times\mathbf{A}^1}$ with respect to a hypersurface, but can be deduced also from the explicit formulas (8.24), it follows from the definition that $b_w = b_{\tau(w)}$ for every $w \in (\iota_g)_+(\mathcal{M})$, in the sense that one exists if and only if the other one exists and in this case they are equal. For example, we see that for every $u \in \mathcal{M}$ and $m \in \mathbf{Z}_{\geq 0}$, if $v = u\otimes\partial_t^m\delta \in (\iota_g)_+(\mathcal{M})$ and $w = u \otimes \partial_t^m\delta \in (\iota_f)_+(\mathcal{M})$, then

$$b_w = b_{\tau(w)} = b_{p^{-m-1}v} = b_v,$$

where the last equality follows from Remark 8.39.

The following result provides the criterion for the existence of V-filtrations in terms of b-functions. We give the proof following an approach due to Sabbah.

THEOREM 8.42. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module.*

i) *If $\mathcal{M}$ has a V-filtration with respect to f, then every section $w \in \iota_+(\mathcal{M})$ has a b-function. Moreover, if $w \in V^\alpha\iota_+(\mathcal{M})$, then all roots of $b_w(s)$ are rational numbers $\gamma \leq -\alpha$ such that $\mathrm{Gr}_V^{-\gamma}(\iota_+\mathcal{M}) \neq 0$.*
ii) *Conversely, if $w_1,\ldots,w_r \in \Gamma(X,\mathcal{M})$ generate $\mathcal{M}$ as a $\mathcal{D}_X$-module and if each $w_i \otimes \delta \in \iota_+(\mathcal{M})$ has a b-function whose roots are all rational, then $\mathcal{M}$ has a V-filtration with respect to f.*

PROOF. In order to prove the assertion in i), note that by applying Remark 8.14 for the induced V-filtration on $\mathcal{D}_{X\times\mathbf{A}^1}\cdot w$, we see that there is β such that

$$V^\beta\iota_+(\mathcal{M}) \cap \mathcal{D}_{X\times\mathbf{A}^1}\cdot w \subseteq V^1\mathcal{D}_{X\times\mathbf{A}^1}\cdot w.$$

Since $(s+\gamma)$ is nilpotent on $\mathrm{Gr}_V^\gamma(\iota_+\mathcal{M})$ for all $\gamma \in \mathbf{Q}$ and since the V-filtration is discrete, it follows that if $\gamma_1,\ldots,\gamma_r$ are the rational numbers γ with $\alpha \leq \gamma < \beta$ and with $\mathrm{Gr}_V^{-\gamma}(\iota_+\mathcal{M}) \neq 0$, then there is $N \geq 1$ such that

$$(s+\gamma_1)^N\cdots(s+\gamma_r)^N w \in V^\beta\iota_+\mathcal{M} \cap \mathcal{D}_{X\times\mathbf{A}^1}\cdot w \subseteq V^1\mathcal{D}_{X\times\mathbf{A}^1}\cdot w.$$

This implies that $b_w(s)$ divides $\prod_{i=1}^r(s+\gamma_i)^N$, proving the assertion in i).

In order to prove the statement in ii), we consider the following type of filtrations, that we call *pre-V-filtrations* (associated to f): these are decreasing, exhaustive filtrations $W^\bullet\iota_+(\mathcal{M})$ parametrized by integers, by quasi-coherent $\mathcal{O}_X$-modules, that satisfy the following properties:

a) We have $V^i\mathcal{D}_{X\times\mathbf{A}^1}\cdot W^j\iota_+(\mathcal{M}) \subseteq W^{i+j}\iota_+(\mathcal{M})$ for all $i,j \in \mathbf{Z}$.
b) $W^m\iota_+(\mathcal{M})$ is locally finitely generated over $V^0\mathcal{D}_{X\times\mathbf{A}^1}$ for every $m \in \mathbf{Z}$.

c) We have $W^{m+1}\iota_+(\mathcal{M}) = t\cdot W^m\iota_+(\mathcal{M})$ for $m\gg 0$.
d) There is a polynomial $p=p_W\in \mathbf{Q}[x]$, with all roots in $\mathbf{Q}$, such that $p(\partial_t t-m)\cdot W^m\iota_+(\mathcal{M})\subseteq W^{m+1}\iota_+(\mathcal{M})$ for all $m\in\mathbf{Z}$.

Our first goal is to show that there is a pre-V-filtration such that the polynomial p_W in d) above has all its roots in $[0,1)$. Note to start with that we have pre-V-filtrations. Indeed, for every $j\in\mathbf{Z}$, let

$$W^j\iota_+(\mathcal{M}) := \sum_{\ell=1}^r V^j\mathcal{D}_{X\times\mathbf{A}^1}\cdot(w_\ell\otimes\delta)\subseteq \iota_+(\mathcal{M}).$$

Since $w_1,\ldots,w_r$ generate $\mathcal{M}$ over $\mathcal{D}_X$, it follows that $w_1\otimes\delta,\ldots,w_r\otimes\delta$ generate $\iota_+(\mathcal{M})$ over $\mathcal{D}_{X\times\mathbf{A}^1}$, hence $W^\bullet\iota_+(\mathcal{M})$ is exhaustive. It is also clear that it satisfies conditions a), b), and c) in the definition. In order to check condition d), let b_ℓ be the b-function of $w_\ell\otimes\delta$, so we have $b_\ell(-\partial_t t)\cdot V^0\mathcal{D}_{X\times\mathbf{A}^1}(w_\ell\otimes\delta)\subseteq V^1\mathcal{D}_{X\times\mathbf{A}^1}(w_\ell\otimes\delta)$ for all ℓ (see Remark 8.37). This implies that if $p(x)=\prod_{\ell=1}^r b_\ell(-x)$, then $p(\partial_t t)\cdot W^0\iota_+(\mathcal{M})\subseteq W^1\iota_+(\mathcal{M})$. Moreover, it follows from the definition of $W^\bullet\iota_+(\mathcal{M})$ and the formulas (8.16) and (8.17) that for all $m\geq 0$, we have $W^m\iota_+(\mathcal{M})=t^m\cdot W^0\iota_+(\mathcal{M})$ and $W^{-m}\iota_+(\mathcal{M})=\sum_{j=0}^m\partial_t^j\cdot W^0\iota_+(\mathcal{M})$. Since we have $p(\partial_t t-m)t^m = t^m p(\partial_t t)$ by Lemma 8.2, we deduce that for all $m\geq 0$, we have

$$p(\partial_t t-m)\cdot W^m\iota_+(\mathcal{M})\subseteq t^m p(\partial_t t)W^0\iota_+(\mathcal{M})\subseteq t^m\cdot W^1\iota_+(\mathcal{M})\subseteq W^{m+1}\iota_+(\mathcal{M}).$$

On the other hand, Lemma 8.2 gives $p(\partial_t t+m)\partial_t^j=\partial_t^j p(\partial_t t+m-j)$, hence for every $m\geq 0$, we have

$$p(\partial_t t+m)\partial_t^m\cdot W^0\iota_+(\mathcal{M})=\partial_t^m p(\partial_t t)\cdot W^0\iota_+(\mathcal{M})\subseteq\partial_t^m\cdot W^1\iota_+(\mathcal{M})\subseteq W^{-m+1}\iota_+(\mathcal{M}),$$

while for $0\leq j\leq m-1$, we clearly have

$$p(\partial_t t+m)\partial_t^j\cdot W^0\iota_+(\mathcal{M})\subseteq W^{-j}\iota_+(\mathcal{M})\subseteq W^{-m+1}\iota_+(\mathcal{M}).$$

We thus conclude that $p(\partial_t t+m)\cdot W^{-m}\iota_+(\mathcal{M})\subseteq W^{-m+1}\iota_+(\mathcal{M})$ for all $m\geq 0$, concluding the proof for the fact that $W^\bullet\mathcal{M}$ satisfies condition d).

Note that if $W^\bullet\iota_+(\mathcal{M})$ is a pre-V-filtration with polynomial p_W and for an integer q we put $\widetilde{W}^m\iota_+(\mathcal{M})=W^{m+q}\iota_+(\mathcal{M})$, then $\widetilde{W}^\bullet\iota_+(\mathcal{M})$ is again a pre-V-filtration with corresponding polynomial $p_{\widetilde{W}}(x)=p_W(x-q)$. By taking a suitable q (small enough) and replacing $W^\bullet\iota_+(\mathcal{M})$ by $\widetilde{W}^\bullet\iota_+(\mathcal{M})$, we see that we may assume that all roots of p_W are <1.

Suppose now that λ is a root of p_W and let us write $p_W=(x-\lambda)^d q(x)$, where $q(\lambda)\neq 0$. We define a new filtration $U^\bullet\iota_+(\mathcal{M})$ by the formula

$$U^m\iota_+(\mathcal{M}):=W^{m+1}\iota_+(\mathcal{M})+(\partial_t t-m-\lambda)^d\cdot W^m\iota_+(\mathcal{M})\quad\text{for all}\quad m\in\mathbf{Z}.$$

It is clear that this is a decreasing, exhaustive filtration and it is an easy exercise to see, using Lemma 8.2, that since $W^\bullet\iota_+(\mathcal{M})$ satisfies conditions a), b), and c), so does $U^\bullet\iota_+(\mathcal{M})$. Let us show that we may take $p_U(x)=(x-\lambda-1)^d q(x)$. Indeed, for every $m\in\mathbf{Z}$, we have

$$(\partial_t t-\lambda-m-1)^d q(\partial_t t-m)\cdot W^{m+1}\iota_+(\mathcal{M})\subseteq q(\partial_t t-m)\cdot U^{m+1}\iota_+(\mathcal{M})\subseteq U^{m+1}\iota_+(\mathcal{M})$$

by definition of $U^\bullet\iota_+(\mathcal{M})$ and

$$\begin{aligned}&(\partial_t t-\lambda-m-1)^d q(\partial_t t-m)\cdot(\partial_t t-m-\lambda)^d\cdot W^m\iota_+(\mathcal{M})\\ &=(\partial_t t-\lambda-m-1)^d p_W(\partial_t t-m)\cdot W^m\iota_+(\mathcal{M})\subseteq(\partial_t t-\lambda-m-1)^d\cdot W^{m+1}\iota_+(\mathcal{M})\\ &\subseteq U^{m+1}\iota_+(\mathcal{M}).\end{aligned}$$

We thus conclude that $U^\bullet\iota_+(\mathcal{M})$ is a pre-V-filtration and we may take $p_U(x) = (x-\lambda-1)^d q(x)$. After applying this construction finitely many times, we may replace λ by a root in $[0,1)$, and after repeating the same process for the other roots of p_W, and replacing $W^\bullet\iota_+(\mathcal{M})$ by the final pre-V-filtration, we see that we may assume that all roots of p_W lie in $[0,1)$.

It is now easy to construct the V-filtration on $\iota_+(\mathcal{M})$ (in order to simplify the notation, we will write V^α and W^m for $V^\alpha\iota_+(\mathcal{M})$ and $W^m\iota_+(\mathcal{M})$). Let $\alpha_1 < \ldots < \alpha_d$ be the distinct roots of $p_W(x)$, which by assumption lie in the interval $[0,1)$. For every $m \in \mathbf{Z}$ and every i, with $1 \leq i \leq d$, let $P_i^{(m)}$, with $W^{m+1} \subseteq P_i^{(m)} \subseteq W^m$ be such that $P_i^{(m)}/W^{m+1}$ is the generalized eigenspace with eigenvalue $\alpha_i + m$ for the action of $\partial_t t$ on W^m/W^{m+1}. It is a standard linear algebra result that we have $W^m/W^{m+1} = \bigoplus_{i=1}^d P_i^{(m)}/W^{m+1}$. Since $\partial_t t = -s$ is a $\mathcal{D}_X[s]$-linear operator and since $t \cdot P_i^{(m)} \subseteq W^{m+1} \subseteq P_i^{(m)}$, it follows that each P_i is a $V^0\mathcal{D}_{X\times\mathbf{A}^1}$-submodule of $\iota_+(\mathcal{M})$. We put $V^m = W^m$ and for $1 \leq i \leq d$, we define the $V^0\mathcal{D}_{X\times\mathbf{A}^1}$-module $V^{m+\alpha_i}$ such that $W^{m+1} \subseteq V^{m+\alpha_i}$ and $V^{m+\alpha_i}/W^{m+1} = P_i \oplus \ldots \oplus P_d$. Note that by definition we have

$$W^m = V^m = V^{m+\alpha_1} \supseteq V^{m+\alpha_2} \supseteq \ldots \supseteq V^{m+\alpha_d} \supseteq W^{m+1} = V^{m+1}.$$

We extend this filtration to all rational numbers such that V^λ takes constant value for λ in each interval $(m, m+\alpha_1], (m+\alpha_1, m+\alpha_2], \ldots, (m+\alpha_d, m+1]$. Checking that this is indeed a V-filtration corresponding to f is a straightforward exercise. $\square$

In fact, once we know that a $\mathcal{D}$-module has a V-filtration, this can be characterized via b-functions by the condition in Theorem 8.42i).

PROPOSITION 8.43. If a coherent $\mathcal{D}_X$-module $\mathcal{M}$ has a V-filtration with respect to f, then for every $\alpha \in \mathbf{Q}$, $V^\alpha\iota_+(\mathcal{M})$ consists of those sections $w \in \iota_+(\mathcal{M})$ with the property that all roots of b_w are $\leq -\alpha$.

PROOF. We have already seen in Theorem 8.42i) that if $w \in V^\alpha\iota_+(\mathcal{M})$, then all roots of b_w are rational numbers $\leq -\alpha$. Conversely, suppose that all roots of b_w are $\leq -\alpha$. Let $\beta \in \mathbf{Q}$ be such that $w \in V^\beta\iota_+(\mathcal{M})$. If $\beta \geq \alpha$, then we are done. If $\beta < \alpha$, since the V-filtration is discrete, we may assume that $w \not\in V^{>\beta}\iota_+(\mathcal{M})$ and aim for a contradiction. Since $b_w(s)w \in V^1\mathcal{D}_{X\times\mathbf{A}^1} \cdot w \subseteq V^{\beta+1}\iota_+(\mathcal{M})$, we have $b_w(s)\overline{w} = 0$ in $\mathrm{Gr}_V^\beta\big(\iota_+(\mathcal{M})\big)$. Since $s+\beta$ is nilpotent on $\mathrm{Gr}_V^\beta(\iota_+\mathcal{M})$, while $b_w(s)\overline{w} = 0$ and $b_w(s) = \prod_{i=1}^r (s+\alpha_i)$ with $\alpha_i \geq \alpha > \beta$ for all i, we conclude that $\overline{w} = 0$ in $\mathrm{Gr}_V^\beta(\iota_+\mathcal{M})$, a contradiction. $\square$

REMARK 8.44. With the notation in Theorem 8.42ii), if

$$R = \bigcup_{i=1}^r \big\{\lambda \in \mathbf{Q} \mid b_{w_i}(-\lambda) = 0\big\},$$

then

$$R \subseteq \big\{\alpha \in \mathbf{Q} \mid \mathrm{Gr}_V^\alpha(\iota_+\mathcal{M}) \neq 0\big\} \subseteq R + \mathbf{Z}.$$

Indeed, the first inclusion follows from assertion i) in the theorem, while the second inclusion follows from the proof of assertion ii) in the theorem.

EXAMPLE 8.45. If $f \in \mathcal{O}_X(X)$ is invertible and $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ is the corresponding graph embedding, then it follows from Remark 8.31 that for every coherent

$\mathcal{D}_X$-module $\mathcal{M}$, we have a V-filtration given by $V^\alpha\iota_+(\mathcal{M}) = \iota_+(\mathcal{M})$ for all $\alpha \in \mathbf{Q}$. In this case it follows from Proposition 8.43 that $b_w = 1$ for every $w \in \iota_+(\mathcal{M})$.

A consequence of Theorem 8.42 is that the subcategory of $\mathcal{D}_X$-modules that have a V-filtration with respect to f is closed under extensions:

Corollary 8.46. Given a short exact sequence of coherent $\mathcal{D}_X$-modules

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0,$$

if $\mathcal{M}'$ and $\mathcal{M}''$ have a V-filtration with respect to f, then so does $\mathcal{M}$.

Proof. By Corollary 8.17, we may and will assume that X is affine. After applying ι_+, we get an exact sequence

$$0 \to \iota_+(\mathcal{M}') \hookrightarrow \iota_+(\mathcal{M}) \xrightarrow{p} \iota_+(\mathcal{M}'') \to 0.$$

We use the criterion for the existence of V-filtrations with respect to f in Theorem 8.42. Therefore it is enough to show that every element $u \in \iota_+(\mathcal{M})$ has a b-function with roots in $\mathbf{Q}$. Since $\mathcal{M}''$ has a V-filtration with respect to f, we have a polynomial $b_1(s)$ with roots in $\mathbf{Q}$ and $Q_1 \in V^1\mathcal{D}_{X\times\mathbf{A}^1}$ such that $b_1(s)u - Q_1u \in \iota_+(\mathcal{M}')$. Since $\mathcal{M}'$ has a V-filtration with respect to f, it follows that we have a polynomial $b_2(s)$ with roots in $\mathbf{Q}$ and $Q_2 \in V^1\mathcal{D}_{X\times\mathbf{A}^1}$ such that

$$b_2(s)\big(b_1(s)u - Q_1u\big) = Q_2\big(b_1(s)u - Q_1u\big).$$

Therefore $b_1(s)b_2(s)u \in V^1\mathcal{D}_{X\times\mathbf{A}^1}u$, hence u has a b-function that divides b_1b_2, and thus has rational roots. □

Corollary 8.47. If $j\colon U \hookrightarrow X$ is the inclusion of the complement of the hypersurface defined by f and if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module such that $\mathcal{M}|_U$ is holonomic, then $\mathcal{M}$ has a V-filtration with respect to f if and only if $j_+(\mathcal{M}|_U)$ has a V-filtration with respect to f.

Proof. Since $\mathcal{M}|_U$ is holonomic, we know that $j_+(\mathcal{M}|_U)$ is holonomic, hence coherent. The assertion in the statement then follows from Corollaries 8.21 and 8.46, using the fact that the kernel and the cokernel of the canonical morphism $\mathcal{M} \to j_+(\mathcal{M}|_U)$ are supported on the hypersurface defined by f, and thus have a V-filtration with respect to f by Example 8.32. □

Remark 8.48. If $j\colon U \hookrightarrow X$ is the inclusion of the complement of the hypersurface defined by f, then it follows from Theorem 6.45 that if $\mathcal{N}$ is a holonomic $\mathcal{D}_U$-module, then every local section of $j_+(\mathcal{N})$ has a b-function. We then deduce from Theorem 8.42 that $j_+(\mathcal{N})$ has a V-filtration with respect to f if and only if all the roots of these b-functions are rational. This is not necessarily the case even if $\mathcal{N}$ is regular holonomic: for example, if $\mathcal{M} = \mathcal{D}_{\mathbf{A}^1}/\mathcal{D}_{\mathbf{A}^1}\cdot(\partial_x x - \lambda)$, for some $\lambda \notin \mathbf{Q}$, and $f = x$, then $\mathcal{M} \simeq j_+(\mathcal{M}|_U)$ and it is easy to see that the b-function of $\overline{1}\otimes\delta$ is $b(s) = (s+\lambda)$.

We note that it is possible to make sense of the notion of V-filtration with respect to f for arbitrary holonomic $\mathcal{D}_X$-modules by allowing filtrations indexed by complex numbers (when $k = \mathbf{C}$). However, we do not pursue this more general version here.

So far, the only examples in which we could show that V-filtrations exist were the ones in which we were able to construct them explicitly. However, once we will prove in Section 8.4 that all roots of the Bernstein-Sato polynomial $b_f(s)$ are rational, we will be able to deduce from Theorem 8.42 that $\mathcal{O}_X$ has a V-filtration with respect to any nonzero $f \in \mathcal{O}_X(X)$.

8.2.1. V-filtrations and nearby and vanishing cycles. V-filtrations have been introduced by Malgrange [**Mal83**] in the case of the $\mathcal{D}$-module $\mathcal{O}_X$ and were generalized by Kashiwara [**Kas83**] to the case of arbitrary holonomic $\mathcal{D}$-modules. The original indexing was by integers, the indexing by rational numbers (when it exists) was introduced by Saito [**Sai84**].

The motivation for introducing the V-filtration came from the theory of vanishing and nearby cycles in the analytic setting. We just give a quick overview of the basic notions involved, for a detailed discussion we refer to [**Sch03**]. Suppose that X is a smooth complex algebraic variety and $f\colon X \to \mathbf{C}$ is a regular function (or, more generally, a holomorphic function). Let $i\colon Z = f^{-1}(0) \hookrightarrow X$ and $j\colon U = X \smallsetminus Z \hookrightarrow X$ be the inclusion maps and consider the Cartesian diagram

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\widetilde{j}"] \arrow[d, "\widehat{\pi}"'] & \widetilde{\mathbf{C}^*} \arrow[d, "\pi"] \\ U \arrow[r, "f"] & \mathbf{C}^* \end{tikzcd},$$

where π is the universal cover of $\mathbf{C}^*$. The *nearby cycle* functor is given by

$$\psi_f\colon \mathcal{D}^b_c(\underline{\mathbf{C}}_X) \to \mathcal{D}^b_c(\underline{\mathbf{C}}_Z), \quad \psi_f(u) = i^*(j\circ\widehat{\pi})_*(j\circ\widehat{\pi})^*(u).$$

We note that since the morphism $\widehat{\pi}$ is not a morphism of algebraic varieties, we are in a different framework than that discussed in Chapter 7.2, hence the fact that $\psi_f(u)$ has constructible cohomology requires different arguments.

The canonical functorial transformation $\mathrm{Id} \to (j\circ\widehat{\pi})_*(j\circ\widehat{\pi})^*$ induces a morphism $\mathrm{sp}\colon i^*(u) \to \psi_f(u)$ (the *specialization morphism*) and $\varphi_f(u)$ is defined by the exact triangle

$$i^*(u) \xrightarrow{\mathrm{sp}} \psi_f(u) \xrightarrow{\mathrm{can}} \varphi_f(u) \xrightarrow{+1}.$$

In fact, one can define an exact functor $\varphi_f\colon \mathcal{D}^b_c(\underline{\mathbf{C}}_X) \to \mathcal{D}^b_c(\underline{\mathbf{C}}_Z)$, the *vanishing cycle* functor, such that the above exact triangle is functorial.

The automorphism of $\widetilde{\mathbf{C}^*}$ over $\mathbf{C}$ induced by a path in $\mathbf{C}^*$ going once counterclockwise around the origin induces an automorphism of $\widetilde{U}$ over U, and we get a functorial automorphism, the *monodromy automorphism* $T\colon \psi_f(u) \to \psi_f(u)$ such that the diagram

$$\begin{tikzcd} i^*(u) \arrow[r, "\mathrm{sp}"] \arrow[d, "\mathrm{Id}"'] & \psi_f(u) \arrow[d, "T"] \\ i^*(u) \arrow[r, "\mathrm{sp}"] & \psi_f(u) \end{tikzcd}$$

is commutative. We thus get an induced automorphism $\widetilde{T}\colon \varphi_f(u) \to \varphi_f(u)$ such that we have a morphism of exact triangles

$$\begin{tikzcd} i^*(u) \arrow[r, "\mathrm{sp}"] \arrow[d, "\mathrm{Id}"'] & \psi_f(u) \arrow[r, "\mathrm{can}"] \arrow[d, "T"'] & \varphi_f(u) \arrow[r, "+1"] \arrow[d, "\widetilde{T}"'] & {} \\ i^*(u) \arrow[r, "\mathrm{sp}"] & \psi_f(u) \arrow[r, "\mathrm{can}"] & \varphi_f(u) \arrow[r, "+1"] & {} \end{tikzcd}.$$

In this setting one defines also a functorial transformation var: $\varphi_f(u) \to \psi_f(u)$ such that $\mathrm{can} \circ \mathrm{var} = \widetilde{T} - \mathrm{Id}$ and $\mathrm{var} \circ \mathrm{can} = T - \mathrm{Id}$.

The nearby cycle functor provides a global and functorial incarnation for the cohomology of the Milnor fiber. Recall that if $x \in Z$, $0 < \epsilon \ll 1$, and $B_\epsilon(x)$ is a ball of radius ϵ with center x (with respect to a system of coordinates centered at x), then the induced map $B_\epsilon(x) \cap f^{-1}\big(B_\delta(0) \smallsetminus \{0\}\big) \to B_\delta(0) \smallsetminus \{0\}$ is a fibration if $0 < \delta \ll \epsilon$; the fiber is the *Milnor fiber* of f at x, denoted by F_x (one can show that this is independent of choices, up to diffeomorphism). Note that associated to this fibration, we have an automorphism of $H^*(F_x, \mathbf{C})$, the *monodromy*, that corresponds to the loop going once counterclockwise around 0 in $B_\delta(0)$. In this case, we have

$$H^i(F_x, \mathbf{C}) \simeq \mathcal{H}^i\big(\psi_f(\underline{\mathbf{C}}_X)\big)_x \quad \text{for every} \quad i \in \mathbf{Z}_{\geq 0},$$

such that the monodromy corresponds to the automorphism induced by T.

It is an important result that if u is a perverse sheaf on X, then $\psi_f(u)[-1]$ and $\varphi_f(u)[-1]$ are again perverse sheaves (viewed either on Z or on X). By the Riemann-Hilbert correspondence, the functor $\psi_f[-1]$ corresponds to a functor on regular holonomic $\mathcal{D}_X$-modules and it is natural to ask for a direct description of this functor. This was achieved by Malgrange [**Mal83**] and Kashiwara [**Kas83**], see also Saito [**Sai88**, Chapter 3.4].

Suppose that $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module that has a V-filtration with respect to f and let $u = \mathrm{DR}_X^{\mathrm{an}}(\mathcal{M})$, so u is a perverse sheaf by Theorem 7.37, and thus $\psi_f(u)[-1]$ and $\varphi_f(u)[-1]$ are perverse sheaves as well. One can show that we have a decomposition $T = T_s \cdot T_u$ on $\psi_f(u)[-1]$ in semisimple and unipotent part and we get a decomposition

$$\psi_f(u)[-1] = \bigoplus_\lambda \psi_{f,\lambda}(u)[-1],$$

where $\psi_{f,\lambda}(u)[-1] = \mathrm{Ker}\big((T - \lambda\mathrm{Id})^N\big)$ for $N \gg 0$. Moreover, each $\psi_{f,\lambda}(u)$ is preserved by the action of T_u. We get similar decompositions $\widetilde{T} = \widetilde{T}_s\widetilde{T}_u$ and $\varphi_f(u)[-1] = \bigoplus_\lambda \varphi_{f,\lambda}(u)[-1]$. Note that for every $\lambda \neq 1$, we have an isomorphism $\psi_{f,\lambda}[-1] \simeq \varphi_{f,\lambda}[-1]$ induced by $\mathrm{can}[-1]$.

The main result here is that each $\mathrm{Gr}_V^\alpha(\iota_+\mathcal{M})$ is a holonomic $\mathcal{D}_X$-module and that we have

$$\psi_f(u)[-1] \simeq \bigoplus_{\alpha \in (0,1]} \mathrm{DR}_X^{\mathrm{an}}\big(\mathrm{Gr}_V^\alpha(\iota_+\mathcal{M})\big) \quad \text{and} \quad \varphi_f(u)[-1] \simeq \bigoplus_{\alpha \in [0,1)} \mathrm{DR}_X^{\mathrm{an}}\big(\mathrm{Gr}_V^\alpha(\iota_+\mathcal{M})\big),$$

such that

$$\psi_{f,\lambda}(u)[-1] \simeq \mathrm{DR}_X^{\mathrm{an}}\big(\mathrm{Gr}_V^\alpha(\iota_+\mathcal{M})\big) \quad \text{and} \quad \varphi_{f,\lambda}(u)[-1] \simeq \mathrm{DR}_X^{\mathrm{an}}\big(\mathrm{Gr}_V^\alpha(\iota_+\mathcal{M})\big),$$

where $\lambda = \exp(-2\pi i\alpha)$, and the actions of T_u and $\widetilde{T}_u$ are obtained after applying $\mathrm{DR}_X^{\mathrm{an}}$ to the map $\exp\big(2\pi i(\partial_t t - \alpha)\big)$. Furthermore, the maps

$$\psi_{f,1}(u)[-1] \to \varphi_{f,1}(u)[-1] \quad \text{and} \quad \varphi_{f,1}(u)[-1] \to \psi_{f,1}(u)[-1]$$

induced by can and var, respectively, are obtained by applying $\mathrm{DR}_X^{\mathrm{an}}$ to the maps

$$\mathrm{Gr}_V^1(\iota_+\mathcal{M}) \xrightarrow{-\partial_t \cdot} \mathrm{Gr}_V^0(\iota_+\mathcal{M}) \quad \text{and} \quad \mathrm{Gr}_V^0(\iota_+\mathcal{M}) \xrightarrow{t \cdot} \mathrm{Gr}_V^1(\iota_+\mathcal{M}).$$

8.3. The Bernstein-Sato polynomial: first properties and examples

Let X be a smooth irreducible variety and $f \in \mathcal{O}_X(X)$ nonzero. Recall that, by definition, $b_f(s) \in k[s]$ is the monic polynomial of minimal degree such that $b_f(s)f^s \in \mathcal{D}_X[s] \cdot f^{s+1}$ in $\mathcal{O}_X[1/f, s]f^s$ (for every $m \in \mathbf{Z}$, it is common to write f^{s+m} for $f^m \cdot f^s$).

Remark 8.49. If $X = U_1 \cup \ldots \cup U_r$ is an open cover and $f_i = f|_{U_i}$, then

$$b_f = \mathrm{lcm}\{b_{f_i} \mid 1 \leq i \leq r\}.$$

This is clear from the definition (and it is a special case of the assertion in Remark 8.38).

Proposition 8.50. If X is affine, with $R = \mathcal{O}_X(X)$, then a polynomial $b(s) \in k[s]$ is divisible by $b_f(s)$ if and only if there is $P \in D_R[s]$ such that

$$b(m)f^m = P(m) \cdot f^{m+1} \text{ in } R_f \tag{8.25}$$

for infinitely many $m \in \mathbf{Z}$ (equivalently, for all $m \in \mathbf{Z}$).

Proof. We only need to show that if (8.25) holds for infinitely many $m \in \mathbf{Z}$, then

$$b(s)f^s = P(s) \cdot f^{s+1} \tag{8.26}$$

(the converse follows by Remark 6.48). Note that if we write $P = \sum_{i=0}^{d} P_i s^i$, for some $P_i \in D_R$, then for every i, we can write

$$P_i \cdot f^{s+1} = Q_i(s) f^s$$

for some $Q_i \in R_f[s]$. Therefore the condition in (8.26) is equivalent to having $\sum_{i=0}^{d} s^i Q_i(s) = b(s)$ in $R_f[s]$. On the other hand, condition (8.25) says precisely that $\sum_{i=0}^{d} m^i Q_i(m) = b(m)$. Since R_f is a domain containing $\mathbf{Q}$ and a nonzero polynomial with coefficients in a domain can have at most finitely many roots, we conclude that if (8.25) holds for infinitely many $m \in \mathbf{Z}$, then in fact $\sum_{i=0}^{d} s^i Q_i(s) = b(s)$. □

Proposition 8.51. If f is not invertible, then $b_f(-1) = 0$.

Proof. Specializing s to -1 in (6.4) (see Remark 6.48), we conclude that

$$b_f(-1)\tfrac{1}{f} \in \mathcal{D}_X \cdot 1 \subseteq \mathcal{O}_X.$$

Since f is not invertible, this implies $b_f(-1) = 0$. □

Definition 8.52. If $f \in \mathcal{O}_X(X)$ is nonzero and noninvertible, it follows from the above proposition that $b_f(s)$ is divisible by $(s+1)$. The *reduced Bernstein-Sato polynomial* of f is $\widetilde{b}_f(s) = b_f(s)/(s+1)$.

Proposition 8.53. The Bernstein-Sato polynomial $b_f(s)$ only depends on the hypersurface defined by f.

Proof. This is a special case of the assertion in Remark 8.24.

□

Example 8.54. If $f \in \mathcal{O}_X(X)$ is invertible, then $b_f = 1$. Indeed, it follows from Proposition 8.53 that we may take $f = 1$. In this case we have $f^m = 1 \cdot f^{m+1}$ for all $m \in \mathbf{Z}$, hence it follows from Proposition 8.50 that b_f divides 1, hence $b_f = 1$.

EXAMPLE 8.55. Suppose that $f \in \mathcal{O}_X(X)$ defines a nonempty smooth hypersurface in X. In this case we have $b_f(s) = s+1$. Indeed, it follows from Remark 8.49 that it is enough to prove the assertion when we have coordinates $x_1, \ldots, x_n$ on X such that $f = x_1$ (note that if $U \subseteq X$ is an open subset such that $f|_U$ is invertible, then $b_{f|_U}(s) = 1$ by Example 8.54). In this case we have

$$\partial_{x_1} \cdot x_1^{s+1} = (s+1)x_1^s,$$

hence $b_f(s)$ divides $(s+1)$. The fact that $b_f(s) = s+1$ now follows from Proposition 8.51.

In general, the Bernstein-Sato polynomial b_f is a measure of the singularities of the hypersurface defined by f. In Section 8.5.1 we will discuss its connection with other invariants of singularities.

DEFINITION 8.56. If Z is a hypersurface in X, we can define the Bernstein-Sato polynomial $b_Z \in k[s]$ as follows. We consider an open cover $X = U_1 \cup \ldots \cup U_r$ such that $Z \cap U_i$ is defined in U_i by $f_i \in \mathcal{O}_X(U_i)$ and put

$$b_Z = \mathrm{lcm}\big\{b_{f_i} \mid 1 \leq i \leq r\big\}.$$

It is an easy consequence of Remark 8.49 and Proposition 8.53 that this definition does not depend on the choice of cover or the choice of the functions f_i. Furthermore, if Z is nonempty, then it follows from Proposition 8.51 that b_Z is divisible by $(s+1)$ and we put $\widetilde{b}_Z = b_Z/(s+1)$.

It is convenient to also consider a local version of the Bernstein-Sato polynomial.

REMARK 8.57. If $f \in \mathcal{O}_X(X)$ is nonzero, then it is clear that for every nonempty open subsets $U \subseteq V$ in X, the polynomial $b_{f|_U}$ divides $b_{f|_V}$. This implies that for any $P \in X$, there is an open neighborhood V of P such that $b_{f|_U} = b_{f|_V}$ for every open neighborhood U of P with $U \subseteq V$. The polynomial $b_{f|_V} \in k[s]$ is the *Bernstein-Sato polynomial of f at P*, denoted $b_{f,P}$. Note that by Proposition 8.51 and Example 8.54, we have $b_{f,P} \neq 1$ if and only if $f(P) = 0$ and in this case $(s+1)$ divides $b_{f,P}$ and we put we put $\widetilde{b}_{f,P}(s) := b_{f,P}(s)/(s+1) \in k[s]$; this is the *reduced Bernstein-Sato polynomial of f at P*. Note that by Proposition 8.49, we have

$$b_f = \mathrm{lcm}\{b_{f,P} \mid P \in X\}.$$

If Z is a hypersurface in X, then we define $b_{Z,P}$ for every $P \in X$ and $\widetilde{b}_{Z,P}$ for every $P \in Z$ in the obvious way.

We next discuss two easy examples. We note that even in such easy examples, while it is easy to show that the Bernstein-Sato polynomial divides a certain polynomial, it is not easy to show that equality holds.

EXAMPLE 8.58. Let $f = x_1^{a_1} \cdots x_r^{a_r} \in k[x_1, \ldots, x_n]$, for some $r \leq n$ and some positive integers $a_1, \ldots, a_r$. Note that we have

$$\partial_1^{a_1} \cdots \partial_r^{a_r} \cdot f^{s+1} = \prod_{i=1}^{r} \prod_{j=1}^{a_i} (a_i s + j) f^s,$$

hence b_f divides $\prod_{i=1}^{r} \prod_{j=1}^{a_i} \left(s + \frac{j}{a_i}\right)$.

EXAMPLE 8.59. Let $f = \sum_{i=1}^n x_i^2 \in k[x_1, \ldots, x_n]$. Note that we have

$$\partial_i \cdot f^{s+1} = 2(s+1)x_i f^s,$$

and thus

$$\partial_i^2 \cdot f^{s+1} = 2(s+1)f^s + 4s(s+1)\tfrac{x_i^2}{f}f^s.$$

It follows that if $\Delta = \sum_{i=1}^n \partial_i^2$, then

$$\Delta \cdot f^{s+1} = (s+1)(4s+2n),$$

and thus b_f divides $(s+1)\big(s+\frac{n}{2}\big)$. In fact, we have equality, but this is not entirely trivial: we will prove this in a more general setting in Theorem 8.61 below.

8.3.1. The Bernstein-Sato polynomial of weighted homogeneous isolated singularities. If X is a smooth, irreducible variety and $f \in \mathcal{O}_X(X)$ is nonzero, then the *Jacobian ideal* J_f of f is defined as follows. If $x_1, \ldots, x_n$ are algebraic coordinates in an open subset U of X, then $J_f|_U$ is generated by $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$. It is easy to see that the definition is independent of the choice of coordinates, hence the definitions glue to give a coherent sheaf of ideals $J_f \subseteq \mathcal{O}_X$. We note that the ideal J_f depends on the choice of equation f and not just on the hypersurface Z defined by f.

Since $f\colon X \to \mathbf{A}^1$ is generically smooth, it follows that in a suitable neighborhood of Z the zero-locus $V(J_f)$ of J_f is contained in Z, and thus the singular locus of Z is equal to $V(J_f)$. We say that $P \in Z$ is an *isolated singular point* if there is an open neighborhood U of P such that $\big(U \smallsetminus \{P\}\big) \cap Z$ is smooth (note that P might be a smooth point of Z, too). In this case either $J_f = \mathcal{O}_X$ in a neighborhood of P (if $P \in Z$ is a smooth point) or, if $x_1, \ldots, x_n$ are algebraic coordinates in a neighborhood of P, the generators $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ of J_f form a regular sequence at P.

One way to approach the description of the Bernstein-Sato polynomial is by describing the ideal $\mathrm{Ann}_{\mathcal{D}_X[s]}(f^s)$. In the next result, we give a description of $\mathrm{Ann}_{\mathcal{D}_X}(f^s)$ in the case of isolated singularities, following [**Yan78**, Theorem 2.19].

PROPOSITION 8.60. With the above notation, if Z has an isolated singular point P and if $x_1, \ldots, x_n$ are algebraic coordinates in a neighborhood of P, then the left ideal

$$\mathrm{Ann}_{\mathcal{D}_X}(f^s) = \{Q \in \mathcal{D}_X \mid Q \cdot f^s = 0\}$$

is generated in a neighborhood of P by $\frac{\partial f}{\partial x_i}\partial_{x_j} - \frac{\partial f}{\partial x_j}\partial_{x_i}$, for $1 \leq i < j \leq n$. In particular, we have

$$\{Q \in \mathcal{D}_X \mid Q \cdot f^s \in \mathcal{D}_X \cdot J_f f^s\} \subseteq \mathcal{D}_X \cdot J_f \quad \text{in} \quad \mathcal{D}_X.$$

PROOF. It is clear that the left ideal I of $\mathcal{D}_X$ generated by $\frac{\partial f}{\partial x_i}\partial_{x_j} - \frac{\partial f}{\partial x_j}\partial_{x_i}$, for $1 \leq i < j \leq n$, is contained in $\mathrm{Ann}_{\mathcal{D}_X}(f^s)$, hence we need to prove the reverse inclusion. The assertion is easy to check when P is a smooth point of Z. Indeed, if $\frac{\partial f}{\partial x_i}(P) \neq 0$ and $Q \in \mathrm{Ann}_{\mathcal{D}_X}(f^s)$, after writing Q modulo the left ideal generated by $\partial_{x_j} - \big(\frac{\partial f}{\partial x_i}\big)^{-1}\frac{\partial f}{\partial x_j}\partial x_i$, we may assume that $Q \in \mathcal{O}_X[\partial_{x_i}]$ and we need to show that $Q = 0$. This follows from the fact that $\partial_{x_i}^m f^s = Q_m f^s$, where $Q_m \in \mathcal{O}_X[1/f, s]$ is a polynomial of degree m in s.

From now on we assume that P is a singular point of Z. After possibly replacing X by a suitable open neighborhood of P, we may assume that X is affine, with

$\mathcal{O}_X(X) = R$, that $x_1, \ldots, x_n$ are defined on X, and that $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ form a regular sequence in R. We need to show that if $Q \in D_R$ is such that $Q \cdot f^s = 0$, then Q lies in I. We argue by induction on the order q of Q, the case $q = 0$ being trivial: in this case it is clear that $Q = 0$. Note that for every $\alpha \in \mathbf{Z}^n_{\geq 0}$, we can write $\partial_x^\alpha f^s = g_\alpha f^{s-|\alpha|}$, where $g_\alpha \in R[s]$ has degree $|\alpha|$ in s, with the coefficient of the top degree term equal to $\prod_i \big(\frac{\partial f}{\partial x_i}\big)^{\alpha_i}$. By Theorem 2.11, we can write $Q = \sum_{|\alpha| \leq q} Q_\alpha \partial_x^\alpha$, with $Q_\alpha \in \mathcal{O}_X$ for all α, and we see that

$$\sum_{|\alpha|=q} Q_\alpha \cdot \prod_{i=1}^n \big(\tfrac{\partial f}{\partial x_i}\big)^{\alpha_i} = 0.$$

Since $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ is a regular sequence in R, it follows that in the ring $R[y_1, \ldots, y_n]$ we can write

$$\sum_{|\alpha|=q} Q_\alpha y_1^{\alpha_1} \cdots y_n^{\alpha_n} = \sum_{i<j} h_{i,j} \big(\tfrac{\partial f}{\partial x_i} y_j - \tfrac{\partial f}{\partial x_j} y_i\big),$$

for some $h_{i,j} \in R[y_1, \ldots, y_n]$ that are homogeneous in $y_1, \ldots, y_n$, of degree $q-1$. If $h_{i,j} = \sum_\beta h_{i,j,\beta} y^\beta$, with $h_{i,j,\beta} \in R$, then the difference

$$\widetilde{Q} = Q - \sum_{i,j,\beta} h_{i,j,\beta} \partial_x^\beta \big(\tfrac{\partial f}{\partial x_i} \partial_{x_j} - \tfrac{\partial f}{\partial x_j} \partial_{x_i}\big) \in D_R$$

has order $q-1$ and $\widetilde{Q} f^s = 0$. We conclude by induction that $\widetilde{Q} \in I$ and thus $Q \in I$, too.

The last assertion in the proposition is clear once we note that

$$\tfrac{\partial f}{\partial x_i} \partial_{x_j} - \tfrac{\partial f}{\partial x_j} \partial_{x_i} = \partial_{x_j} \tfrac{\partial f}{\partial x_i} - \tfrac{\partial^2 f}{\partial x_j \partial x_i} - \partial_{x_i} \tfrac{\partial f}{\partial x_j} + \tfrac{\partial^2 f}{\partial x_i \partial x_j} = \partial_{x_j} \tfrac{\partial f}{\partial x_i} - \partial_{x_i} \tfrac{\partial f}{\partial x_j} \in \mathcal{D}_X \cdot J_f.$$

$\square$

Suppose now that $X = \mathbf{A}^n$, with $\mathcal{O}_X(X) = R = k[x_1, \ldots, x_n]$. A polynomial $f \in R$ is *weighted homogeneous* if the following condition holds: there are $w_1, \ldots, w_n \in \mathbf{Q}_{>0}$ such that if for a monomial $x^u = x_1^{u_1} \cdots x_n^{u_n} \in R$ we put $\rho(x^u) = \sum_{i=1}^n u_i w_i$, then there is d such that $f = \sum_{\rho(x^u)=d} c_u x^u$ (once the w_i are fixed, we will refer to such polynomials as being *w-homogeneous of degree d* and write $\rho(h) = d$). After possibly rescaling all the w_i by the same positive rational number, we may and will assume that $\rho(f) = 1$. We put $|w| := \sum_{i=1}^n w_i$.

A special role in this setting is played by the operator $\theta = \sum_{i=1}^n w_i x_i \partial_i$ and by $\sum_{i=1}^n w_i \partial_i x_i = \theta + |w|$. Note that if h is w-homogeneous, then $\theta(h) = \rho(h) h$, hence an easy computation gives

(8.27)

$$\theta \cdot h f^s = \big(s + \rho(h)\big) h f^s \quad \text{and} \quad (w_1 \partial_1 x_1 + \ldots + w_n \partial_n x_n) \cdot h f^s = \big(s + \rho(h) + |w|\big) h f^s.$$

Note that since $f = \theta(f)$, it follows that $f \in J_f$, hence the zero-locus of $V(J_f)$ is precisely the singular locus of the hypersurface Z defined by f. Furthermore, Z has an isolated singularity at 0 if and only if $V(J_f) \subseteq \{0\}$. Indeed, if m is a positive integer such that $m w_i \in \mathbf{Z}$ for all i, then the k^*-action on X given by $\lambda \cdot (u_1, \ldots, u_n) = (\lambda^{m w_1} u_1, \ldots, \lambda^{m w_n} u_n)$ preserves Z; if P is a singular point of Z different from the origin, then $k^* \cdot P$ is a 1-dimensional subset of the singular locus of Z whose closure contains 0.

From now on we assume that f is w-homogeneous, of degree 1, with an isolated singularity at 0. By the above discussion, we see that R/J_f is a finite-dimensional k-vector space. We put

$$\Sigma(f) = \{\rho(g) \mid g \in R \smallsetminus J_f,\, g \text{ is } w-\text{homogeneous}\}.$$

Note that this is a finite set: if $x^{u_1}, \ldots, x^{u_r}$ are monomials whose classes in R/J_f form a basis, then $\Sigma(f) = \big\{\rho(u_1), \ldots, \rho(u_r)\big\}$. Indeed, the only thing to note is that if $h \not\in J_f$ is w-homogeneous and if we write $h = \sum_{i=1}^r c_i x^{u_i} + g$, where $c_i \in k$ for all i and $g \in J_f$, since J_f is generated by w-homogeneous elements, we may assume that g is w-homogeneous, with $\rho(h) = \rho(g) = \rho(x^{u_i})$ for all i such that $c_i \neq 0$ (note also that some c_i must be nonzero since $g \not\in J_f$).

In this case, the formula for the Bernstein-Sato polynomial is given by the following result:

Theorem 8.61. *If $w_1, \ldots, w_n \in \mathbf{Q}_{>0}$ are such that $f \in R = k[x_1, \ldots, x_n]$ is w-homogeneous, of degree 1, and if f has an isolated singularity at 0, then*

$$b_f(s) = (s+1) \cdot \prod_{\lambda \in \Sigma(f)} \big(s + \lambda + |w|\big). \tag{8.28}$$

Proof. We give an argument following [**BGM86**]. We first show that $b_f(s)$ divides $(s+1) \cdot \prod_{\lambda \in \Sigma(f)} \big(s + \lambda + |w|\big)$, or equivalently, that

$$(s+1) \cdot \prod_{\lambda \in \Sigma(f)} \big(s + \lambda + |w|\big) f^s \in A_n[s] f^{s+1}. \tag{8.29}$$

It is convenient to prove, more generally, that if $h \in R$ is w-homogeneous, then

$$\prod_{\lambda \in \Sigma(f), \lambda \geq \rho(h)} \big(s + \lambda + |w|\big) h f^s \in A_n[s] \cdot J_f f^s. \tag{8.30}$$

If we know this for $h = 1$, then we can write

$$\prod_{\lambda \in \Sigma(f), \lambda \geq 0} \big(s + \lambda + |w|\big) f^s = \sum_{i=1}^n P_i \cdot \tfrac{\partial f}{\partial x_i} f^s$$

for some $P_1, \ldots, P_n \in A_n[s]$, in which case, we have

$$(s+1) \cdot \prod_{\lambda \in \Sigma(f)} \big(s + \lambda + |w|\big) f^s = \sum_{i=1}^n (s+1) P_i \tfrac{\partial f}{\partial x_i} f^s = (P_1 \partial_1 + \ldots + P_n \partial_n) \cdot f^{s+1},$$

hence (8.29) holds.

Note first that (8.30) clearly holds if $h \in J_f$, hence from now on we assume that $h \not\in J_f$ and argue by descending induction on $\rho(h)$. Suppose first that h is such that $\rho(h)$ is the largest element of $\Sigma(f)$, in which case we have $x_i f \in J_f$ for $1 \leq i \leq n$. In this case it follows from (8.27) that we have

$$\big(s + \rho(h) + |w|\big) h f^s = \sum_{i=1}^n w_i \partial_i (x_i h) f^s \in A_n \cdot J_f f^s.$$

Suppose now that $h \not\in J_f$ and we know that (8.30) holds for all h' that are w-homogeneous, with $\rho(h') > \rho(h)$. In particular, it holds for $x_i h$ for all i, hence

$$\prod_{\lambda' \in \Sigma(f), \lambda' > \rho(h)} \big(s + \lambda' + |w|\big) x_i h f^s \in A_n \cdot J_f f^s \quad \text{for} \quad 1 \leq i \leq n.$$

We thus deduce using (8.27) that

$$\prod_{\lambda\in\Sigma(f),\lambda\geq\rho(h)}\big(s+\lambda+|w|\big)hf^s=\prod_{\lambda\in\Sigma(f),\lambda>\rho(h)}\big(s+\lambda+|w|\big)\cdot\big(s+\rho(h)+|w|\big)hf^s$$

$$=\prod_{\lambda\in\Sigma(f),\lambda>\rho(h)}\big(s+\lambda+|w|\big)\cdot\sum_{i=1}^n w_i\partial_i(x_ih)f^s\subseteq A_n[s]\cdot J_ff^s.$$

This completes the proof of the induction step and thus that of (8.29).

We next need to show that $(s+1)\cdot\prod_{\lambda\in\Sigma(f)}\big(s+\lambda+|w|\big)$ divides $b_f(s)$, or equivalently, that for every $h\in R\smallsetminus J_f$ that is w-homogeneous, if $\lambda=\rho(h)$, then $\widetilde{b}_f\big(-\lambda-|w|\big)=0$. By definition of the reduced Bernstein-Sato polynomial, we can write

$$(s+1)\widetilde{b}_f(s)f^s=P\cdot f^{s+1}, \tag{8.31}$$

for some $P\in A_n[s]$. By Theorem 2.11, we may write $P=\sum_{i=1}^n P_i\partial_i+Q$, where $Q\in R[s]$ and $P_i\in A_n[s]$. The equality (8.31) becomes

$$(s+1)\widetilde{b}_f(s)f^s=Q\cdot f^{s+1}+(s+1)\cdot\sum_{i=1}^n P_i\tfrac{\partial f}{\partial x_i}\cdot f^s.$$

We can thus write $Q=(s+1)T$ for some $T\in R[s]$ and we have

$$\widetilde{b}_f(s)f^s=Q_0\cdot f^{s+1}+\sum_{i=1}^n P_i\tfrac{\partial f}{\partial x_i}\cdot f^s\in A_n[s]\cdot J_ff^s,$$

where we use the fact that $f\in J_f$.

Note also that

$$A_n[s]\cdot J_ff^s\subseteq A_n\cdot J_ff^s. \tag{8.32}$$

Indeed, we have $sf^s=\theta f^s$ and $\big[\theta,\frac{\partial f}{\partial x_i}\big]=\theta\big(\frac{\partial f}{\partial x_i}\big)=(1-w_i)\frac{\partial f}{\partial x_i}$ (due to the fact that $\frac{\partial f}{\partial x_i}$ is w-homogeneous with $\rho\big(\frac{\partial f}{\partial x_i}\big)=1-w_i$), hence

$$s\tfrac{\partial f}{\partial x_i}f^s=(\theta-1+w_i)\tfrac{\partial f}{\partial x_i}f^s.$$

We thus conclude that $\widetilde{b}_f(s)f^s\in A_n\cdot J_ff^s$, hence also

$$\widetilde{b}_f(s)hf^s\in A_n\cdot J_ff^s. \tag{8.33}$$

Note now that by (8.27), we have

$$\left(\sum_{i=1}^n w_i\partial_ix_i\right)\cdot hf^s=\big(s+\lambda+|w|\big)hf^s.$$

Let us write $\widetilde{b}_f(s)=\big(s+\lambda+|w|\big)q(s)+a$, for some $q\in k[s]$ and some $a\in k$. Our goal is to show that $a=0$. Since

$$\widetilde{b}_f(s)hf^s=ahf^s+\left(\sum_{i=1}^n w_i\partial_ix_i\right)\cdot q(s)hf^s\in A_n\cdot J_ff^s$$

and since $q(s)f^s\in A_n\cdot f^s$ (this follows using again $sf^s=\theta f^s$), we conclude that there is $P\in A_n$ such that

$$ahf^s+\left(\sum_{i=1}^n w_i\partial_ix_i\right)\cdot Pf^s\in A_n\cdot J_ff^s,$$

in which case, we conclude from Proposition 8.60 that

$$ah+\left(\sum_{i=1}^{n} w_i\partial_i x_i\right)\cdot P\in A_n\cdot J_f.$$

Since we have a direct sum decomposition $A_n=\bigoplus_{\alpha\in\mathbf{Z}^n_{\geq 0}}\partial^\alpha R$ that induces the decomposition $A_n\cdot J_f=\bigoplus_{\alpha\in\mathbf{Z}^n_{\geq 0}}\partial^\alpha J_f$, we conclude that $ah\in J_f$. Since $h\not\in J_f$, it follows that $a=0$, hence $\widetilde{b}_f\big(-\lambda-|w|\big)=0$. This completes the proof of the theorem. $\square$

EXAMPLE 8.62. If $f=\sum_{i=1}^n x_i^{a_i}$, with $a_i\geq 2$ for all i, then

$$J_f=(x_1^{a_1-1},\ldots,x_n^{a_n-1}),$$

so $k[x_1,\ldots,x_n]/J_f$ has a basis given by $x_1^{i_1}\cdots x_n^{i_n}$, with $0\leq i_p\leq a_p-2$ for $1\leq p\leq n$. Note that f is w-homogeneous, with $\rho(f)=1$, where $w_i=\frac{1}{a_i}$ for all i. We thus conclude that

$$\Lambda:=\big\{\lambda+|w|\mid \lambda\in\Sigma(f)\big\}=\big\{\tfrac{i_1}{a_1}+\ldots+\tfrac{i_n}{a_n}\mid 1\leq i_p\leq a_p-1\text{ for }1\leq p\leq n\big\}.$$

By Theorem 8.61, we have $b_f(s)=(s+1)\cdot\prod_{\lambda\in\Lambda}(s+\lambda)$.

8.3.2. An application to complex powers. We next briefly discuss the original application of b-functions to the meromorphic continuation of complex powers. We consider the following setting: suppose that X is a smooth affine complex algebraic variety. We assume that we have coordinates $z_1,\ldots,z_n$ on X and write $dz=dz_1\wedge\ldots\wedge dz_n$. The classical case is that when $X=\mathbf{C}^n$.

Suppose that $f\in\mathcal{O}_X(X)$ is a regular function. For every real function φ on X, which is smooth (that is, $\mathcal{C}^\infty$) and with compact support, and every $s\in\mathbf{C}$ with $\mathrm{Re}(s)>0$, we consider

$$Z_{f,\Phi}(s):=\int_{X^{\mathrm{an}}}|f|^{2s}\Phi\, dzd\overline{z}.$$

Recall that for $a\in\mathbf{R}_{>0}$ and $\lambda\in\mathbf{C}$, we have $a^\lambda=\exp\big(\lambda\cdot\log(a)\big)$. Of course, in the above integral we can ignore those $P\in X^{\mathrm{an}}$ with $f(P)=0$, which form a set of measure 0.

PROPOSITION 8.63. The function $Z_{f,\Phi}$ is well-defined and holomorphic in the half-plane $H_0=\{s\mid \mathrm{Re}(s)>0\}$.

PROOF. Note that if $s\in H_0$, then $||f(x)|^{2s}|=|f(x)|^{2\mathrm{Re}(s)}$, so if $|f(x)|\leq M$ for $x\in\mathrm{Supp}(\Phi)$, we have

$$||f(x)|^{2s}\Phi(x)|\leq M^{2\mathrm{Re}(s)}\mathrm{sup}|\Phi|.$$

Since we integrate on the support of Φ, which is compact, it follows that $Z_{f,\Phi}(s)$ is well-defined for $s\in H_0$.

By Morera's theorem, in order to show that $Z_{f,\Phi}$ is holomorphic in H_0, it is enough to show that $Z_{f,\Phi}$ is continuous on H_0 and for every closed smooth curve γ in H_0, we have $\int_\gamma Z_{f,\Phi}(s)ds=0$. Both these assertions follow easily using Lebesgue's dominated convergence theorem. $\square$

It was a question of I. Gel'fand (ICM, Amsterdam, 1954) whether $Z_{f,\Phi}$ admits a meromorphic extension to $\mathbf{C}$. In fact, one would like to do this uniformly in Φ (more precisely, given any $s_0\in\mathbf{C}$, one would like to find $N=N(s_0)$ such that

$(s-s_0)^N Z_{f,\Phi}$ is holomorphic in a neighborhood of s_0 for all Φ). This is easy to do, using an argument via integration by parts, when f is a monomial $z_1^{a_1}\cdots z_n^{a_n}$. A few years after Hironaka's proof of resolution of singularities in [**Hir64**], an affirmative answer to I. Gel'fand's question was given independently by Bernstein-S. Gel'fand [**BG69**] and Atiyah [**Ati70**], based on Hironaka's result. The idea is to use resolution of singularities and the Change of Variable formula to reduce the assertion to the monomial case.

A second solution to I. Gel'fand's question was given shortly afterwards by Bernstein [**Ber72**], directly extending the integration by parts argument mentioned above, using the existence of the b-function. We sketch the argument in what follows.

By definition of b_f, we have a relation of the form

$$b_f(s)f^s = P\cdot f^{s+1}.$$

Note that if we are in an open subset U of X where a branch of $\log(f)$ is defined, so $f^s=\exp\big(s\cdot\log(f)\big)$ is defined, then we can write

$$b_f(s)\overline{b_f}(s)|f|^{2s} = b_f(s)f^s\cdot\overline{b_f}(s)\overline{f}^s = (P\cdot f^{s+1})\cdot(\overline{P}\cdot\overline{f}^{s+1}) = Q\cdot|f|^{2(s+1)},$$

where $Q=P\cdot\overline{P}$. The last equality follows from the fact that if $P\in\mathcal{O}_X[\partial_{z_1},\ldots,\partial_{z_n}]$ and g and h are holomorphic functions, then $P\cdot(g\cdot\overline{h})=\overline{h}(P\cdot g)$ and $\overline{P}\cdot(g\overline{h})=g(\overline{P}\cdot\overline{h})=g\cdot\overline{P\cdot h}$. Using the Stokes theorem, we thus see that if $s\in H_0$, then we can write

$$b_f(s)\overline{b_f}(s)\cdot Z_{f,\Phi}(s)=\int_{X^{\mathrm{an}}}\big(Q\cdot|f|^{2(s+1)}\big)\cdot\Phi dzd\overline{z}=\int_{X^{\mathrm{an}}}|f|^{2(s+1)}\Psi\, dzd\overline{z}, \tag{8.34}$$

with $\Psi=\widetilde{Q}\cdot\Phi$, where $\widetilde{Q}$ is the adjoint of Q (we use here the fact that Φ has compact support). Note that the right-hand side of (8.34) is in fact holomorphic in the half-space $\big\{s\mid \mathrm{Re}(s)>-1\big\}$ by Proposition 8.63, since Ψ is a smooth function, with compact support. We can now multiply by $b_f(s+1)\overline{b_f}(s+1)$ and repeat. The conclusion is that indeed, $Z_{f,\Phi}$ admits a meromorphic extension such that for every positive integer m, in the half-space $\big\{s\mid\mathrm{Re}(s)>-m\big\}$, the function $\prod_{i=0}^{m-1}b_f(s+i)\overline{b_f}(s+i)\cdot Z_{f,\Phi}(s)$ is holomorphic. In particular, we see that every pole of $Z_{f,\Phi}$ is of the form $\lambda-j$ or $\overline{\lambda}-j$, for some root λ of b_f and some nonnegative integer j (we will see in the next chapter that, in fact, such λ is a negative rational number).

8.4. Rationality of the roots of the Bernstein-Sato polynomial

We fix a smooth, irreducible n-dimensional algebraic variety X over an algebraically closed field k of characteristic 0. Our main goal in this section is to prove the following theorem of Kashiwara [**Kas77**]:

THEOREM 8.64. *If $f\in\mathcal{O}_X(X)$ is nonzero, then all roots of the Bernstein-Sato polynomial $b_f(s)$ are negative rational numbers.*

By combining this result with Theorem 8.42, we obtain the following

COROLLARY 8.65. The $\mathcal{D}_X$-module $\mathcal{O}_X$ has a V-filtration with respect to any nonzero $f\in\mathcal{O}_X(X)$.

In order to relate the roots of the Bernstein-Sato polynomial to other invariants of singularities, it is useful to give an estimate of these roots in terms of a log resolution of the pair (X, H), where H is the hypersurface defined by f. Such an estimate was also given by Kashiwara in [**Kas77**], but the optimal one was given by Lichtin [**Lic89**], using a slight modification of Kashiwara's argument. We begin by reviewing some terminology related to log resolutions.

DEFINITION 8.66. Let X be a smooth variety and H a hypersurface in X. A *log resolution* of (X, H) is a projective morphism $\pi\colon Y \to X$ such that the following conditions hold:

i) π is an isomorphism over $X \smallsetminus H$ (it is often enough to only assume that π is birational, but for us it will be important to put this stronger condition).
ii) Y is a smooth variety.
iii) The divisor $\pi^*(H)$ on Y has simple normal crossings[5].

Log resolutions as above exist by Hironaka's fundamental result. In the setting of the above definition, we will use the following notation: we write $\pi^*(H) = \sum_{i=1}^N a_i E_i$, where the E_i are mutually distinct prime divisors on Y. Note that the largest open subset U of X with the property that π is an isomorphism over U satisfies $\mathrm{codim}_X(X \smallsetminus U) \geq 2$ (this is a consequence of the valuative criterion for properness, since X is normal and π is proper). It follows that if Z is a prime divisor on X, then $Z \cap U \neq \emptyset$; the *strict transform* $\widetilde{Z}$ of Z is $\overline{\pi^{-1}(Z \cap U)}$. Note that a prime divisor E on Y either intersects $\pi^{-1}(U)$ (in which case it is the strict transform of a prime divisor on X) or satisfies $\dim\big(\pi(E)\big) \leq n-2$ (in which case we say that E is an *exceptional* divisor). In our setting, we see that every exceptional divisor is an irreducible component of $\pi^*(H)$; moreover, the irreducible components of $\pi^*(H)$ that are not exceptional are strict transforms of the irreducible components of H.

We will also consider the *relative canonical divisor* $K_{Y/X}$. This is defined as follows: the canonical morphism $\pi^*(\Omega_X) \to \Omega_Y$ induces a morphism of line bundles $\pi^*(\omega_X) \to \omega_Y$ and $K_{Y/X}$ is the effective divisor defined by the corresponding (nonzero) section of $\omega_Y \otimes_{\mathcal{O}_Y} \pi^*(\omega_X)^{-1}$. If π is an isomorphism over U, then it is clear that $\mathrm{Supp}(K_{Y/X}) \subseteq \pi^{-1}(X \smallsetminus U)$. In particular, we see that we can write $K_{Y/X} = \sum_{i=1}^N k_i E_i$. Moreover, if E_i is not exceptional, then $k_i = 0$. One can also show that if E_i is an exceptional divisor, then $k_i > 0$; this is a consequence of Zariski's Main Theorem that we leave as an exercise for the interested reader since we will not not need it.

If H is defined by $f \in \mathcal{O}_X(X)$ and $\pi\colon Y \to X$ is a log resolution of (X, H) as above, then it was shown in [**Kas77**] that every root of b_f is of the form $-\frac{\ell}{a_i}$ for some i and some positive integer ℓ, while the result in [**Lic89**] says that it is of the form $-\frac{k_i+\ell}{a_i}$ for some i and some positive integer ℓ. In fact, we will prove a similar result for more general elements of $\mathcal{O}_X\big[\frac{1}{f}, s\big]f^s \simeq \iota_+\big(\mathcal{O}_X[1/f]\big)$, following [**DM22**]. Recall that $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ is the graph embedding given by $\iota(x) = \big(x, f(x)\big)$. While the more general assertions are more technical, we will use them in the next section to relate the Bernstein-Sato polynomial and the V-filtration to other

[5]We have defined this notion for reduced divisors in Chapter 7.4.1. An arbitrary effective divisor D on the smooth variety Y has simple normal crossings if D_{red} does; explicitly, around every point on Y, there are algebraic coordinates $y_1, \ldots, y_n$ such that D is defined by $\prod_{i=1}^n y_i^{a_i}$ for some nonnegative integers $a_1, \ldots, a_n$.

invariants of singularities. We are interested in the b-function of elements of the form $g\partial_t^m\delta \in \iota_+(\mathcal{O}_X)$. The result that we prove is the following:

THEOREM 8.67. *Let X be a smooth, irreducible algebraic variety and $f \in \mathcal{O}_X(X)$ nonzero, defining the hypersurface H. Given a log resolution $\pi\colon Y \to X$ of (X,H), if we write*

$$\pi^*(H) = \sum_{i=1}^N a_i E_i \quad \textit{and} \quad K_{Y/X} = \sum_{i=1}^N k_i E_i,$$

then the following hold:

i) *Every root of b_f is of the form $-\frac{k_i+\ell}{a_i}$, for some i with $1 \le i \le N$ and some positive integer ℓ. In particular, for every $u \in \iota_+\big(\mathcal{O}_X[1/f]\big)$, all roots of b_u are rational numbers.*

   From now on, suppose also that m is a nonnegative integer and $g \in \mathcal{O}_X(X)$ is nonzero. We denote by b_i the coefficient of E_i in $\pi^*\big(\mathrm{div}(g)\big)$.

ii) *Every root of $b_{g\partial_t^m\delta}$ is $\le -\min\big\{1, \frac{k_i+b_i+1}{a_i} - m \mid 1 \le i \le N\big\}$.*

iii) *Every root of $b_{g\delta}$ is $\le -\min\big\{\frac{k_i+b_i+1}{a_i} \mid 1 \le i \le N\big\}$.*

iv) *Every root of $b_{\partial_t^m\delta}$ is either a negative integer or it is of the form $m-\frac{k_i+\ell}{a_i}$, for some i with $1 \le i \le N$ and some positive integer ℓ. Furthermore, if H is reduced and the strict transforms of the components of H on X are disjoint*[6]*, then we may take i such that the divisor E_i is exceptional.*

We note that the second assertion in i) is a consequence of the first one: indeed, the fact that b_f has rational roots implies, by Theorem 8.42, that we have a V-filtration on $\iota_+(\mathcal{O}_X)$, and thus on $\iota_+\big(\mathcal{O}_X[1/f]\big)$ by Corollary 8.47. Another application of Theorem 8.42 leads to the desired conclusion.

8.4.1. Preliminary results I: The SNC case. The argument for the proof of Theorem 8.67 has two parts. On one hand, we need to treat the case when f defines a simple normal crossing divisor (SNC, for short). On the other hand, we need to relate the setup on the log resolution to the original setup on our variety X. In this section we treat the SNC case. We work in the following setup: X is a smooth, irreducible, n-dimensional variety, and $f \in \mathcal{O}_X(X)$ is a nonzero regular function. We recall that we have a canonical isomorphism $\mathcal{O}_X[1/f,s]f^s \simeq \iota_+\big(\mathcal{O}_X[1/f]\big)$ that maps f^s to δ (see Proposition 8.35) and we have $\iota_+(\mathcal{O}_X) \subseteq \iota_+\big(\mathcal{O}_X[1/f]\big)$. Note that for an element $u \in \iota_+(\mathcal{O}_X)$, the b-function b_u does not depend on whether we consider u as an element of $\iota_+(\mathcal{O}_X)$ or of $\iota_+\big(\mathcal{O}_X[1/f]\big)$.

We begin with two general results. The first one is an extension of Proposition 8.51 when we have an auxiliary function g.

PROPOSITION 8.68. If $f,g \in \mathcal{O}_X(X)$ are nonzero, $g/f \notin \mathcal{O}_X(X)$, and $u = g\delta \in \iota_+(\mathcal{O}_X)$, then $(s+1)$ divides $b_u(s)$.

We note that the existence of b_u is guaranteed by Theorem 6.45.

PROOF. Let $U \subseteq X$ be an affine open subset such that $g/f \notin \mathcal{O}_X(U)$. By Remark 8.38, we may replace X by U and thus assume that $X = \mathrm{Spec}(R)$ is affine. Using Proposition 8.35 and the definition of $b_u(s)$, we can write

$$b_u(s)gf^s = P(s)\cdot fgf^s$$

[6]This condition can always be achieved after performing finitely many blow-ups.

in $R[1/f, s]f^s$, for some $P \in D_R[s]$. After specializing to $s = -1$ (see Remark 6.48), we obtain

$$b_u(-1)\tfrac{g}{f} = P(-1) \cdot g \in R,$$

hence $b_u(-1) = 0$. $\square$

In the setting of Lemma 8.68, we write $\widetilde{b}_u(s) = b_u(s)/(s+1)$. We next prove a result that relates the b-functions of $g\delta$ and $g\partial_t^m\delta$.

Proposition 8.69. With the notation in Proposition 8.68, for every nonnegative integer m, the b-function $b_{g\partial_t^m\delta}$ exists and we have

$$b_{g\partial_t^m\delta} | (s+1)\widetilde{b}_{g\delta}(s-m).$$

Proof. The case $m = 0$ is clear, hence from now on we assume $m \geq 1$. We put $\widetilde{b} = \widetilde{b}_{g\delta}(s)$. By Remark 8.38, it is enough to show that for every affine open subset $U \simeq \mathrm{Spec}(R)$ of X, the b-function of $g\partial_t^m\delta|_U$ divides $(s+1)\widetilde{b}(s-m)$. After replacing X by U, we may thus assume that there is $P \in D_R[s]$ such that

$$(s+1)\widetilde{b}(s)g\delta = P \cdot gf\delta.$$

Since $s + 1 = -t\partial_t$ and $P(s) \cdot gf\delta = P(s)t \cdot g\delta = tP(s-1)g\delta$ by Lemma 8.2, we have

$$-t\partial_t\widetilde{b}(s)g\delta = tP(s-1) \cdot g\delta.$$

Since the action of t on $\iota_+(\mathcal{O}_X)$ is injective (see Remark 8.30), we deduce that

$$\partial_t\widetilde{b}(s)g\delta = -P(s-1) \cdot g\delta.$$

Using again Lemma 8.2, we have

$$(s+1)\widetilde{b}(s-m)g\partial_t^m\delta = (s+1)\partial_t^m\widetilde{b}(s)gf^s = -(s+1)\partial_t^{m-1}P(s-1) \cdot g\delta$$
$$= t\partial_t^m P(s-1) \cdot g\delta = P(s-m)t \cdot \partial_t^m g\delta.$$

Since $P(s-m)t \in V^1\mathcal{D}_{X\times\mathbf{A}^1}$, we conclude that $b_{g\partial_t^m\delta}$ exists and divides the polynomial $(s+1)\widetilde{b}(s-m)$. $\square$

Remark 8.70. We will show in Theorem 8.99 below that when $g = 1$, the divisibility in the above proposition is, in fact, an equality.

We next consider the case of the b-function $b_{g\partial_t^m\delta}$ when both f and g are given by monomials. This extends the computation in Example 8.58.

Proposition 8.71. Suppose that we have algebraic coordinates $x_1, \ldots, x_n$ on X and $f = v\prod_{i=1}^n x_i^{a_i}$ and $g = w\prod_{i=1}^n x_i^{b_i}$ for nonnegative integers $a_1, \ldots, a_n, b_1, \ldots, b_n$ and invertible functions $v, w \in \mathcal{O}_X(X)$. If $u = g\partial_t^m\delta$, for a nonnegative integer m, then b_u exists and the following assertions hold:

i) If $m = 0$, then b_u divides $\prod_{i=1}^n\prod_{\ell=1}^{a_i}\left(s + \frac{b_i+\ell}{a_i}\right)$.

ii) For every m, the polynomial b_u divides $(s+1)\cdot\prod_{i=1}^n\prod_{\ell=1}^{a_i}\left(s - m + \frac{b_i+\ell}{a_i}\right)$.

iii) If $a_1 = 1$ and $b_1 = 0$, then b_u divides $(s+1)\cdot\prod_{i=2}^n\prod_{\ell=1}^{a_i}\left(s - m + \frac{b_i+\ell}{a_i}\right)$.

PROOF. Note first that we may and will assume that $v = w = 1$: this is a consequence of the way the b-function changes when scaling f by an invertible function (see Remark 8.24) and of the fact that $b_u = b_{hu}$ if h is an invertible function. Writing $gf^{s+1} = \prod_{i=1}^n x_i^{a_i s + a_i + b_i}$, we are led to the formula

$$\partial_1^{a_1} \cdots \partial_n^{a_n} \cdot gf^{s+1} = \prod_{i=1}^{n} \prod_{\ell=1}^{a_i} (a_i s + b_i + \ell) gf^s,$$

which is easy to check. This gives the assertion in i).

In order to prove the assertion in ii), note that by using Lemma 8.2, we can write

$$(s+1)\partial_t^m t = -t\partial_t^{m+1} t = t\partial_t^m s = t(s-m)\partial_t^m = (s-m+1)t\partial_t^m.$$

We thus deduce that if $b(s)g\delta = P(s) \cdot tg\delta$, for some $P \in \mathcal{D}_X$, then

$$\begin{aligned}(s+1)b(s-m)u = (s+1)\partial_t^m b(s)g\delta &= P(s-m)(s+1)\partial_t^m tg\delta \\ &= (s-m+1)P(s-m)t\partial_t^m g\delta \in V^1\mathcal{D}_{X\times\mathbf{A}^1}u.\end{aligned}$$

Therefore the assertion in ii) follows from that in i).

Suppose now that we are in the setting of iii). Note that in this case $g/f \notin \mathcal{O}_X(X)$, hence Proposition 8.68 applies, and we conclude from i) that $\widetilde{b}_{g\delta}$ divides $\prod_{i=2}^n \prod_{\ell=1}^{a_i} \left(s + \frac{b_i+\ell}{a_i}\right)$. The assertion in iii) is then a consequence of Proposition 8.69. $\square$

For every nonzero $f \in \mathcal{O}_X(X)$, we put $\mathcal{N}_{f,m} := \mathcal{D}_X\langle s,t\rangle \cdot \partial_t^m \delta \subseteq \iota_+(\mathcal{O}_X)$. We note that if f is nonconstant, then by generic smoothness we may replace X by an open neighborhood of the hypersurface H defined by f to assume that the induced morphism $X \smallsetminus H \to \mathbf{A}^1 \smallsetminus \{0\}$ is smooth. Equivalently, we have $V(J_f) \subseteq \mathrm{Supp}(H)$, where J_f is the Jacobian ideal of f. Under this assumption, we also consider the subvariety W_f of T^*X which is the closure of

$$W_f^\circ := \big\{(P, \lambda\, df(P) \mid P \in X \smallsetminus H, \lambda \in k\big\}.$$

We note that by our assumption, $df(P) \neq 0$ for every $P \in X \smallsetminus H$. It is clear that W_f° is a rank 1 geometric subbundle of $T^*X|_{X\smallsetminus H}$, and thus W_f is an irreducible $(n+1)$-dimensional subvariety of T^*X that dominates X.

PROPOSITION 8.72. Let $f \in \mathcal{O}_X(X)$ define the nonempty hypersurface H in X such that $V(J_f) \subseteq \mathrm{Supp}(H)$ and let m be a nonnegative integer. If H has simple normal crossings, then the following hold:

i) The $\mathcal{D}_X$-module $\mathcal{N}_{f,m}$ is coherent and $\mathrm{Char}(\mathcal{N}_{f,m}) = W_f$. In particular, $\mathcal{N}_{f,m}$ is subholonomic[7] and $\mathcal{N}_{f,m}/t \cdot \mathcal{N}_{f,m}$ is holonomic.

ii) If $\pi\colon T^*X \to X$ is the canonical projection, then $W_f \cap \pi^{-1}(H)$ is an isotropic subvariety of $T^*(X)$.

PROOF. The proof is elementary, though somewhat tedious. Both assertions can be checked locally, hence we may and will assume that we have algebraic coordinates $x_1, \ldots, x_n$ on X such that $f = ug$, where $u \in \mathcal{O}_X(X)$ is invertible and $g = \prod_{i=1}^n x_i^{a_i}$. We assume that the coordinates are indexed such that $a_i > 0$ if and only if $i \leq r$, so $\mathrm{Supp}(H) = V(x_1 \ldots x_r)$. On T^*X we consider the corresponding coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$.

[7] This means that $\dim(\mathcal{N}_{f,m}) = n + 1$.

We begin by treating the case $m = 1$, when we simply write $\mathcal{N}_f$ for $\mathcal{N}_{f,0} = \mathcal{D}_X[s] \cdot f^s$. We first note that $\mathcal{N}_f$ is not holonomic. Indeed, we have the following decreasing sequence of $\mathcal{D}_X$-modules:

$$\mathcal{N}_f \supseteq t\mathcal{N}_f \supseteq \ldots t^q\mathcal{N}_f \supseteq \ldots.$$

If $\mathcal{N}_f$ is holonomic, then the above sequence is stationary by Proposition 6.37, hence there is $q \geq 0$ such that $t^q\mathcal{N}_f = t^{q+1}\mathcal{N}_f$. Since the action of t on $\iota_+(\mathcal{O}_X)$ is injective, we conclude that $\mathcal{N}_f = t\mathcal{N}_f$, hence $\delta \in V^1\mathcal{D}_{X\times\mathbf{A}^1} \cdot \delta$, so $b_f(s) = 1$. Since we assume that f is not invertible, this contradicts Proposition 8.51. Since W_f is irreducible, of dimension $n+1$, if we show that $\mathrm{Char}(\mathcal{N}_f) \subseteq W_f$, then in fact we have equality (otherwise $\mathcal{N}_f$ would be holonomic). Note that for every i, we have

$$x_i\tfrac{\partial f}{\partial x_i} = \big(x_i\tfrac{\partial u}{\partial x_i} + a_iu\big) \cdot g. \tag{8.35}$$

We put $h_i = x_i\frac{\partial u}{\partial x_i} + a_iu$ for $i \leq r$ and $h_i = \frac{\partial u}{\partial x_i}$ for $i > r$. Note first that

$$V\big(h_i \mid 1 \leq i \leq n\big) = \emptyset. \tag{8.36}$$

Indeed, if $h_i(P) = 0$ for all i, then it follows from (8.35) that $x_i\frac{\partial f}{\partial x_i}(P) = 0$ for $i \leq r$ and $\frac{\partial f}{\partial x_i}(P) = 0$ for all $i > r$. Since $V(J_f) \subseteq \mathrm{Supp}(H)$, it follows that there is $i \leq r$ such that $x_i(P) = 0$; then $h_i(P) = 0$ implies $a_iu(P) = 0$, a contradiction.

We show that $\mathcal{N}_f$ is coherent and $\mathrm{Char}(\mathcal{N}_f) \subseteq W_f$ separately on each $U_i = (h_i \neq 0)$. Suppose first that $i \leq r$. It follows from (8.35) that

$$x_j\partial_{x_j}f^s = sh_ju^{-1}f^s \quad \text{for} \quad j \leq r \text{ and } \quad \partial_{x_k}f^s = sh_ku^{-1}f^s \quad \text{for } k > r. \tag{8.37}$$

In particular, we have $s - h_i^{-1}ux_i\partial_{x_i} \in \mathrm{Ann}(f^s)$, hence $\mathcal{N}_f|_{U_i}$ is generated over $\mathcal{D}_{U_i}$ by f^s, and thus it is coherent. Moreover, all

$$x_j\partial_{x_j} - h_jh_i^{-1}x_i\partial_{x_i} \quad \text{for} \quad j \leq r \quad \text{and} \quad \partial_{x_k} - h_kh_i^{-1}x_i\partial_{x_i} \quad \text{for} \quad k > r$$

lie in $\mathrm{Ann}(f^s)$. We thus conclude that $\mathrm{Char}(\mathcal{N}_f|_{U_i}) \subseteq W_i$, where W_i is the subvariety of T^*U_i defined by the equations $x_jy_j - h_jh_i^{-1}x_iy_i$ for $j \leq r$, with $j \neq i$, and $y_k - h_kh_i^{-1}x_iy_i$ for $k > r$. Note that no irreducible component of W_i is contained in $\pi^{-1}(H)$. Indeed, if a general point (P,Q) on such a component satisfies $x_j(P) = 0$ for some $j \leq r$, then $h_j(P) \neq 0$ (since $a_ju(P) \neq 0$), and thus $x_i(P)y_i(Q) = 0$, which in turn implies $x_\ell(P)y_\ell(Q) = 0$ for all $\ell \leq r$ and $y_k(Q)$ for all $k > r$. Therefore this irreducible component of W_i has dimension $\leq n$, contradicting the fact that W_i is defined by $n-1$ equations.

Since no irreducible component of W_i is contained in $\pi^{-1}(H)$, in order to show that $W_i \subseteq W_f$, it is enough to show that $W_i \cap \pi^{-1}(X \smallsetminus H) \subseteq W_f^\circ$. This is easy: it follows from the definition of W_f° that it is defined in $\pi^{-1}(X \smallsetminus H)$ by

$$\tfrac{\partial f}{\partial x_i}y_j - \tfrac{\partial f}{\partial x_j}y_i = 0 \quad \text{for all} \quad 1 \leq i,j \leq n.$$

The fact that $W_i \cap \pi^{-1}(X \smallsetminus H) \subseteq W_f^\circ$ follows directly from the equations of W_i and the fact that

$$h_jh_i^{-1} = \tfrac{x_j\cdot\partial f/\partial x_j}{x_i\cdot\partial f/\partial x_i} \quad \text{for} \quad j \leq r \quad \text{and} \quad h_kh_i^{-1} = \tfrac{\partial f/\partial x_k}{x_i\cdot\partial f/\partial x_i} \quad \text{for} \quad k > r.$$

Moreover, we have seen that

$$W_i \cap \pi^{-1}(H \cap U_i) \subseteq \bigcap_{i=1}^{r} V(x_iy_i) \cap \bigcap_{i=r+1}^{n} V(y_i)$$

and the irreducible components of the right-hand side are smooth, isotropic subvarieties of T^*U_i. Therefore $W_i \cap \pi^{-1}(H \cap U_i)$ is an isotropic subvariety of T^*U_i.

We argue similarly on a subset U_i with $i > r$. Using again (8.37), we see that $s - h_i^{-1}u\partial_{x_i} \in \mathrm{Ann}(f^s)$, hence $\mathcal{N}_f|_{U_i}$ is generated over $\mathcal{D}_{U_i}$ by f^s, so $\mathcal{N}_f|_{U_i}$ is coherent. Furthermore, we see that $\mathrm{Char}(\mathcal{N}_f|_{U_i})$ is contained in the subvariety W_i of T^*U_i defined by the equations $x_jy_j - h_jh_i^{-1}y_i$, for $j \leq r$, and $y_k - h_kh_i^{-1}y_i$, for $k > r$, $k \neq i$. As before, we see that if for a general point (P, Q) on an irreducible component of W_i we have $x_j(P) = 0$ for some $j \leq r$, then $h_j(P) \neq 0$, $y_i(Q) = 0$, and then $x_\ell(P)y_\ell(Q) = 0$ for all $\ell \leq r$ and $y_k(Q) = 0$ for all $k > r$. We thus have an irreducible component of a subset cut out by $n - 1$ equations which has codimension $\geq n$, a contradiction. The fact that also in this case we have $W_i \subseteq W_f$ and $W_i \cap \pi^{-1}(H \cap U_i)$ is isotropic now follows as before.

We thus conclude that $\mathrm{Char}(\mathcal{N}_f) \subseteq W_f$; as we have seen, this must be an equality. Moreover, the above argument also shows that for every i we have $W_i = W_f \cap \pi^{-1}(U_i)$, which is an isotropic subvariety of T^*U_i, hence the assertion in ii) holds.

We next prove the first assertion in i) for $m \geq 1$. Note that $t \cdot \partial_t^j\delta = f\partial_t^j\delta - j\partial_t^{j-1}\delta$, hence it follows by descending induction on j that $\partial_t^j\delta \in \mathcal{N}_{f,m}$ for $0 \leq j \leq m$. Therefore we have

$$\mathcal{N}_{f,m} = \mathcal{D}_X[s] \cdot \{\partial_t^j\delta \mid 0 \leq j \leq m\}.$$

Since $P(s) \cdot \partial_t^j\delta = \partial_t^jP(s+m)\delta$ and since we have already seen that $\mathcal{N}_f$ is generated over $\mathcal{D}_X$ by 1, it follows that $\mathcal{N}_f = \mathcal{D}_X \cdot \{\partial_t^j\delta \mid 0 \leq j \leq m\}$. Therefore we have an exact sequence of $\mathcal{D}_X$-modules

$$0 \to \mathcal{N}_{f,m-1} \to \mathcal{N}_{f,m} \to \overline{\mathcal{N}}_{f,m} \to 0,$$

and $\overline{\mathcal{N}}_{f,m}$ is generated over $\mathcal{D}_X$ by $\overline{\partial_t^m\delta}$. If $P \in \mathcal{D}_X$ is such that $P \cdot \delta = 0$, then $P \cdot \partial_t^m\delta = \partial_t^mP \cdot \delta = 0$, hence we have a surjective morphism of $\mathcal{D}_X$-modules $\mathcal{N}_f \to \overline{\mathcal{N}}_{f,m}$, given by $P \cdot \delta \mapsto P \cdot \overline{\partial_t^m\delta}$. We conclude using Proposition 3.33 that

$$\mathrm{Char}(\mathcal{N}_{f,m}) = \mathrm{Char}(\mathcal{N}_{f,m-1}) \cup \mathrm{Char}\big(\overline{\mathcal{N}}_{f,m}\big) \subseteq \mathrm{Char}(\mathcal{N}_{f,m-1}) \cup \mathrm{Char}(\mathcal{N}_f),$$

and we obtain $\mathrm{Char}(\mathcal{N}_{f,m}) = W_f$ by induction on m.

It is now clear that $\mathcal{N}_{f,m}$ is a subholonomic $\mathcal{D}_X$-module. Since t acts injectively on $\iota_+(\mathcal{O}_X)$, multiplication by t induces an isomorphism $\mathcal{N}_{f,m} \simeq t \cdot \mathcal{N}_{f,m}$. Using the exact sequence

$$0 \to t \cdot \mathcal{N}_{f,m} \to \mathcal{N}_{f,m} \to \mathcal{N}_{f,m}/t \cdot \mathcal{N}_{f,m} \to 0,$$

by considering the multiplicities of the characteristic varieties at the generic point of W_f, we conclude that all components of $\mathrm{Char}(\mathcal{N}_{f,m}/t \cdot \mathcal{N}_{f,m})$ are proper subvarieties of W_f, hence have dimension $\leq n$. Therefore $\mathcal{N}_{f,m}/t \cdot \mathcal{N}_{f,m}$ is a holonomic $\mathcal{D}_X$-module. This completes the proof of i). ☐

8.4.2. Preliminary results II: general behavior of the direct image functor. We have seen in Theorem 6.54 that if $\pi\colon Y \to X$ is a proper morphism of smooth, irreducible varieties and $\mathcal{M}$ is a coherent $\mathcal{D}_Y$-module, then $\mathcal{H}^i\big(\pi_+(\mathcal{M})\big)$ is a coherent $\mathcal{D}_X$-module for every $i \in \mathbf{Z}$. The following important result of Kashiwara, see [**Kas77**, Theorem 4.2], gives an estimate for the characteristic varieties of these $\mathcal{D}_X$-modules in terms of the characteristic variety of $\mathcal{M}$. Let $\alpha_\pi\colon \pi^*(T^*X) \to T^*Y$ be the morphism induced by $\pi^*(\Omega_X) \to \Omega_Y$ and let $\beta_\pi\colon \pi^*(T^*X) \to T^*X$ be the morphism corresponding to π via base-change.

THEOREM 8.73. *If $\pi\colon Y\to X$ is a proper morphism of smooth, irreducible varieties and $\mathcal{M}$ is a coherent $\mathcal{D}_Y$-module, with $V=\mathrm{Char}(\mathcal{M})\subseteq T^*Y$, then for every $i\in\mathbf{Z}$, we have*

$$\mathrm{Char}\big(\mathcal{H}^i(\pi_+(\mathcal{M}))\big)\subseteq\beta_\pi\big(\alpha_\pi^{-1}(V)\big).$$

We do not give the proof of this theorem that requires different tools (though hopefully the proof will be included in a later version of these notes). The argument in [**Kas77**] makes use of microdifferential operators. An algebraic argument, using the push-forward for graded modules over Rees rings of the form $\bigoplus_{m\geq 0}F_m\mathcal{D}_Xz^m$, was given by Laumon [**Lau85**]. A nice outline of this argument can be found in [**Sab11**, Chapter 3.5].

LEMMA 8.74. *If $\pi\colon Y\to X$ is a morphism of smooth, irreducible varieties and γ_X and γ_Y are the canonical symplectic forms on T^*X and T^*Y, respectively, then $\alpha_\pi^*(\gamma_Y)=\beta_\pi^*(\gamma_X)$.*

PROOF. Since the assertion can be checked locally, we may and will assume that we have algebraic coordinates $x_1,\ldots,x_n$ on X and $y_1,\ldots,y_m$ on Y. We get corresponding isomorphisms $T^*X\simeq X\times\mathbf{A}^n$ and $T^*Y\simeq Y\times\mathbf{A}^m$, giving coordinates $x_1,\ldots,x_n,u_1,\ldots,u_n$ on T^*X and $y_1,\ldots,y_m,v_1,\ldots,v_m$ on T^*Y. We also have an isomorphism $\pi^*(T^*X)\simeq Y\times\mathbf{A}^n$, so on $\pi^*(T^*X)$ we have coordinates $y_1,\ldots,y_m,u_1,\ldots,u_n$ (note that, with a slight abuse of notation, we sometimes denote by the same letter a function and its pull-back). Let $\pi_i=x_i\circ\pi$ for $1\leq i\leq n$. We have

$$\beta_\pi^*(\gamma_X)=-\sum_{i=1}^n\beta_\pi^*(du_i\wedge dx_i)=-\sum_{i=1}^n\sum_{j=1}^m\tfrac{\partial\pi_i}{\partial y_j}du_i\wedge dy_j.$$

On the other hand, α_π is given by

$$\alpha_\pi(P,\lambda_1,\ldots,\lambda_n)=\big(P,\sum_i\tfrac{\partial\pi_i}{\partial y_1}(P)\lambda_i,\ldots,\sum_i\tfrac{\partial\pi_i}{\partial y_m}(P)\lambda_i\big),$$

hence $v_j\circ\alpha_\pi=\sum_{i=1}^n\frac{\partial\pi_i}{\partial y_j}u_i$, so

$$\alpha_\pi^*(dv_j)=\sum_{i=1}^n\sum_{k=1}^m\tfrac{\partial^2\pi_i}{\partial y_j\partial y_k}u_idy_k+\sum_{i=1}^n\tfrac{\partial\pi_i}{\partial y_j}du_i.$$

Therefore we have

$$\alpha_\pi^*(\gamma_Y)=-\sum_{j=1}^m\alpha_\pi^*(dv_j\wedge dy_j)$$

$$=-\sum_{i=1}^n\sum_{j,k=1}^m\tfrac{\partial^2\pi_i}{\partial y_j\partial y_k}v_idy_k\wedge dy_j-\sum_{i=1}^n\sum_{j=1}^m\tfrac{\partial\pi_i}{\partial y_j}du_i\wedge dy_j$$

$$=-\sum_{i=1}^n\sum_{j=1}^m\tfrac{\partial\pi_i}{\partial y_j}du_i\wedge dy_j=\beta_\pi^*(\gamma_X).$$

$\square$

COROLLARY 8.75. With the notation in Lemma 8.74, if $V\subseteq T^*Y$ is an isotropic subvariety, then $\beta_\pi\big(\alpha_\pi^{-1}(V)\big)$ is an isotropic subvariety of T^*X.

Proof. Recall that V is an isotropic subvariety of T^*Y if for every smooth point $y \in V$, the subspace $T_yV \subseteq T_yY$ is isotropic with respect to the standard symplectic form γ_Y of Y (that is, we have $\gamma_Y(v_1, v_2) = 0$ for all $v_1, v_2 \in T_yV$). In fact, since this is a closed condition, it is enough to check it for general smooth points of each irreducible component of V.

The main observation is that in this case every closed subvariety Z of V is isotropic. In order to see this, we may assume that both Z and V are irreducible and it is enough to show that for $P \in Z$ general, T_PZ lies in the closure of the tangent bundle of the smooth locus $V_{\rm sm}$ of V. To see this, let $g\colon \widetilde{V} \to V$ be a resolution of singularities that is an isomorphism over $V_{\rm sm}$ and let $\widetilde{Z} \subseteq \widetilde{V}$ be a closed irreducible subvariety such that $g(\widetilde{Z}) = Z$. By Generic Smoothness, we can find nonempty smooth open subsets $\widetilde{Z}_0 \subseteq \widetilde{Z}$ and $Z_0 \subseteq Z$ such that the induced morphism $\widetilde{Z}_0 \to Z_0$ is smooth and surjective. For every $Q \in \widetilde{Z}_0$, it is clear that $T_Q\widetilde{Z}$ lies in the closure of the tangent bundle of $g^{-1}(V_{\rm sm})$, hence for every $P \in Z_0$, the tangent space T_PZ lies in the closure of the tangent bundle of $V_{\rm sm}$.

We can now prove the assertion in the corollary. Note first that $\beta_\pi\big(\alpha_\pi^{-1}(V)\big)$ is a closed subvariety of T^*X: this follows from the fact that π being proper, β_π is proper, too. Let W be an irreducible component of $\beta_\pi\big(\alpha_\pi^{-1}(V)\big)$ and let T be an irreducible component of $\alpha_\pi^{-1}(V)$ with $\beta_\pi(T) = W$ and let $Z = \overline{\alpha_\pi(T)} \subseteq V$. As we have seen, Z is isotropic, and the fact that W is isotropic follows now easily using generic smoothness for the morphism $T \to W$ and the assertion in Lemma 8.74. $\square$

Remark 8.76. In conjunction with the involutivity of the characteristic variety of a coherent $\mathcal{D}$-module (see Theorem 3.44), Theorem 8.73 and the above corollary provide another proof for the fact that holonomic $\mathcal{D}$-modules are preserved by proper push-forward. Indeed, if $\pi\colon Y \to X$ is a proper morphism of smooth, irreducible varieties, and if $\mathcal{M}$ is a holonomic $\mathcal{D}_Y$-module, then $V = \mathrm{Char}(\mathcal{M})$ is an isotropic subvariety of T^*Y (since it is involutive, of dimension $\dim(Y)$). Theorem 8.73 implies that for every i, the characteristic variety of $\mathcal{H}^i\big(\pi_+(\mathcal{M})\big)$ is a subset of $\beta_\pi\big(\alpha_\pi^{-1}(V)\big)$, which is isotropic by Corollary 8.75. Therefore the characteristic variety of $\mathcal{H}^i\big(\pi_+(\mathcal{M})\big)$ is itself isotropic (see the proof of the corollary), and thus $\mathcal{H}^i\big(\pi_+(\mathcal{M})\big)$ is holonomic.

8.4.3. Preliminary results III: $\mathcal{D}_X\langle s,t\rangle$-modules and holonomicity. We begin with a general result about endomorphisms of holonomic $\mathcal{D}$-modules. In this section we fix a smooth, irreducible variety X.

Proposition 8.77. If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module and $u \in \mathrm{End}_{\mathcal{D}_X}(\mathcal{M})$, then there is a nonzero $p \in k[s]$ such that $p(u) = 0$.

Proof. By the Lefschetz Principle, we may and will assume that the ground field is $\mathbf{C}$. Note that if $\mathcal{M}_1 \subseteq \mathcal{M}$ is a $\mathcal{D}_X$-submodule such that $u(\mathcal{M}_1) \subseteq \mathcal{M}_1$ and if the induced endomorphisms $u_1 \in \mathrm{End}_{\mathcal{D}_X}(\mathcal{M}_1)$ and $u_2 \in \mathrm{End}_{\mathcal{D}_X}(\mathcal{M}/\mathcal{M}_1)$ satisfy $p_1(u_1) = 0$ and $p_2(u_2) = 0$ for some nonzero $p_1, p_2 \in \mathbf{C}[s]$, then $p(u) = 0$ for $p = p_1p_2$. Arguing by induction on the length of $\mathcal{M}$, it follows that it is enough to find $\lambda \in \mathbf{C}$ such that $u - \lambda \cdot \mathrm{Id}$ is not injective at some $x \in X$ (we then take $\mathcal{M}_1 = \mathrm{Ker}(u - \lambda \cdot \mathrm{Id})$). We may assume that $\mathcal{M} \neq 0$ and after possibly replacing X by an open subset, we may assume that $\mathrm{Supp}(\mathcal{M})$ is a smooth, irreducible subvariety of X. By Theorem 6.20, we may replace X by $\mathrm{Supp}(\mathcal{M})$, hence we may assume that $\mathrm{Supp}(\mathcal{M}) = X$. By Remark 6.30, after replacing X by an open subset, we may assume that $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module, of rank $r \geq 1$.

In this case we will show the existence of $p \in \mathbf{C}[s]$ such that $p(u) = 0$. If $\lambda_1, \ldots, \lambda_r$ are the roots of p, then some $u - \lambda_i \cdot \mathrm{Id}$ is not injective, completing the proof. The existence of p in turn follows if we show that $\dim_{\mathbf{C}} \mathrm{End}_{\mathcal{D}_X}(\mathcal{M}) < \infty$. Note that we have a $\mathcal{D}_X$-module structure on $\mathcal{E} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M})$ given by

$$(\xi \cdot \varphi)(u) = \xi \cdot \varphi(u) - \varphi(\xi \cdot u) \quad \text{for} \quad \xi \in \mathcal{D}er_{\mathbf{C}}(\mathcal{O}_X),\ \varphi \in \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M}),\ u \in \mathcal{M}$$

such that

$$\mathrm{End}_{\mathcal{D}_X}(\mathcal{M}) = \Gamma(X, \mathcal{E}^{\nabla}),$$

where ∇ is the corresponding integrable connection on $\mathcal{E}$ and $\mathcal{E}^{\nabla} \subseteq \mathcal{E}$ is the subsheaf of sections s, with $\nabla(s) = 0$. Since the canonical morphism $\Gamma(X, \mathcal{E}) \to \Gamma(X^{\mathrm{an}}, \mathcal{E}^{\mathrm{an}})$ is injective (this is due to the fact that the morphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathrm{an}},x}$ is injective for every $x \in X$), it is enough to show that $\Gamma\big(X^{\mathrm{an}}, (\mathcal{E}^{\mathrm{an}})^{\nabla}\big)$ is finite-dimensional. By Theorem 7.5, $\mathcal{L} = (\mathcal{E}^{\mathrm{an}})^{\nabla}$ is a local system on X^{an}. Since X^{an} is connected, it is easy to see that for every $x \in X$, the induced map $\Gamma(X^{\mathrm{an}}, \mathcal{L}) \to \mathcal{L}_x$ is injective. Since $\dim_{\mathbf{C}}(\mathcal{L}_x) < \infty$, we conclude that $\Gamma(X^{\mathrm{an}}, \mathcal{L})$ is finite-dimensional, which completes the proof. $\square$

COROLLARY 8.78. If $\mathcal{M}$ is a $\mathcal{D}_X\langle s,t\rangle$-module that is holonomic as a $\mathcal{D}_X$-module, then there is $N \geq 0$ such that $t^N \cdot \mathcal{M} = 0$.

PROOF. Since $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, the decreasing sequence of $\mathcal{D}_X$-submodules

$$\mathcal{M} \supseteq t \cdot \mathcal{M} \supseteq \ldots \supseteq t^n \cdot \mathcal{M} \supseteq \ldots$$

is stationary. After replacing $\mathcal{M}$ by $t^N \cdot \mathcal{M}$, we may thus assume that $\mathcal{M} = t \cdot \mathcal{M}$ and we need to show that $\mathcal{M} = 0$ (note that $t^N \cdot \mathcal{M}$ is a $\mathcal{D}_X\langle s,t\rangle$-submodule of $\mathcal{M}$ since $st^N \cdot \mathcal{M} \subseteq t^N(s-m) \cdot \mathcal{M} \subseteq t^N \cdot \mathcal{M}$ by Lemma 8.2). By Proposition 8.77, there is a nonzero polynomial $p \in k[x]$ such that $p(s) \cdot \mathcal{M} = 0$. Another application of Lemma 8.2 gives

$$p(s+\ell) \cdot \mathcal{M} = p(s+\ell)t^{\ell} \cdot \mathcal{M} = t^{\ell} p(s) \cdot \mathcal{M} = 0.$$

For $\ell \gg 0$, the polynomials $p(x)$ and $p(x+\ell)$ are relatively prime, so from $p(s) \cdot \mathcal{M} = 0$ and $p(s+\ell) \cdot \mathcal{M} = 0$, we conclude that $\mathcal{M} = 0$. $\square$

PROPOSITION 8.79. Let $\mathcal{M}$ be a $\mathcal{D}_X\langle s,t\rangle$-module and $\mathcal{M}' \subseteq \mathcal{M}$ a $\mathcal{D}_X\langle s,t\rangle$-submodule such that $\mathcal{M}/\mathcal{M}'$ is holonomic as a $\mathcal{D}_X$-module. If $b \in k[x]$ is such that $b(s) \cdot \mathcal{M} \subseteq t \cdot \mathcal{M}$, then there is $N \geq 0$ such that

$$b(s)b(s+1)\cdots b(s+N) \cdot \mathcal{M}' \subseteq t \cdot \mathcal{M}'.$$

PROOF. It follows from Corollary 8.78 that there $N \geq 0$ such that $t^N \cdot (\mathcal{M}/\mathcal{M}') = 0$. By successively applying Lemma 8.2, we obtain

$$b(s)b(s+1)\cdots b(s+N) \cdot \mathcal{M}' \subseteq b(s)b(s+1)\cdots b(s+N) \cdot \mathcal{M} \subseteq b(s+1)\cdots b(s+N)t \cdot \mathcal{M}$$

$$= tb(s)\cdots b(s+N-1) \cdot \mathcal{M} \subseteq \ldots \subseteq t^{N+1} \cdot \mathcal{M} \subseteq t \cdot (t^N \cdot \mathcal{M}) \subseteq t \cdot \mathcal{M}'.$$

$\square$

REMARK 8.80. While we stated the results in this section for left $\mathcal{D}_X\langle s,t\rangle$-modules, which is our usual setting, in the proof of Theorem 8.67 we will make use of Proposition 8.79 for a right $\mathcal{D}_X\langle s,t\rangle$-module. Recall that we have an equivalence of categories, described in Chapter 3.6, between left and right $\mathcal{D}_X$-modules, that associates to a left $\mathcal{D}_X$-module $\mathcal{M}$ a right $\mathcal{D}_X$-module $\mathcal{M}^r$ with underlying $\mathcal{O}_X$-module $\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$. The local description of this equivalence in terms of local coordinates shows that we have a similar equivalence of categories between left and right $\mathcal{D}_X\langle s,t\rangle$-modules, such that if $\mathcal{M}^r$ is the right $\mathcal{D}_X$-module associated to the left $\mathcal{D}_X\langle s,t\rangle$-module $\mathcal{M}$, then t and s act on $\mathcal{M}^r$ by

$$(\alpha\otimes w)\cdot t=\alpha\otimes tw \quad\text{and}\quad (\alpha\otimes w)\cdot s=-(s+1)\cdot(\alpha\otimes w)$$

for every sections w of $\mathcal{M}$ and α of ω_X (note that $-(s+1)=t\partial_t$). By switching between right and left $\mathcal{D}_X\langle s,t\rangle$-modules, we deduce from Proposition 8.79 that if $\mathcal{M}'\subseteq\mathcal{M}$ are right $\mathcal{D}\langle s,t\rangle$-modules and $\mathcal{M}\cdot p(s)\subseteq\mathcal{M}\cdot t$, then there is $N\geq 0$ such that

$$\mathcal{M}'\cdot p(s)p(s-1)\cdots p(s-N)\subseteq\mathcal{M}'\cdot t.$$

8.4.4. Proof of rationality and estimates via a log resolution. We can now prove the main result of this section:

PROOF OF THEOREM 8.67. We put $F=f\circ\pi$ and $G=g\circ\pi$. All assertions are local, hence we may and will assume that X is affine and we have algebraic coordinates $x_1,\ldots,x_n\in\mathcal{O}_X(X)$.

Note first that if f is invertible, then the b-functions in the theorem are all 1 by Remark 8.45 and there is nothing to prove. From now on, we assume that f is not invertible, hence the hypersurface H defined by f is nonempty. Furthermore, it follows from Remarks 8.38 and 8.45 that the b-functions in the statement do not change if we replace X by an open neighborhood of H. Since f is nonconstant, by Generic Smoothness we may and will assume that the induced morphism $X\smallsetminus H\to\mathbf{A}^1\smallsetminus\{0\}$ is smooth, so $V(J_f)\subseteq\mathrm{Supp}(H)$. Since π is an isomorphism over $X\smallsetminus H$, we also have $V(J_F)\subseteq\pi^{-1}\big(\mathrm{Supp}(H)\big)$.

We put $u=g\partial_t^m f^s\in\mathcal{O}_X[1/f,s]f^s\simeq(\iota_f)_+\big(\mathcal{O}_X[1/f]\big)$, where $\iota_f\colon X\hookrightarrow X\times\mathbf{A}^1$ is the graph embedding corresponding to f. By Remark 8.37, the b-function b_u is the minimal polynomial for the action of $s=-\partial_t t$ on $\mathcal{N}_{f,m}(g)/t\mathcal{N}_{f,m}(g)$, where $\mathcal{N}_{f,m}(g)=\mathcal{D}_X\langle s,t\rangle\cdot u\subseteq\mathcal{O}_X[1/f,s]f^s$. We similarly define the $\mathcal{D}_Y$-module $\mathcal{N}_{F,m}(G)$. We note that since $\mathcal{N}_{F,m}(G)\subseteq\mathcal{N}_{F,m}:=\mathcal{N}_{F,m}(1)$, it follows from Lemma 8.72 that $\mathcal{N}_{F,m}(G)$ is a coherent $\mathcal{D}_Y$-module and

$$\mathrm{Char}\big(\mathcal{N}_{F,m}(G)\big)\subseteq\mathrm{Char}(\mathcal{N}_{F,m})=W_F.$$

The improvement in [**Lic89**] as opposed to the argument in [**Kas77**] is the use of right $\mathcal{D}$-modules, that allows bringing in the picture the relative canonical class. Let $\mathcal{N}^r_{f,m}(g)$ be the right $\mathcal{D}_X\langle s,t\rangle$-module corresponding to $\mathcal{N}_{f,m}(g)$ and let $\mathcal{N}^r_{F,m}(G)$ be the right $\mathcal{D}_Y\langle s,t\rangle$-module corresponding to $\mathcal{N}_{F,m}(G)$ (see Remark 8.80). Let η be the image of $\pi^*(dx_1\wedge\ldots\wedge dx_n)$ via the canonical morphism $\pi^*(\omega_X)\to\omega_Y$ and let

$$u^*=(dx_1\wedge\ldots\wedge dx_n)\otimes g\partial_t^m f^s\in\Gamma\big(X,\mathcal{N}_{f,m}(g)\big)\quad\text{and}$$
$$v^*=\eta\otimes G\partial_t^m F^s\in\Gamma\big(Y,\mathcal{N}_{F,m}(G)\big).$$

We will be interested in the following right $\mathcal{D}_Y\langle s,t\rangle$-module:

$$\mathcal{M}:=v^*\cdot\mathcal{D}_Y\langle s,t\rangle\subseteq\mathcal{N}_{F,m}(G).$$

Note that $b_u(s) = p(-s-1)$, where p is the minimal polynomial of the right action of s on $\mathcal{N}^r_{f,m}(g)/\mathcal{N}^r_{f,m}(g)t$. Let us consider now the right action of s on $\mathcal{M}/\mathcal{M}t$ and let us determine a polynomial q such that $(\mathcal{M}/\mathcal{M}t)q(s) = 0$. For this, it is enough to take $q(s) = b(-s-1)$, where b satisfies the following property: we can cover Y by open subsets U on which we have coordinates $y_1, \ldots, y_n$ such that $F|_U = v_1 \cdot \prod_{i=1}^n y_i^{a_i}$ and $\eta|_U = v_2 \cdot \prod_{i=1}^n y_i^{k_i} dy_1 \wedge \ldots \wedge dy_n$ for some invertible functions v_1 and v_2, and

$$b(s)Gv_2 \prod_{i=1}^n y_i^{k_i} \cdot \partial_t^m F^s \in \mathcal{D}_U\langle s,t\rangle tGv_2 \prod_{i=1}^n y_i^{k_i} \cdot \partial_t^m F^s.$$

We can find such a polynomial $b(s)$ such that the following conditions, corresponding to the different assertions in the theorem, hold:

i) If $u = f^s$, then every root of $b(s)$ is of the form $-\frac{k_i+j}{a_i}$, for some i with $1 \leq i \leq N$ and some j, with $1 \leq j \leq a_i$.
ii) Given g and m, every root of $b(s)$ is $\leq -\min\left\{1, \frac{k_i+b_i+1}{a_i} - m \mid 1 \leq i \leq N\right\}$, where b_i is the coefficient of E_i in $\mathrm{div}(G)$.
iii) If $m = 0$, then every root of $b(s)$ is $\leq -\min\left\{\frac{k_i+b_i+1}{a_i} \mid 1 \leq i \leq N\right\}$.
iv) If $g = 1$, then every root of $b(s)$ is either -1 or it is of the form $\frac{k_i+\ell}{a_i} - m$ for some i, with $1 \leq i \leq N$, and some positive integer ℓ. Moreover, if H is reduced and the strict transforms of the components of H are disjoint, then we may assume that E_i is an exceptional divisor.

Indeed, the assertions in i) and iv) follow directly from Proposition 8.71 (note that under the assumptions in the last assertion in iv), we may assume that in every chart we have at most one y_i that defines the strict transform of a component of H, and for this i we have $a_i = 1$ and $k_i = 0$). The assertions in ii) and iii) also follow from Proposition 8.71 if in each chart U as above we have $G|_U = \prod_{i=1}^n y_i^{b_i}$. While we only know that $G|_U = h \cdot \prod_{i=1}^n y_i^{b_i}$, the sections of $(\iota_F)_+(\mathcal{O}_Y)$ whose b-function has all roots $\leq \gamma$ (for some fixed γ) form an $\mathcal{O}_Y$-submodule by Proposition 8.43 (in order to see that $\mathcal{O}_Y$ has a V-filtration with respect to F, we may either first prove assertion i) in the theorem or use the simple normal crossing case that was explicitly discussed in Example 8.34).

Our goal is to show that there is $N \neq 0$ such that b_u divides the polynomial $b(s)b(s+1)\cdots b(s+N)$. By Properties i)-iv) above, this will gives the corresponding assertions in the theorem. In order to do relate $\mathcal{M}$ and $\mathcal{N}^r_f$, we consider

$$\mathcal{M}_X := \mathcal{H}^0\big(\pi_+(\mathcal{M})\big).$$

This is not just a right $\mathcal{D}_X$-module, but a right $\mathcal{D}_X\langle s,t\rangle$-module: indeed, the morphisms induced by multiplication with s and t on $\mathcal{M}$ induce by functoriality corresponding morphisms on $\mathcal{M}_X$, that satisfy the correct commutation relation. Furthermore, since $\mathcal{M} \cdot q(s) \subseteq \mathcal{M} \cdot t$, we also have $\mathcal{M}_X \cdot q(s) \subseteq \mathcal{M}_X \cdot t$. Indeed, since multiplication by t on $\mathcal{M}$ (in fact, on $\mathcal{O}_Y[1/F,s]F^s$) is injective, we have a morphism of $\mathcal{D}_Y$-modules $\varphi\colon \mathcal{M} \to \mathcal{M}$ such that $\varphi(v) \cdot t = vq(s)$ for all $v \in \mathcal{M}$. By functoriality, if $\psi = \mathcal{H}^0\big(\pi_+(\varphi)\big)$, it follows that $\psi(w) \cdot t = wq(s)$ for all $w \in \mathcal{M}_X$.

We next show that we have a canonical section $\sigma \in \Gamma(X, \mathcal{M}_X)$. Recall that, by definition, we have

$$\mathcal{M}_X = R^0\pi_*(\mathcal{M} \otimes^L_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}).$$

On Y, the section v^* induces a morphism $\mathcal{D}_Y \to \mathcal{M}$, which after tensoring with $\mathcal{D}_{Y\to X}$ induces a morphism of right $f^{-1}(\mathcal{D}_X)$-modules

$$\mathcal{D}_{Y\to X} \to \mathcal{M} \otimes^L_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}.$$

By composing this with the morphism of $f^{-1}(\mathcal{O}_X)$-modules $f^{-1}(\mathcal{O}_X) \to \mathcal{D}_{Y\to X}$ that maps $f^{-1}(1)$ to $f^*(1)$ and applying $R^0\pi_*$, we obtain a morphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \to R^0\pi_*\big(f^{-1}(\mathcal{O}_X)\big) \to R^0\pi_*(\mathcal{M} \otimes^L_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}) = \mathcal{M}_X.$$

The image of 1 is the section σ. We note that, by assumption, π induces an isomorphism $Y \smallsetminus \pi^{-1}(H) \to X \smallsetminus H$, hence we have an isomorphism

$$\mathcal{M}_X|_{X\smallsetminus H} \simeq \mathcal{N}^r_{f,m}(g)|_{X\smallsetminus H}$$

such that $\sigma|_{X\smallsetminus H}$ corresponds to $u^*|_{X\smallsetminus H}$.

Let $\mathcal{M}^\circ_X := \sigma\cdot\mathcal{D}_X\langle s,t\rangle \subseteq \mathcal{M}_X$. We have a morphism of right $\mathcal{D}_X\langle s,t\rangle$-modules $\tau\colon \mathcal{M}^\circ_X \to \mathcal{N}^r_{f,m}(g)$ such that $\tau(\sigma) = u^*$. Indeed, suppose that $P \in \mathcal{D}_X\langle s,t\rangle$ is such that $\sigma\cdot P = 0$. By restricting to $X \smallsetminus H$, we see that $(u^*\cdot P)|_{X\smallsetminus H} = 0$. Since $\mathcal{N}^r_{f,m}(g)$ has no f-torsion, it follows that $u^*\cdot P = 0$. This shows that we have indeed a morphism τ, which is surjective since $\mathcal{N}^r_{f,m}(g)$ is generated over $\mathcal{D}_X\langle s,t\rangle$ by u^*.

The key point is that $\mathcal{M}_X/\mathcal{M}^\circ_X$ is a holonomic $\mathcal{D}_X$-module. Note first that since $\mathcal{M}$ is coherent and $\mathrm{Char}(\mathcal{M}) \subseteq W_F$, it follows from Theorem 6.54 that $\mathcal{M}_X$ is coherent and Theorem 8.73 implies that $\mathrm{Char}(\mathcal{M}_X) \subseteq \beta_\pi\big(\alpha_\pi^{-1}(W_F)\big)$. Moreover, since $\mathcal{M}_X|_{X\smallsetminus H} = \mathcal{M}^\circ_X|_{X\smallsetminus H}$, we have

$$\mathrm{Char}(\mathcal{M}_X/\mathcal{M}^\circ_X) \subseteq \beta_\pi\big(\alpha_\pi^{-1}(W_F)\big) \times_X H = \beta_\pi\big(\alpha_\pi^{-1}(W_F \times_Y \pi^{-1}(H))\big).$$

On the other hand, it follows from Proposition 8.72 that $W_F \times_Y \pi^{-1}(H)$ is isotropic, hence $\beta_\pi\big(\alpha_\pi^{-1}(W_F \times_Y \pi^{-1}(H))\big)$ is isotropic by Corollary 8.75. This implies that every irreducible component of $\mathrm{Char}(\mathcal{M}_X/\mathcal{M}^\circ_X)$ has dimension $\leq \dim(X)$, hence $\mathcal{M}_X/\mathcal{M}^\circ_X$ is holonomic.

In this case, since $\mathcal{M}_X\cdot q(s) \subseteq \mathcal{M}_X\cdot t$, it follows from Proposition 8.79 (see also Remark 8.80) that there is $N \geq 0$ such that

$$\mathcal{M}^\circ_X\cdot q(s)q(s-1)\cdots q(s-N) \subseteq \mathcal{M}^\circ_X\cdot t.$$

Since we have a surjective morphism of right $\mathcal{D}_X\langle s,t\rangle$-modules $\mathcal{M}^\circ_X \to \mathcal{N}^r_{f,m}(g)$, it follows that we also have

$$\mathcal{N}^r_{f,m}(g)\cdot q(s)q(s-1)\cdots q(s-N) \subseteq \mathcal{N}^r_{f,m}(g)\cdot t$$

and by passing to left $\mathcal{D}$-modules, we obtain

$$b(s)b(s+1)\cdots b(s+N)\cdot\mathcal{N}_{f,m}(g) \subseteq t\cdot\mathcal{N}_{f,m}(g).$$

As we have already seen, this implies the assertions in the theorem. ☐

REMARK 8.81. In the setting of Theorem 8.67, if g is such that the hypersurface H' defined by g has simple normal crossings on $X \smallsetminus H$, then there is a log resolution $\pi\colon Y \to X$ of (X,H) such that $\pi^*(H+H')$ has simple normal crossings. In this case, we can be more precise regarding the assertions ii) and iii) in the theorem, namely the following hold:

- ii') Every root of $b_{g\partial_t^m\delta}$ is either a negative integer or it is of the form $m - \frac{k_i+b_i+\ell}{a_i}$ for some i, with $1 \leq i \leq N$, and some positive integer ℓ.

iii) Every root of $b_{g\delta}$ is of the form $-\frac{k_i+b_i+\ell}{a_i}$ for some i, with $1 \leq i \leq N$, and some positive integer ℓ.

This follows as in the proof of the theorem, using Proposition 8.71 and the fact that we have local coordinates $y_1, \ldots, y_n$ on Y such that $f \circ \pi = vy_1^{a_1} \cdots y_n^{a_n}$ and $g \circ \pi = wy_1^{b_1} \cdots y_n^{b_n}$ for some invertible functions v and w.

8.5. Invariants of singularities related to the V-filtration

We now discuss some connections between the invariants of singularities that we have seen so far (the roots of the Bernstein-Sato polynomial of f and the V-filtration of $\mathcal{O}_X$ with respect to f) and other invariants of singularities.

8.5.1. The V-filtration and multiplier ideals. In this section we describe the connection between the V-filtration and the Bernstein-Sato polynomial, on one side, and multiplier ideals, on the other side. We begin with a quick introduction to multiplier ideals. For a more detailed discussion and for the proofs of some of the results we state, we refer to [**Laz04**, Chapter 9].

Let X be a smooth, irreducible algebraic variety over an algebraically closed field k of characteristic 0. Suppose that H is a (nonempty) hypersurface on X. We consider a log resolution $\pi\colon Y \to X$ of the pair (X, H) and write

$$\pi^*(H) = \sum_{i=1}^{N} a_i E_i \quad \text{and} \quad K_{Y/X} = \sum_{i=1}^{N} k_i E_i. \tag{8.38}$$

Definition 8.82. For every $\lambda \in \mathbf{Q}_{\geq 0}$, the *multiplier ideal* $\mathcal{J}(X, \lambda H)$ is given by

$$\mathcal{J}(X, \lambda H) = \pi_*\mathcal{O}_Y\big(K_{Y/X} - \lfloor \lambda\pi^*(H) \rfloor\big).$$

We note that, by definition, we have $\lfloor \lambda\pi^*(H) \rfloor = \sum_{i=1}^{N} \lfloor \lambda a_i \rfloor E_i$. We also note that since $\lfloor \lambda\pi^*(H) \rfloor$ is an effective divisor and since $K_{Y/X}$ is an effective exceptional divisor, we have

$$\mathcal{J}(X, \lambda H) \subseteq \pi_*\mathcal{O}_Y(K_{Y/X}) = \mathcal{O}_X,$$

hence $\mathcal{J}(X, \lambda H)$ is indeed a coherent ideal of $\mathcal{O}_X$. It is a basic fact that the definition of multiplier ideals is independent of the choice of log resolution (see [**Laz04**, Theorem 9.2.18]).

In what follows we list a few properties of multiplier ideals. Most of these follow in a straightforward way from the definition:

1) If $\lambda \geq \mu$, then
$$\mathcal{J}(X, \lambda H) \subseteq \mathcal{J}(X, \mu H).$$
This is a consequence of the fact that the divisor $\lfloor \lambda\pi^*(H) \rfloor - \lfloor \mu\pi^*(H) \rfloor$ is effective.
2) It is an immediate consequence of the properties of the round-down function that for every $\lambda \in \mathbf{Q}_{\geq 0}$, there is $\epsilon > 0$ such that
$$\mathcal{J}(X, \lambda H) = \mathcal{J}(X, \mu H) \quad \text{for} \quad \lambda \leq \mu \leq \lambda + \epsilon.$$
3) In particular, we have $\mathcal{J}(X, \mu H) = \mathcal{O}_X$ for $0 \leq \mu \ll 1$.
4) We say that $\lambda \in \mathbf{Q}_{>0}$ is a *jumping number* of (X, H) if
$$\mathcal{J}(X, \lambda H) \subsetneq \mathcal{J}\big(X, (\lambda - \epsilon)H\big) \quad \text{for all} \quad \epsilon > 0.$$

Note that in this case we have $a_i\lambda \in \mathbf{Z}$ for some i with $1 \leq i \leq N$. Therefore the set of jumping numbers of (X, H) is contained in $\frac{1}{\ell}\mathbf{Z}_{>0}$ for some positive integer ℓ.

5) The smallest jumping number of (X, H) is the *log canonical threshold* $\mathrm{lct}(X, H)$ (also written $\mathrm{lct}(H)$ when X is understood):

$$\mathrm{lct}(X, H) = \min\big\{\lambda > 0 \mid \mathcal{J}(X, \lambda H) \neq \mathcal{O}_X\big\}.$$

Note that, by definition, we have $1 \in \mathcal{J}(X, \lambda H)$ if and only if $k_i \geq \lfloor \lambda a_i \rfloor$ for all i; equivalently, $k_i > \lambda a_i - 1$ for all i. We thus deduce that

$$\mathrm{lct}(X, H) = \min_i \tfrac{k_i+1}{a_i}. \tag{8.39}$$

6) 1 is always a jumping number of (X, H). In order to see this, we may replace X by any open subset U such that $U \cap H \neq \emptyset$. We may thus assume that $H = mZ$ for some smooth hypersurface Z and some positive integer m. In this case we may take π to be the identity and we see that

$$\mathcal{J}(X, H) = \mathcal{O}_X(-mZ) \quad \text{and} \quad \mathcal{J}\big(X, (1-\epsilon)H\big) = \mathcal{O}_X\big(-(m-1)Z\big) \text{ for } 0 < \epsilon \ll 1.$$

7) For every $\lambda \geq 1$, it follows from the definition and the projection formula that

$$\mathcal{J}(X, \lambda H) = \mathcal{O}_X(-H) \cdot \mathcal{J}\big(X, (\lambda - 1)H\big).$$

In particular, we see that $\lambda > 1$ is a jumping number of (X, H) if and only if $\lambda - 1$ has this property. This means that as invariants of singularities, it is enough to consider the multiplier ideals $\mathcal{J}(X, \lambda H)$ for $\lambda < 1$.

8) If $k = \mathbf{C}$, then there is an analytic description of $\mathcal{J}(X, \lambda H)$ that is more intuitive than the algebraic one that we gave. Suppose, for simplicity, that H is defined by $f \in \mathcal{O}_X(X)$. In this case we have

$$\mathcal{J}(X, \lambda H) = \big\{g \in \mathcal{O}_X(X) \mid \tfrac{|g|^2}{|f|^{2\lambda}} \text{ is locally integrable}\big\}.$$

The local integrability condition means that for every $P \in X$, if $z_1, \ldots, z_n$ are local coordinates around P, then there is an open neighborhood U of P such that $\int_U \frac{|g|^2}{|f|^{2\lambda}} dz d\overline{z} < \infty$. The equivalence with the formula in the algebraic definition is shown using the Change of Variable formula and the fact that, in one variable, $\frac{1}{|z|^{2\lambda}} dz d\overline{z}$ is locally integrable if and only if $\lambda < 1$ (see [**Laz04**, Chapter 9.3.D] for details).

Suppose now that the hypersurface H in X is defined by $f \in \mathcal{O}_X(X)$. The following 3 results relate the $\mathcal{D}$-module theoretic invariants of f to the multiplier ideals of H. The description of the log canonical threshold in the following theorem was proved by Kollár [**Kol97**], by making use of Lichtin's upper bound [**Lic89**] for the roots of the Bernstein-Sato polynomial that we discussed in the previous chapter.

THEOREM 8.83. *The largest root of* b_f *is* $-\mathrm{lct}(X, H)$.

Partially generalizing this to higher jumping numbers, we have the following result due to Ein, Lazarsfeld, Smith, and Varolin [**ELSV04**].

THEOREM 8.84. *If* $\lambda \leq 1$ *is a jumping number of* (X, H)*, then* $b_f(-\lambda) = 0$.

As we will see, both the above results follow from the following theorem of Budur and Saito [**BS05**], that describes the multiplier ideals of (X,H) via the V-filtration of f. Recall that $\iota\colon X\hookrightarrow X\times\mathbf{A}^1$ is the graph embedding associated to f.

THEOREM 8.85. *For every $\lambda\in\mathbf{Q}_{\geq 0}$, we have*

$$\mathcal{J}(X,\lambda H)=\big\{g\in\mathcal{O}_X\mid g\delta\in V^{>\alpha}\iota_+(\mathcal{O}_X)\big\}.$$

The proof of this result in [**BS05**] makes use of results in Saito's theory of mixed Hodge modules [**Sai90**]. Here we give a more elementary proof following [**DM22**].

PROOF OF THEOREM 8.85. We may and will assume that X is affine, with $R=\mathcal{O}_X(X)$ and let $\pi\colon Y\to X$ be a log resolution of (X,H). Let $g\in R$ be nonzero. We use the notation in (8.38) and denote by b_i the coefficient of E_i in $\pi^*\big(\mathrm{div}(g)\big)$. By definition, we have $g\in\mathcal{J}(X,\lambda H)$ if and only if $b_i+k_i\geq\lfloor\lambda a_i\rfloor$ for all i, which is the case if and only if $\lambda<\min_i\frac{b_i+k_i+1}{a_i}=:\mathrm{lct}_g(X,H)$. On the other hand, it follows from Proposition 8.43 that $g\delta\in V^{>\lambda}\iota_+(\mathcal{O}_X)$ if and only if all roots of $b_{g\delta}$ are $<-\lambda$. We thus conclude that the assertion in the theorem is equivalent to the fact that for every nonzero $g\in\mathcal{O}_X(X)$, the largest root of $b_{g\delta}$ is $-\mathrm{lct}_g(X,H)$.

The fact that every root of $b_{g\delta}$ is $\leq-\mathrm{lct}_g(X,H)$ follows from Theorem 8.67iii). In order to complete the proof, it is enough to show that if $g\in\mathcal{J}\big(X,(\lambda-\epsilon)H\big)$ for every $\epsilon>0$, but $g\notin\mathcal{J}(X,\lambda H)$, then $b_{g\delta}(-\lambda)=0$. For this, we use the Lefschetz Principle to reduce to the case when the ground field is $\mathbf{C}$, when we can use the analytic description of multiplier ideals. The argument is similar in spirit to Bernstein's argument for the meromorphic extension of complex powers that we discussed in Section 8.3.2 and follows closely the argument for the proof of Theorem 8.84 in [**ELSV04**].

We may assume that we have coordinates $z_1,\ldots,z_n$ on X. Since $g\notin\mathcal{J}(X,\lambda H)$, it follows that there is a point $x_0\in X$ such that $\frac{|g|^2}{|f|^{2\lambda}}$ is not integrable in any neighborhood of x_0. On the other hand, since $\frac{|g|^2}{|f|^{2\mu}}$ is locally integrable at x_0 for all $\mu<\lambda$, we can choose an open ball B around x_0 (with respect to our coordinates) such that $\int_B\frac{|g|^2}{|f|^{2\mu}}dzd\overline{z}<\infty$ for all $\mu<\lambda$ (the fact that we can choose B independently of μ follows from the proof of the analytic characterization of the multiplier ideal, see [**Laz04**, Chapter 9.3.D]).

By definition of the b-function, if we put $b=b_{g\delta}$, then there is $P\in D_R[s]$ such that

$$b(s)gf^s=P\cdot gf^{s+1}.$$

Suppose that we are in an open subset U of X^{an} where a branch of $\log(f)$ is defined, hence $f^\mu=\exp\big(\mu\cdot\log(f)\big)$ is defined for every $\mu\in\mathbf{R}$. We choose μ such that $0<\lambda-\mu\ll 1$ and specialize as in Remark 6.48 to $s=-\mu$, to get

$$b(-\mu)gf^{-\mu}=P(-\mu)\cdot gf^{1-\mu}.$$

Applying complex conjugation, we obtain

$$b(-\mu)\overline{g}\overline{f}^{-\mu}=\overline{P}(-\mu)\overline{g}\overline{f}^{1-\mu}.$$

Using the fact that that

$$P \cdot (\overline{h_1}h_2) = \overline{h_1}P \cdot h_2 \quad \text{and} \quad \overline{P} \cdot (\overline{h_1}h_2) = h_2\overline{P} \cdot \overline{h_1}$$

for every holomorphic functions h_1 and h_2, we conclude that if $R = P\overline{P}$, then

$$b(-\mu)^2 \cdot \tfrac{|g|^2}{|f|^{2\mu}} = R(-\mu) \cdot \tfrac{|g|^2}{|f|^{2(\mu-1)}}. \tag{8.40}$$

Note that this formula does not depend on U and it makes sense on $X \smallsetminus H$.

Since both sides of (8.40) are integrable on B, if φ is a smooth function with compact support on B, then the following integrals are finite

$$\int_B b(-\mu)^2 \cdot \tfrac{|g|}{|f|^{2\mu}} \varphi dz d\overline{z} = \int_B \big(R(-\mu) \cdot \tfrac{|g|^2}{|f|^{2(\mu-1)}}\big)\varphi dz d\overline{z} = \int_B \tfrac{|g|^2}{|f|^{2(\mu-1)}} \psi dz d\overline{z},$$

where $\psi = \widetilde{R}(-\mu)$ (here $\widetilde{R}$ is the classical adjoint of R) and the last equality in the displayed formula is a consequence of the Stokes Theorem. In particular, by choosing φ to be nonnegative and with $\varphi = 1$ on a smaller ball $B' \subsetneq B$, we get

$$b(-\mu)^2 \cdot \int_{B'} \tfrac{|g|^2}{|f|^{2\mu}} dz d\overline{z} \leq \int_B \tfrac{|g|^2}{|f|^{2(\mu-1)}} \psi dz d\overline{z} \leq M,$$

for some constant M that is independent of μ. If $b(-\lambda) \neq 0$, then we conclude that $\int_{B'} \tfrac{|g|^2}{|f|^{2\mu}} dz d\overline{z}$ is bounded for $\mu \to \lambda$, hence $\tfrac{|g|^2}{|f|^{2\lambda}}$ is integrable on B' by the monotone convergence theorem. This contradiction completes the proof of the theorem. $\square$

REMARK 8.86. Suppose that X and H are as in Theorem 8.85. We have seen in the proof of the theorem that its statement is equivalent to the fact that for every $g \in \mathcal{O}_X(X)$ nonzero, the largest root of $b_{g\delta}$ is $-\operatorname{lct}_g(X, H)$. The special case $g = 1$ corresponds to the assertion in Theorem 8.83

PROOF OF THEOREM 8.84. We may and will assume that X is an affine variety, with $R = \mathcal{O}_X(X)$. Since λ is a jumping number, it follows that there is $g \in R$ such that $g \in \mathcal{J}\big(X, (\lambda - \epsilon)H\big)$ for $0 < \epsilon \ll 1$, but $g \notin \mathcal{J}(X, \lambda H)$. For simplicity, we write V^α for $V^\alpha\iota_+(\mathcal{O}_X)$. By Theorem 8.85, we have $g\delta \in V^\lambda \smallsetminus V^{>\lambda}$. On the other hand, by definition of b_f, there is $P \in D_R[s]$ such that

$$b_f(s)\delta = P \cdot t\delta.$$

Note that $\delta \in V^{>0}$: this follows from Theorem 8.85 since $\mathcal{J}(X, \lambda H) = \mathcal{O}_X$ for $\lambda = 0$; alternatively, it follows from the description of the V-filtration in Proposition 8.43 and the fact that the roots of b_f are negative by Theorem 8.67i). Therefore $t\delta \in V^{>1} \subseteq V^{>\lambda}$. Since $b_f(s)g\delta = gP \cdot t\delta \in V^{>\lambda}$, it follows that $b_f(s)$ annihilates a nonzero element in $\operatorname{Gr}_V^\lambda$. Since $(s+\lambda)^N$ annihilates $\operatorname{Gr}_V^\lambda$ for some N, it follows that $\gcd\big(b_f, (s + \lambda)^N\big)$ annihilates a nonzero element, hence $(s + \lambda)$ divides b_f. $\square$

8.5.2. The V-filtration and the minimal exponent. Our goal in this section is to discuss a refinement of the notions of log canonical threshold and multiplier ideals due to Saito. The idea is to use the V-filtration in order to define a version of multiplier ideals that give new information also for $\lambda \geq 1$.

We fix a smooth, irreducible algebraic variety X and a (nonempty) hypersurface H in X. To begin with, we assume that H is defined by $f \in \mathcal{O}_X(X)$. For simplicity, we write V^α for $V^\alpha\iota_+(\mathcal{O}_X)$, where $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ is the graph embedding corresponding to f.

DEFINITION 8.87. For every $\lambda \in \mathbf{Q}_{\geq 0}$, we write $\lambda = \alpha + q$, where $q \in \mathbf{Z}_{\geq 0}$ and $\alpha \in [0,1)$, and we denote by $\widetilde{\mathcal{J}}(X, \lambda H)$ the coherent ideal of $\mathcal{O}_X$ consisting of those $h \in \mathcal{O}_X$ with the property that there are $h_0, \ldots, h_{q-1} \in \mathcal{O}_X$ such that $h_0\delta + \ldots + h_{q-1}\partial_t^{q-1}\delta + h\partial_t^q\delta \in V^{>\alpha}$.

These ideals have been introduced by Saito in [**Sai16**] as *microlocal multiplier ideals*, due to the fact that the definition was expressed in terms of the so-called *microlocal V-filtration*. However, we will not use this terminology.

REMARK 8.88. It is a consequence of Theorem 8.85 that for $\lambda < 1$, we have

$$\widetilde{\mathcal{J}}(X, \lambda H) = \mathcal{J}(X, \lambda H).$$

REMARK 8.89. If $g \in \mathcal{O}_X(X)$ defines the same hypersurface, then we can write $g = pf$, for some invertible $p \in \mathcal{O}_X(X)$. If ι_f and ι_g are the graph embeddings corresponding to f and g, respectively, then it follows from Remark 8.24 that

$$V^{>\alpha}(\iota_g)_+(\mathcal{O}_X) = \Big\{ \sum_{j=0}^{q} p^{j+1}u_j\partial_t^j\delta \mid q \in \mathbf{Z}_{\geq 0},\ \sum_{j=0}^{q} u_j\partial_t^j\delta \in V^{>\alpha}(\iota_f)_+(\mathcal{O}_X)\Big\}.$$

This immediately implies that $\widetilde{\mathcal{J}}(X, \lambda H)$ does not depend on the choice of f. We can thus define $\widetilde{\mathcal{J}}(X, \lambda H)$ for every hypersurface H, by glueing the corresponding ideals on a suitable open cover such that H is a principal divisor in each of these open subsets.

REMARK 8.90. For every H, there is a positive integer ℓ such that $\widetilde{\mathcal{J}}(\lambda H)$ is constant for $\lambda \in \big[\frac{i}{\ell}, \frac{i+1}{\ell}\big)$ for every $i \in \mathbf{Z}_{\geq 0}$. Indeed, it is enough to check this when H is defined by $f \in \mathcal{O}_X(X)$, in which case the assertion follows from the fact that $V^{\bullet}\iota_+(\mathcal{O}_X)$ is discrete and left continuous. In particular, we see for every $\lambda \in \mathbf{Q}_{\geq 0}$, there is $\lambda' > \lambda$ such that $\widetilde{\mathcal{J}}(X, \lambda H) = \widetilde{\mathcal{J}}(X, \mu H)$ for every $\lambda \leq \mu \leq \lambda'$.

PROPOSITION 8.91. For every hypersurface H on the smooth, irreducible variety X and every $\lambda \geq \mu$, we have

$$\widetilde{\mathcal{J}}(X, \lambda H) \subseteq \widetilde{\mathcal{J}}(X, \mu H).$$

PROOF. We may assume that X is affine and H is defined by $f \in \mathcal{O}_X(X)$. It follows directly from the definition that for every $q \in \mathbf{Z}_{\geq 0}$, we have

$$\widetilde{\mathcal{J}}(X, \lambda H) \subseteq \widetilde{\mathcal{J}}(X, \mu H)$$

for $q \leq \mu \leq \lambda < q+1$. Therefore, in order to prove the proposition, it is enough to show that for every $q \in \mathbf{Z}_{>0}$ and every μ, with $q-1 \leq \mu < q$, we have

$$\widetilde{\mathcal{J}}(X, qH) \subseteq \widetilde{\mathcal{J}}\big(X, \mu H).$$

In order to prove this, let h be a global section of $\widetilde{\mathcal{J}}(X, qH)$, hence there are $h_0, \ldots, h_{q-1} \in \mathcal{O}_X(X)$ such that $u = h_0\delta + \ldots + h_{q-1}\partial_t^{q-1}\delta + h\partial_t^q\delta \in V^{>0}$. Note that $\delta \in V^{>0}$: this follows, using Proposition 8.43, from the fact that b_f has negative roots, see Theorem 8.67i). Therefore we have

$$\partial_t \cdot (h_1\delta + \ldots + h\partial_t^{q-1}\delta) \in V^{>0} \subseteq V^0.$$

We deduce from Corollary 8.12ii) that $h_1\delta + \ldots + h\partial_t^{q-1}\delta \in V^1$, hence $h \in \widetilde{\mathcal{J}}(X, \mu H)$ for every μ, with $q-1 \leq \mu < q$. This completes the proof. $\square$

PROPOSITION 8.92. Let X be a smooth, irreducible variety and H the hypersurface defined by $f \in \mathcal{O}_X(X)$. If $\lambda \in \mathbf{Q}_{\geq 0}$ is written as $\lambda = \alpha + q$, where $q \in \mathbf{Z}_{\geq 0}$ and $\alpha \in [0, 1)$, then $\widetilde{\mathcal{J}}(X, \lambda H) = \mathcal{O}_X$ if and only if $\partial_t^q \delta \in V^{>\alpha}$.

PROOF. The "if" part is clear from the definition, so we only need to prove the converse. We argue by induction on q, the case $q = 0$ being clear. We may and will assume that X is affine. Suppose that $1 \in \Gamma\big(X, \widetilde{\mathcal{J}}(X, \lambda H)\big)$, so there are $h_0, \ldots, h_{q-1} \in \mathcal{O}_X(X)$ such that

$$h_0\delta + \ldots + h_{q-1}\partial_t^{q-1}\delta + \partial_t^q\delta \in V^{>\alpha}. \tag{8.41}$$

By Proposition 8.91, we know that $\widetilde{\mathcal{J}}\big(X, (\lambda - i)H\big) = \mathcal{O}_X$ for $1 \leq i \leq q$, hence by induction we know that $\partial_t^j \delta \in V^{>\alpha}$ for $j < q$. We thus deduce from (8.41) that $\partial_t^q \delta \in V^{>\alpha}$, completing the proof of the induction step. $\square$

EXAMPLE 8.93. If H is a smooth hypersurface of X, then $\widetilde{\mathcal{J}}(X, \lambda H) = \mathcal{O}_X$ for all $\lambda \geq 0$. Indeed, it follows from Remark 8.33 that $\partial_t^j \delta \in V^1\iota_+(\mathcal{O}_X)$ for all $j \geq 0$. In fact, the converse also holds: of $\widetilde{\mathcal{J}}(X, \lambda H) = \mathcal{O}_X$ for all $\lambda \geq 0$, then H is smooth. However, the proof of this fact requires techniques beyond those covered in these notes.

Our next goal is to describe $\min\big\{\lambda \in \mathbf{Q}_{\geq 0} \mid \widetilde{\mathcal{J}}(X, \lambda H) \neq \mathcal{O}_X\big\}$. Note that by Remark 8.90 and Proposition 8.91, it makes sense to consider this minimum (though this may be infinite, as seen in the above example).

Let H be a nonempty hypersurface in the smooth, irreducible variety X. Recall that in this case $b_H(s)$ is divisible by $(s + 1)$ and $\widetilde{b}_H(s) = b_H(s)/(s + 1)$ (see Remark 8.56).

DEFINITION 8.94. The *minimal exponent* $\widetilde{\alpha}(X, H)$ (also written $\widetilde{\alpha}(H)$ when X is understood) of the hypersurface H is the negative of the largest root of $\widetilde{b}_H$, with the convention that this is infinite if $\widetilde{b}_H = 1$. Similarly, if $P \in Z$, then we define the local version $\widetilde{\alpha}_P(X, H)$ as the negative of the largest root of $\widetilde{b}_{H,P}$.

REMARK 8.95. It follows from the relation between the global and local (reduced) Bernstein-Sato polynomials that

$$\widetilde{\alpha}(X, H) = \min_{P \in H} \widetilde{\alpha}_P(X, H) \quad \text{and}$$

$$\widetilde{\alpha}_P(X, H) = \max_{U \ni P} \widetilde{\alpha}(U, H \cap U).$$

REMARK 8.96. It is a consequence of Theorem 8.83 that

$$\operatorname{lct}(X, H) = \min\big\{\widetilde{\alpha}(X, H), 1\big\}.$$

EXAMPLE 8.97. It follows from Theorem 8.61 that if $H \subset \mathbf{A}^n$ is a singular hypersurface defined by a quasi-homogeneous polynomial $f \in k[x_1, \ldots, x_n]$, with isolated singularities, and $w_1, \ldots, w_n \in \mathbf{Q}_{>0}$ are weights such that f has w-degree 1, then $\widetilde{\alpha}(X, H) = \sum_{i=1}^n w_i$. Indeed, since H is singular, we have $1 \notin J_f$ and it is clear that the smallest element in $\Sigma(f)$ is $\rho(1) + |w| = \sum_{i=1}^n w_i$.

The minimal exponent was introduced by Saito in [**Sai94**], where it was related to the microlocal V-filtration. For the general properties of this invariant, that parallel some of the standard properties of the log canonical threshold, see [**MP20**]. We now show that the minimal exponent governs the triviality of the refined multiplier ideals.

THEOREM 8.98. *If H is a nonempty hypersurface in the smooth, irreducible, algebraic variety X, then*

$$\min\left\{\lambda\in\mathbf{Q}_{\geq 0}\mid \widetilde{\mathcal{J}}(X,\lambda H)\neq\mathcal{O}_X\right\}=\widetilde{\alpha}(X,H).$$

This result was proved by Saito in [**Sai16**] using the connection with the microlocal V-filtration. We give a proof using the following result, which is of independent interest:

THEOREM 8.99. *If X is a smooth, irreducible algebraic variety and $f\in\mathcal{O}_X(X)$ is not invertible, then for every $q\in\mathbf{Z}_{\geq 0}$, we have*

$$b_{\partial_t^q\delta}(s)=(s+1)\cdot\widetilde{b}_f(s-q).$$

A slightly weaker assertion (the fact that $\widetilde{b}_f(s-q)$ divides $b_{\partial_t^q\delta}$, which divides $(s+1)\cdot\widetilde{b}_f(s-q)$) was proved in [**MP20**, Proposition 6.11], also making use of the microlocal V-filtration. In what follows we give a direct proof.

PROOF OF THEOREM 8.99. The fact that $b_{\partial_t^q\delta}(s)$ divides $(s+1)\cdot\widetilde{b}_f(s-q)$ is a special case of Proposition 8.69, hence we only need to prove that $(s+1)\cdot\widetilde{b}_f(s-q)$ divides $b_{\partial_t^q\delta}(s)$. We may and will assume that X is affine, with $R=\mathcal{O}_X(X)$, and that we have coordinates $x_1,\ldots,x_n$ on X.

Let $p=b_{\partial_t^q\delta}$. By definition of b-functions, we have

$$p(s)\partial_t^q\delta\in V^1\mathcal{D}_{X\times\mathbf{A}^1}\cdot\partial_t^q\delta.$$

It will be convenient to use the isomorphism of $\mathcal{D}_R\langle t,\partial_t\rangle$-modules $\iota_+\big(\mathcal{O}_X[1/f]\big)\simeq\mathcal{O}_X[1/f,s]f^s$ (see Proposition 8.35). Note that it follows from Lemma 8.2ii) that

$$\partial_t^q f^s=\partial_t^q t^t f^{s-q}=(-1)^q q!\binom{s}{q}f^{s-q},$$

and thus

$$t^j\partial_t^q f^s=(-1)^q q!\binom{s+j}{q}f^{s-q+j}\quad\text{for all}\quad j\geq 0.$$

Since $V^1\mathcal{D}_{X\times\mathbf{A}^1}=\sum_{j\geq 1}\mathcal{D}_X[s]t^j$, we conclude that we have a positive integer d and $A_1,\ldots,A_d\in D_R[s]$ such that

$$p(s)\binom{s}{q}f^{s-q}=\sum_{j=1}^d A_j(s)\binom{s+j}{q}f^{s-q+j}. \tag{8.42}$$

We first show by descending induction on $\ell\geq 2$ that we may assume that $A_j\in R[s]$ for all $j\geq\ell$. This is trivial if $p>d$, hence it is enough to show that if $A_j\in R[s]$ for all $j\geq\ell+1$, with $\ell\geq 2$, then we can modify A_ℓ and $A_{\ell-1}$ so that also $A_\ell\in R[s]$. Let us write

$$A_\ell=A_{\ell,0}+\sum_{i=1}^n A_{\ell,i}\partial_{x_i},\quad\text{with}\quad A_{\ell,0}\in R[s].$$

Since $\partial_{x_i}\cdot f^{s-q+\ell}=(s-q+\ell)\frac{\partial f}{\partial x_i}f^{s-q+\ell-1}$ and $(s-q+\ell)\binom{s+\ell}{q}=(s+\ell)\binom{s+\ell-1}{q}$, it follows that

$$A_{\ell,i}\partial_{x_i}\binom{s+\ell}{q}f^{s-q+\ell}\in D_R[s]\binom{s+\ell-1}{q}f^{s-q+\ell-1}\quad\text{for all}\quad 1\leq i\leq n.$$

We thus see that after modifying $A_{\ell-1}$, we may assume that $A_\ell = A_{\ell,0} \in R[s]$. The conclusion is that we may and will assume that $A_2, \ldots, A_d \in R[s]$.

The next step is to show, by descending induction on $\ell \geq 2$, that we may assume that $A_j \in R$ for all $j \geq \ell$. Again, this is clear if $\ell > d$. We assume that $A_j \in R$ for all $j \geq \ell + 1$, with $\ell \geq 2$, and show that we can modify A_ℓ and $A_{\ell-1}$ so that $A_\ell \in R$. Note that we can write

$$A_\ell(s) = B_\ell(s - q + \ell) + C_\ell \quad \text{with} \quad B_\ell \in R[s],\ C_\ell \in R$$

and we have

$$B_\ell(s - q + \ell)\binom{s+\ell}{q} f^{s-q+\ell} \in R[s]\binom{s+\ell-1}{q} f^{s-q+\ell-1}.$$

We can thus modify $A_{\ell-1}$ so that $A_\ell = C_\ell \in R$. We conclude that we may and will assume that $A_j \in R$ for all $j \geq 2$.

We next specialize s to s_0 (see Remark 6.48), where $s_0 \in \{-1, 0, \ldots, q-2\}$. In this case, since $\binom{s_0+1}{q} = 0$, equation (8.42) becomes

$$p(s_0)\binom{s_0}{q} f^{s_0-q} = \sum_{j=2}^{d} A_j \binom{s_0+j}{q} f^{s_0-q+j}. \tag{8.43}$$

Note that if we have $\lambda + \sum_{j=1}^{d} g_j f^j = 0$ for some $\lambda \in k$ and $g_1, \ldots, g_d \in R$, by evaluating at a point in H, we conclude that $\lambda = 0$. We deduce from (8.43) that $p(-1) = 0$ and

$$\sum_{j=2}^{d} A_j \binom{s_0+j}{q} f^j = 0$$

for every $s_0 \in \{-1, 0, \ldots, q-2\}$. Let us write $p = (s+1)\widetilde{p}$. Note that

$$\sum_{j=2}^{d} A_j \binom{s+j}{q} f^{j-2} \in R[s]$$

has degree $\leq q$ in s and it vanishes when $s = s_0 \in \{-1, 0, \ldots, q-2\}$. We thus conclude that $\sum_{j=2}^{d} A_j \binom{s+j}{q} f^{j-2} = Q \cdot \binom{s+1}{q}$ for some $Q \in R$. Therefore we have

$$p(s)\binom{s}{q} f^{s-q} = A_1\binom{s+1}{q} f^{s-q+1} + Q\binom{s+1}{q} f^{s-q+2} = (A_1 + Qf)\binom{s+1}{q} f^{s-q+1}.$$

Since $p = (s+1)\widetilde{p}$ and $\mathcal{O}_X[1/f, s]f^s$ is a free $\mathcal{O}_X[1/f, s]$-module, we conclude that

$$(s - q + 1)\widetilde{p}(s) f^{s-q} \in \mathcal{D}_X[s] \cdot f^{s-q+1}.$$

Multiplying on the left by t^q (which has the effect of replacing s by $s+q$), we obtain

$$(s+1)\widetilde{p}(s+q) f^s \in \mathcal{D}_X[s] f^{s+1},$$

hence $b_f(s) = (s+1)\widetilde{b}_f(s)$ divides $(s+1)\widetilde{p}(s+q)$. We thus conclude that, indeed, $(s+1)\widetilde{b}_f(s-q)$ divides $p(s)$, completing the proof of the theorem. $\square$

The connection between the minimal exponent and the refined version of multiplier ideals now follows easily:

Proof of Theorem 8.98. By taking a suitable open cover of X, we may and will assume that H is defined by $f \in \mathcal{O}_X(X)$. By Proposition 8.92, the assertion in the theorem is equivalent to the fact that for every $q \in \mathbf{Z}_{\geq 0}$ and $\alpha \in \mathbf{Q} \cap [0,1)$, we have $\partial_t^q \delta \in V^{>\alpha}$ if and only if $q+\alpha < \widetilde{\alpha}(X,H)$. Note also that by Proposition 8.43, we have $\partial_t^q \delta \in V^{>\alpha}$ if and only if all roots of $b_{\partial_t^q \delta}$ are $< -\alpha$. Since $\alpha < 1$, it follows from Theorem 8.99 that this is the case if and only if all roots of $\widetilde{b}_f$ are $< -q-\alpha$, or equivalently, $\widetilde{\alpha}(X,H) > q+\alpha$. □

We end with a bound for $\widetilde{\alpha}(X,H)$ in terms of a log resolution. This was proved in [**MP20**] using techniques involving mixed Hodge modules; the more elementary proof that we give here, based on the Kashiwara-Lichtin estimate for roots of b-functions, is from [**DM22**]. Note that if H is not reduced, then $\mathrm{lct}(X,H) < 1$, hence $\widetilde{\alpha}(X,H) = \mathrm{lct}(X,H)$ can be computed in terms of a log resolution via (8.39).

Theorem 8.100. *Let X be a smooth, irreducible algebraic variety and H a (nonempty) reduced hypersurface in X. Let $\pi\colon Y \to X$ be a log resolution of (X,H) such that the strict transforms of the irreducible components of H on Y are disjoint. If we write $\pi^*(H) = \sum_{i=1}^N a_i E_i$ and $K_{Y/X} = \sum_{i=1}^N k_i E_i$, then*

$$\widetilde{\alpha}(X,H) \geq \min\left\{ \tfrac{k_i+1}{a_i} \mid E_i \text{ exceptional}\right\}. \tag{8.44}$$

Proof. After taking a suitable open cover of X, we may and will assume that H is defined by $f \in \mathcal{O}_X(X)$. Let λ be the right-hand side of (8.44) and let us write $\lambda = q+\beta$, with $q \in \mathbf{Z}_{\geq 0}$ and $\beta \in (0,1]$. We have $\widetilde{\alpha}(X,H) \geq \lambda$ if and only if every root of $\widetilde{b}_f(s-q)$ is $\leq -\beta$; by Theorem 8.99, this is equivalent to $b_{\partial_t^q \delta}(s)$ having all roots $\leq -\beta$. However, by Theorem 8.67 that such a root is either $-m$, for some positive integer m (in which case it is clearly $\leq -\beta$) or of the form $q - \frac{k_i+\ell}{a_i}$, for some positive integer ℓ and some i with E_i exceptional (in which case, it is $\leq q-\lambda = -\beta$). This completes the proof. □

Remark 8.101. Unlike in the case of formula (8.39) that computes $\mathrm{lct}(X,H)$ in terms of any log resolution, we can't hope that the inequality in (8.44) is an equality for an arbitrary resolution. Indeed, if the minimum on the right-hand side of (8.44) is > 1 (so $\widetilde{\alpha}(X,H) > 1$), then one can construct a sequence of blow-ups with smooth centers of codimension 2, such that the corresponding minima converge to 1. Indeed, let E_j be the strict transform of an irreducible component of H (so that $a_j = 1$ and $k_j = 0$) and let E_i be an exceptional divisor that intersects E_j. We consider the blow-up $Y_1 \to Y$ of Y along a connected component of $E_i \cap E_j$, and let G_1 be the exceptional divisor. We consider next the blow-up $Y_2 \to Y_1$ of Y_1 along the intersection of G_1 with the strict transform of E_j and let G_2 be the exceptional divisor, etc. An easy computation shows that the coefficient of G_ℓ in the inverse image of H on Y_ℓ is $a_i + \ell$ and its coefficient in $K_{Y_\ell/X}$ is $k_i + \ell$. Note that $\lim_{\ell\to\infty} \frac{k_i+1+\ell}{a_i+\ell} = 1$.

However, it is an interesting question whether given a reduced hypersurface H on X, whether there is a log resolution $\pi\colon Y \to X$ of (X,H) such that we have equality in (8.44).

APPENDIX A

A very brief introduction to derived categories

In this appendix we review a few basic facts about derived categories that are used in the main text. We will be very brief; for a detailed introduction to derived categories we refer to [**GM03**].

A.1. The derived category

Let $\mathcal{A}$ be an Abelian category (the ones to keep in mind are the categories of left or right modules over a given ring or, more generally, the category of $\mathcal{R}$-modules, where $\mathcal{R}$ is a sheaf of rings on a topological space).

We denote by $\mathrm{Kom}(\mathcal{A})$ the category of complexes $X^\bullet$ of objects in $\mathcal{A}$ (written in cohomological notation), with the morphisms being morphisms of complexes. It is straightforward to see that $\mathrm{Kom}(\mathcal{A})$ is an abelian category, with the kernels and cokernels computed component-wise.

Definition A.1. A morphism of complexes $u\colon X^\bullet \to Y^\bullet$ is a *quasi-isomorphism* if it induces isomorphisms in cohomology, that is, $\mathcal{H}^i(f)\colon \mathcal{H}^i(X^\bullet) \to \mathcal{H}^i(Y^\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$.

The derived category of $\mathcal{A}$ can be obtained by a formal process inverting all quasi-isomorphisms in $\mathrm{Kom}(\mathcal{A})$. This is a process similar in spirit to the localization in a (noncommutative) ring.

Definition A.2. The *derived category* of an abelian category $\mathcal{A}$ is a category $\mathcal{D}(\mathcal{A})$, together with a functor $p\colon \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ such that for every quasi-isomorphism u in $\mathrm{Kom}(\mathcal{A})$, the morphism $p(u)$ is an isomorphism; moreover, p is universal with this property: for every other functor $q\colon \mathrm{Kom}(\mathcal{A}) \to \mathcal{C}$ that maps quasi-isomorphisms to isomorphisms, there is a unique functor $\alpha\colon \mathcal{D}(\mathcal{A}) \to \mathcal{C}$ such that we have $\alpha \circ p = q$. Note that this characterizes $\mathcal{D}(\mathcal{A})$ up to a unique isomorphism.

Example A.3. It follows from the definition that we have a unique functor $\mathcal{H}^i\colon \mathcal{D}(\mathcal{A}) \to \mathcal{A}$ that maps $p(X^\bullet)$ to $\mathcal{H}^i(X^\bullet)$.

It is not hard to see that $\mathcal{D}(\mathcal{A})$ exists by formally inverting quasi-isomorphisms. More precisely, $\mathcal{D}(\mathcal{A})$ and $\mathrm{Kom}(\mathcal{A})$ have the same objects and a morphism in $\mathcal{D}(\mathcal{A})$ is given by concatenating morphisms in $\mathrm{Kom}(\mathcal{A})$ and "formal inverses" of quasi-isomorphisms, up to an obvious equivalence relation induced by composition. However, the problem with this construction is that it does not give a handle on the structure of $\mathcal{D}(\mathcal{A})$ (for example, it is not even clear that $\mathcal{D}(\mathcal{A})$ is an additive category).

A.2. Triangulated categories

The derived category of an Abelian category is not an Abelian category anymore: it is a triangulated category. The difference is that instead of short exact sequences we have a class of distinguished triangles that are required to satisfy suitable axioms. We now introduce this notion and derive a few easy properties.

DEFINITION A.4. A *triangulated category* is an additive category $\mathcal{D}$, together with the following data:

i) The *translation functor* $X \rightsquigarrow X[1]$, which is an automorphism[1] $\mathcal{D} \to \mathcal{D}$.
ii) A collection of *exact triangles*, consisting of the following data:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1],$$

that satisfy the following axioms:

TR$_1$) For every object X in $\mathcal{D}$, the following is an exact triangle

$$X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow X[1].$$

Moreover, given a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1], \end{tikzcd}$$

if the vertical arrows are isomorphisms and the top row is an exact triangle, then the bottom row is an exact triangle as well.

TR$_2$) For every morphism $u \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$, there is an exact triangle

$$X \xrightarrow{u} Y \longrightarrow Z \longrightarrow X[1].$$

TR$_3$) For every exact triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1],$$

the triangles

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

and

$$Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z$$

are exact as well.

TR$_4$) For every diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] & X[1] \arrow[d, "u[1]"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

with the rows exact triangles and such that the first square is commutative, there is a morphism $w\colon Z \to Z'$ (not necessarily unique) that gives a morphism of triangles (that is, all squares are commutative).

[1]Note that the translation functor T is required to be an automorphism, not just an equivalence of categories; in other words, $T \circ T^{-1}$ and $T^{-1} \circ T$ are the identity functors, and not just equivalent to the identity functors.

TR$_5$) (Octahedral axiom[2]). For every morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "1_X"'] & Y \arrow[r, "j"] \arrow[d, "g"'] & A \arrow[r] \arrow[d] & X[1] \arrow[d, "1_X[1]"] \\
X \arrow[r, "g\circ f"] \arrow[d, "f"'] & Z \arrow[r] \arrow[d, "1_Z"] & B \arrow[r] \arrow[d] & X[1] \arrow[d, "f[1]"] \\
Y \arrow[r, "g"] & Z \arrow[r] & C \arrow[r] \arrow[d] & Y[1] \arrow[d, "j[1]"] \\
& & A[1] \arrow[r, "1_A[1]"] & A[1]
\end{tikzcd}$$

such that the first three rows and the third column are exact triangles.

Definition A.5. An additive functor $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ between triangulated categories is an *exact functor* if the following conditions hold:

i) F commutes with the translation functors, that is, we have $F \circ T_{\mathcal{D}_1} = T_{\mathcal{D}_2} \circ F$.
ii) F maps exact triangles to exact triangles.

Remark A.6. It is straightforward to see that if $\mathcal{D}$ is a triangulated category, then its dual $\mathcal{D}^\circ$ has an induced structure of triangulated category, with the translation functor being the inverse of the one on $\mathcal{D}$ and the exact triangles being the same.

In order to illustrate how to apply the axioms in the definition of a triangulated category, we deduce a few useful properties. In what follows we assume that $\mathcal{D}$ is a tringulated category.

Proposition A.7. For every exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1],$$

we have $g \circ f = 0$, $h \circ g = 0$, and $f[1] \circ h = 0$.

Proof. It is enough to prove the fact that $g \circ f = 0$, since the other assertions follow from this one using Axiom TR$_3$. In order to see this, consider the diagram

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "g"'] & Z \arrow[r, "h"] \arrow[d, "1_Z"'] & X[1] \arrow[r, "-f[1]"] & Y[1] \arrow[d, "g[1]"'] \\
Z \arrow[r, "1_Z"] & Z \arrow[r] & 0 \arrow[r] & Z[1]
\end{tikzcd},$$

where the first square is commuatative and the rows are exact triangles (the top one by hypothesis and TR$_3$ and the second one by Axiom TR$_1$). It follows from Axiom TR$_4$ that there is a morphism $u\colon X[1] \to 0$ (which has to be the 0 morphism) such that we get a morphism of triangles. The right-most commutative square then implies $g[1] \circ f[1] = 0$, and thus $g \circ f = 0$. □

[2]This axiom is more involved and we only give it for the sake of completeness. It will not play any role in this appendix.

PROPOSITION A.8. Given an exact triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

in a triangulated category $\mathcal{D}$, for every object in $\mathcal{D}$, the induced complexes

$$\ldots \to \mathrm{Hom}_{\mathcal{D}}(X,A) \to \mathrm{Hom}_{\mathcal{D}}(X,B) \to \mathrm{Hom}_{\mathcal{D}}(X,C) \to \mathrm{Hom}_{\mathcal{D}}\big(X,A[1]\big) \to \ldots$$

and

$$\ldots \mathrm{Hom}_{\mathcal{D}}\big(A[1],X\big) \to \mathrm{Hom}_{\mathcal{D}}(C,X) \to \mathrm{Hom}_{\mathcal{D}}(B,X) \to \mathrm{Hom}_{\mathcal{D}}(A,X) \to \ldots$$

are exact.

PROOF. It is enough to prove the first assertion, since the second one follows by applying the first one to the dual category $\mathcal{D}^{\circ}$. Furthermore, using TR_3, we see that it is enough to show that the following sequence

$$\mathrm{Hom}_{\mathcal{D}}(X,A) \to \mathrm{Hom}_{\mathcal{D}}(X,B) \to \mathrm{Hom}_{\mathcal{D}}(X,C)$$

is exact. The composition is clearly 0 by Proposition A.7. Suppose now that $u \in \mathrm{Hom}_{\mathcal{D}}(X,B)$ is such that $gu = 0$. Consider the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & 0 \arrow[r] \arrow[d] & X[1] \arrow[r, "-1_X[1]"] & X[1] \arrow[d, "u[1]"'] \\
B \arrow[r, "g"] & C \arrow[r] & A[1] \arrow[r, "-f[1]"] & YB[1]
\end{tikzcd} ,$$

in which the first square is commutative by our assumption on u and the first row is exact by Axioms TR_1 and TR_3, while the second row is exact by hypothesis and Axiom TR_3. By Axiom TR_4, there is a morphism $v\colon X \to A$ such that $v[1]$ induces a morphism of triangles. The commutativity of the third square implies $u = fv$, which completes the proof. $\square$

COROLLARY A.9. For an exact triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1],$$

the morphism f is an isomorphism if and only if $C = 0$.

PROOF. We have $C = 0$ if and only if for every object X in $\mathcal{D}$, we have $\mathrm{Hom}_{\mathcal{D}}(X,C) = 0$. By the proposition, this is the case if and only if the induced morphism

$$\mathrm{Hom}_{\mathcal{D}}(X,A) \to \mathrm{Hom}_{\mathcal{D}}(X,B)$$

is an isomorphism. It is easy to see that this is the case if and only if f is an isomorphism. $\square$

DEFINITION A.10. If $\mathcal{D}$ is a triangulated category and $\mathcal{A}$ is an Abelian category, then an additive functor $F\colon \mathcal{D} \to \mathcal{A}$ is a *cohomological functor* if for every exact triangle

$$X \to Y \to Z \to X[1],$$

the sequence

$$F(X) \to F(Y) \to F(Z)$$

is exact. In this case it follows from Axiom TR_4 that we have a long exact sequence

$$\ldots F\big(Z[-1]\big) \to F(X) \to F(Y) \to F(Z) \to F\big(X[1]\big) \to F\big(Y[1]\big) \to \ldots.$$

EXAMPLE A.11. With this terminology, Proposition A.8 says that if $\mathcal{D}$ is a triangulated category, then for every object T in $\mathcal{D}$, the functors

$$\mathrm{Hom}_{\mathcal{D}}(T,-)\colon \mathcal{D}\to \mathcal{A}b \quad \text{and} \quad \mathrm{Hom}_{\mathcal{D}}(-,T)\colon \mathcal{D}^{\mathrm{op}}\to \mathcal{A}b,$$

where $\mathcal{A}b$ is the category of all abelian groups, are cohomological functors.

COROLLARY A.12. If $\mathcal{D}$ is a triangulated category and we have a morphism of exact triangles

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & B \arrow[r] \arrow[d, "v"'] & C \arrow[r] \arrow[d, "w"'] & A[1] \arrow[d, "u[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]. \end{tikzcd}$$

such that u and v are isomorphisms, then w is an isomorphism.

PROOF. Note that if u and v are isomorphisms, then also $u[1]$ and $v[1]$ are isomorphisms. For every object X in $\mathcal{D}$, applying $\mathrm{Hom}_{\mathcal{D}}(X,-)$ and using Proposition A.8, we get a commutative diagram with exact rows. An application of the 5-Lemma gives that $\mathrm{Hom}_{\mathcal{D}}(X,C)\to \mathrm{Hom}_{\mathcal{D}}(X,C')$ is an isomorphism. Since this holds for every X, we conclude that w is an isomorphism. $\square$

REMARK A.13. If $u\colon X\to Y$ is a morphism in a triangulated category $\mathcal{D}$, then Axiom TR_2 gives an exact triangle

$$X \xrightarrow{u} Y \longrightarrow Z \longrightarrow X[1].$$

We will sometimes refer to Z as the *cone of* u and denote it by $C(u)$. Note that this is well-defined up to isomorphism: given another such exact triangle

$$X \xrightarrow{u} Y \longrightarrow Z' \longrightarrow X[1],$$

we get from Axiom TR_4 a morphism $\varphi\colon Z\to Z'$ such that we have a morphism of triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "1_X"'] & Y \arrow[r] \arrow[d, "1_Y"'] & Z \arrow[r] \arrow[d, "\varphi"'] & X[1] \arrow[d, "1_X[1]"] \\ X' \arrow[r] & Y \arrow[r] & Z' \arrow[r] & X[1]. \end{tikzcd}$$

In this case it follows Corollary A.12 that φ is an isomorphism. However, there is a subtlety here in the fact that φ is not unique. This leads to various constructions being non-canonical.

We note that with this notion of cone of a morphism, an useful consequence of the Octahedral Axiom is that given two morphisms $X\xrightarrow{u} Y\xrightarrow{v} Z$, there is an exact triangle

$$C(u)\to C(vu)\to C(v)\to C(u)[1]. \tag{A.1}$$

This can be interpreted as an analogue of the Third Isomorphism Theorem in the context of two monomorphisms $X\hookrightarrow Y\hookrightarrow Z$ in an Abelian category. However, we stress that unlike in that case, this exact triangle (A.1) is not canonical.

EXERCISE A.14. Let $\mathcal{D}$ be a triangulated category.

i) Show that for every objects X, $Z \in \mathcal{D}$, if $i\colon X \to X \oplus Z$ and $X \oplus Z \to Z$ are the canonical morphisms, then the triangle

$$X \xrightarrow{i} X \oplus Z \xrightarrow{p} Z \xrightarrow{0} X[1] \tag{A.2}$$

is exact.

ii) Show that conversely, if we have an exact triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

such that $w = 0$, then this triangle is isomorphic to the triangle (A.2). In particular, we have $Y \simeq X \oplus Z$.

We will also make use of the following notion:

Definition A.15. Let $\mathcal{D}$ be a triangulated category. A full[3] subcategory $\mathcal{D}'$ of $\mathcal{D}$ is a triangulated subcategory if the following conditions are satisfied:

i) If $A \in \mathrm{Ob}(\mathcal{D})$ is isomorphic to $A' \in \mathrm{Ob}(\mathcal{D}')$, then $A \in \mathrm{Ob}(\mathcal{D}')$.
ii) If $A \in \mathrm{Ob}(\mathcal{D}')$, then $A[1] \in \mathrm{Ob}(\mathcal{D}')$ and $A[-1] \in \mathrm{Ob}(\mathcal{D}')$.
iii) Given an exact triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

in $\mathcal{D}$, if $A, C \in \mathrm{Ob}(\mathcal{D}')$, then $B \in \mathrm{Ob}(\mathcal{D}')$.

A.3. The derived category as a triangulated category

Returning to the derived category of an Abelian category, we have the following

Theorem A.16. *If $\mathcal{A}$ is an Abelian category, then the derived category $\mathcal{D}(\mathcal{A})$ has a structure of triangulated category.*

The translation functor in $\mathcal{D}(\mathcal{A})$ maps a complex $A^\bullet$ to $A^\bullet[1]$, where $A^\bullet[1]^n = A^{n+1}$ and the differentials are the negative of the original maps. We do not describe the exact triangles, except mentioning the following

Remark A.17. It is a consequence of the description of exact triangles in the derived category that if

$$u \to v \to w \to u[1]$$

is an exact triangle in $\mathcal{D}(\mathcal{A})$, then we get a long exact sequence in $\mathcal{A}$:

$$\ldots \to \mathcal{H}^i(u) \to \mathcal{H}^i(v) \to \mathcal{H}^i(w) \to \mathcal{H}^{i+1}(u) = \mathcal{H}^i\big(u[1]\big) \ldots.$$

In other words, the functors $\mathcal{H}^i(-)\colon \mathcal{D}(\mathcal{A}) \to \mathcal{A}$ are cohomological functors.

Example A.18. Given an exact sequence

$$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$$

in $\mathrm{Kom}(\mathcal{A})$, there is a morphism $h\colon C^\bullet \to A^\bullet[1]$ in $\mathcal{D}(\mathcal{A})$ such that we have an exact triangle in $\mathcal{D}(\mathcal{A})$:

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} A^\bullet[1].$$

Using the previous remark, we recover the long exact sequence associated to a short exact sequence of complexes.

[3] A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is *full* if for every two objects X and Y in $\mathcal{C}'$, we have $\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

EXAMPLE A.19. If $C^\bullet$ is complex in $\mathrm{Kom}(\mathcal{A})$, then for every $i \in \mathbf{Z}$, we consider the "stupid" truncations

$$\sigma^{\leq i}(C^\bullet): \quad \ldots \to C^{i-1} \to C^i \to 0 \to 0 \to \ldots$$

and

$$\sigma^{\geq i+1}(C^\bullet): \quad \ldots 0 \to 0 \to C^{i+1} \to C^{i+2} \to \ldots.$$

It is clear that we have a short exact sequence of complexes

$$0 \to \sigma^{\leq i}(C^\bullet) \to C^\bullet \to \sigma^{\geq i+1}(C^\bullet) \to 0,$$

hence via the previous example we get a corresponding exact triangle in $\mathcal{D}(\mathcal{A})$. However, note that these "stupid" truncation functors are not defined on $\mathcal{D}(\mathcal{A})$, since they don't map quasi-isomorphisms to quasi-isomorphisms.

EXAMPLE A.20. Better behaved truncation functors are defined as follows. If $C^\bullet$ is a complex in $\mathrm{Kom}(\mathcal{A})$ and $i \in \mathbf{Z}$, then we put

$$\tau^{\leq i}(C^\bullet): \to C^{i-1} \to C^i \to \mathrm{Im}(C^i \to C^{i-1}) \to 0 \to 0 \to \ldots$$

and

$$\tau^{\geq i+1}(C^\bullet): \to 0 \to 0 \to C^{i+1}/\mathrm{Im}(C^i \to C^{i+1}) \to C^{i+2} \to C^{i+3} \to \ldots.$$

Note that the canonical morphism $\tau^{\leq i}(C^\bullet) \to C^\bullet$ (respectively $C^\bullet \to \tau^{\geq i+1}(C^\bullet)$) induces isomorphisms after applying $\mathcal{H}^p$ with $p \leq i$ (respectively $p \geq i+1$), while $\mathcal{H}^p\big(\tau^{\leq i}(C^\bullet)\big) = 0$ for $p > i$ and $\mathcal{H}^p\big(\tau^{\geq i+1}(C^\bullet)\big) = 0$ for $p \leq i$. It is easy to deduce that we get induced functors $\tau^{\leq i}, \tau^{\geq i+1}\colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$. Moreover, using Example A.17, we see that for every $u \in \mathcal{D}(\mathcal{A})$ and every $i \in \mathbf{Z}$, we have an exact triangle in $\mathcal{D}(\mathcal{A})$:

$$\tau^{\leq i}(u) \to u \to \tau^{\geq i+1}(u) \to \tau^{\leq i}(u)[1].$$

REMARK A.21. The truncation functors in the previous example enjoy a useful universal property: for every $u, v \in \mathcal{D}(\mathcal{A})$, if $\mathcal{H}^i(v) = 0$ for all $i > m$, then the canonical morphism $\tau^{\leq m}(u) \to u$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}\big(v, \tau^{\leq m}(u)\big) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(v, u).$$

Similarly, if $\mathcal{H}^i(v) = 0$ for all $i \leq m$, then the canonical morphism $u \to \tau^{\geq m+1}(u)$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}\big(\tau^{\geq m+1}(u), v\big) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(u, v).$$

REMARK A.22. By mapping an object M in $\mathcal{A}$ to the complex $C^\bullet(M)$ with $C^0(M) = M$ and $C^i(M) = 0$ for $i \neq 0$ we get a functor $\mathcal{A} \to \mathcal{D}(\mathcal{A})$. This gives an equivalence of categories between $\mathcal{A}$ and the full subcategory of $\mathcal{D}(\mathcal{A})$ given by $\big\{u \in \mathcal{D}(\mathcal{A}) \mid \mathcal{H}^i(u) = 0 \text{ for } i \neq 0\big\}$. We always think of $\mathcal{A}$ as a full subcategory of $\mathcal{D}(\mathcal{A})$ in this way.

REMARK A.23. Inside $\mathcal{D}(\mathcal{A})$ there are several important triangulated subcategories:

$$\mathcal{D}^+(\mathcal{A}) = \big\{u \in \mathcal{D}(\mathcal{A}) \mid \mathcal{H}^i(u) = 0 \text{ for } i \ll 0\big\},$$
$$\mathcal{D}^-(\mathcal{A}) = \big\{u \in \mathcal{D}(\mathcal{A}) \mid \mathcal{H}^i(u) = 0 \text{ for } i \gg 0\big\}, \quad \text{and}$$
$$\mathcal{D}^b(\mathcal{A}) = \big\{u \in \mathcal{D}(\mathcal{A}) \mid \mathcal{H}^i(u) = 0 \text{ for } |i| \gg 0\big\}.$$

Using the truncation functors $\tau^{\leq m}$ and $\tau^{\geq n}$ (see Example A.20), one can easily see that every object in $\mathcal{D}^b(\mathcal{A})$ can be represented by a bounded complex (that is, a complex $A^\bullet$ such that $A^i = 0$ if $|i| \gg 0$). Similar assertions hold for the objects of $\mathcal{D}^+(\mathcal{A})$ and $\mathcal{D}^-(\mathcal{A})$.

Other important examples arise in the presence of finiteness conditions: for example, if X is a Noetherian scheme and $\mathcal{A}$ is the category of $\mathcal{O}_X$-modules, then $\mathcal{D}_{\mathrm{qc}}(X)$ (respectively, $\mathcal{D}_{\mathrm{coh}}(X)$) consists of the objects in $u \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{H}^i(u)$ is quasi-coherent (respectively, coherent) for every i. The subcategories $\mathcal{D}^+_{\mathrm{coh}}(X)$, $\mathcal{D}^-_{\mathrm{coh}}(X)$, and $\mathcal{D}^b_{\mathrm{coh}}(X)$ are then defined in the obvious way.

A.4. Derived functors

We end this appendix with a discussion of derived functors in the setting of derived categories. In order to fix the ideas, suppose that $F\colon \mathcal{A} \to \mathcal{B}$ is a left exact functor between Abelian categories, such that $\mathcal{A}$ has enough injective objects. Recall that in classical homological algebra we associate to F a sequence of functors $(R^iF\colon \mathcal{A} \to \mathcal{B})_{i\geq 0}$, the *derived functors* of F. In the setting of derived categories, we have a *derived functor* $\mathbf{R}F\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ such that $R^iF(M) \simeq \mathcal{H}^i\big(\mathbf{R}F(M)\big)$ for every object M of $\mathcal{A}$. This has several advantages: first, we can apply R^iF to arbitrary objects in $\mathcal{D}^+(\mathcal{A})$, instead of just objects of $\mathcal{A}$ (recovering in this way what is known as *hypercohomology* is the classical setting); second, it allows us to compose such transformations.

The functor $\mathbf{R}F$ is defined as follows. Given a complex $A^\bullet$ with $\mathcal{H}^i(A^\bullet) = 0$ for $i \ll 0$, we consider an *injective resolution* $Q^\bullet$, that is, a quasi-isomorphism $A^\bullet \to Q^\bullet$, where $Q^\bullet$ is a complex of injective objects in $\mathcal{A}$, with $Q^i = 0$ for $i \ll 0$. Then one takes $\mathbf{R}F(A^\bullet) := F(Q^\bullet)$ and one shows, as in the classical case, that this is independent of the injective resolution up to a canonical isomorphism. This induces a functor $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(B)$ and we put $R^iF = \mathcal{H}^i \circ \mathbf{R}F$. Note that we recover the usual derived functors on the objects of $\mathcal{A}$.

One can show that one can compute $\mathbf{R}F(A^\bullet)$ using complexes of F-acyclic objects (these are objects $B \in \mathcal{A}$ such that $R^iF(B) = 0$ for all $i \geq 1$). More precisely, if $B^\bullet$ is a complex of F-acyclic objects, with $B^i = 0$ for $i \ll 0$, then $\mathbf{R}F(B^\bullet) \simeq F(B^\bullet)$.

In the case of a right exact functor $F\colon \mathcal{A} \to \mathcal{B}$, with $\mathcal{A}$ having enough projective objects, dual considerations give the *left derived functor* $\mathbf{L}F\colon \mathcal{D}^-(\mathcal{A}) \to \mathcal{D}^-(\mathcal{B})$.

Remark A.24. If $A^\bullet$ is a complex with $\mathcal{H}^i(A^\bullet) = 0$ for $i \ll 0$, then one can compute $R^iF(A^\bullet)$ using the two *hypercohomology spectral sequences*. One of these makes use of the derived functors of the terms in $A^\bullet$:

$$E_1^{p,q} = R^jF(A^i) \Rightarrow R^{i+j}F(A^\bullet)$$

and the E_2 terms are given by $E_2^{p,q} = \mathcal{H}^i\big(R^jF(A^\bullet)\big)$. The other spectral sequence uses the derived functors of the cohomology of $A^\bullet$:

$$E_2^{p,q} = R^iF\big(\mathcal{H}^j(A^\bullet)\big) \Rightarrow R^{i+j}F(A^\bullet).$$

Remark A.25. Suppose that $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ are left exact functors between Abelian categories, with $\mathcal{A}$ and $\mathcal{B}$ having enough injectives, and $F(Q)$ being G-acyclic for every injective object $Q \in \mathcal{A}$. In the classical case, we have the Grothendieck spectral sequence relating the derived functors of F, G, and $G \circ F$. This is upgraded at the level of derived categories to an equivalence of functors $\mathbf{R}G \circ \mathbf{R}F \simeq \mathbf{R}(G \circ F)$.

REMARK A.26. Suppose that $F\colon \mathcal{A} \to \mathcal{B}$ is a left exact functor and $\mathcal{A}$ has enough injective objects. It is easy to see, using the truncation functors in Example A.20, that in order to show that $\mathbf{R}F$ induces a functor $\mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B})$, it is enough to show that for every object A in $\mathcal{A}$, we have $R^iF(A) = 0$ for $i \gg 0$. For example, this is the case for the functor $\mathbf{R}f_*$, when $f\colon X \to Y$ is a morphism between two algebraic varieties over a field k. Similarly, the fact that in this case R^if_* preserves quasi-coherent sheaves implies that we have an induced functor $\mathcal{D}^b_{\mathrm{qc}}(X) \to \mathcal{D}^b_{\mathrm{qc}}(Y)$. This further induces a functor $\mathcal{D}^b_{\mathrm{coh}}(X) \to \mathcal{D}^b_{\mathrm{coh}}(Y)$ if f is proper.

If $\mathcal{A}$ is an Abelian category with enough injective objects, then for every object M in $\mathcal{A}$, we can construct $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M,-)$ as the derived functor of the left exact functor $\mathrm{Hom}_{\mathcal{A}}(M,-)$. More generally, we have a bifunctor

$$\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(-,-)\colon \mathcal{D}^-(\mathcal{A}) \times \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{A}),$$

exact in each variable, such that if $A^\bullet$ is a complex with $A^i = 0$ for $i \gg 0$ and $B^\bullet \to Q^\bullet$ is a quasi-isomorphism, with $Q^\bullet$ a complex of injective objects with $Q^i = 0$ for $i \ll 0$, then $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(A^\bullet, B^\bullet)$ is given by the complex $\mathrm{Hom}^\bullet_{\mathcal{A}}(A^\bullet, Q^\bullet)$, where

$$\mathrm{Hom}^m_{\mathcal{A}}(A^\bullet, Q^\bullet) = \oplus_{p-q=m}\mathrm{Hom}_{\mathcal{A}}(A^q, Q^p),$$

and the differential is induced, up to suitable signs, by the differentials on the two complexes (note that the above direct sum has only finitely many nonzero terms because of our assumptions on $A^\bullet$ and $Q^\bullet$). We also note that if $A^\bullet$ has a projective resolution $P^\bullet \to A^\bullet$, we also compute $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(A^\bullet, B^\bullet)$ as given by $\mathrm{Hom}^\bullet_{\mathcal{A}}(P^\bullet, B^\bullet)$.

REMARK A.27. For every Abelian category $\mathcal{A}$ and every u, $v \in \mathcal{D}(\mathcal{A})$ and every $i \in \mathbf{Z}$, we put

$$\mathrm{Ext}^i(u,v) := \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}\big(u, v[i]\big).$$

It is a basic fact that if $u \in \mathcal{D}^-(\mathcal{A})$ and $v \in \mathcal{D}^+(\mathcal{A})$, then there is a canonical isomorphism

$$\mathrm{Ext}^i(u,v) \simeq \mathcal{H}^i\big(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(u,v)\big).$$

REMARK A.28. If $\mathcal{R}$ is a sheaf of (possibly noncommutative) rings on a topological space X and $\mathcal{A}$ is the category of $\mathcal{R}$-modules on X (which has enough injectives), if we use the sheaf $\mathcal{H}om_{\mathcal{R}}(-,-)$ instead of $\mathrm{Hom}_{\mathcal{R}}(-,-)$, we obtain as above the bifunctor, exact in each variable,

$$\mathbf{R}\mathcal{H}om_{\mathcal{R}}(-,-)\colon \mathcal{D}^-(\mathcal{A}) \times \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{A}).$$

This can be computed either using injective resolutions in the second entry or using resolutions by locally free $\mathcal{R}$-modules in the first entry (when these are available).

Another useful derived functor is the derived tensor product. Suppose that $\mathcal{R}$ is a sheaf of rings on the topological space X and $\mathcal{A}$ is the category of $\mathcal{R}$-modules on X. While $\mathcal{A}$ does not have enough projectives in general, it turns out that the derived tensor product can be computed using *flat* resolutions (so this constructions falls outside of the general framework we discussed so far). One can show that for every complex $A^\bullet$ of $\mathcal{R}$-modules with $\mathcal{H}^i(A^\bullet) = 0$ for $i \gg 0$, there is a *flat resolution*, that is, a quasi-isomorphism $F^\bullet \to A^\bullet$, with $F^i = 0$ for $i \gg 0$ and F^i a flat $\mathcal{R}$-module for all i. There is a bifunctor

$$- \otimes^L -\colon \mathcal{D}^-(\mathcal{A}) \times \mathcal{D}^-(\mathcal{A}) \to \mathcal{D}^-(\mathcal{A}),$$

exact in each entry, such that if u and v are represented, respectively, by the complexes $A^\bullet$ and $B^\bullet$, and $F^\bullet \to A^\bullet$ and $G^\bullet \to B^\bullet$ are flat resolutions, then

$$u \otimes^L v := F^\bullet \otimes_{\mathcal{R}} G^\bullet \simeq F^\bullet \otimes_{\mathcal{R}} B^\bullet \simeq A^\bullet \otimes_{\mathcal{R}} G^\bullet$$

(note that the isomorphisms hold in the derived category). Recall that the tensor product of two complexes $F^\bullet$ and $G^\bullet$ is given by the complex $T^\bullet$, with

$$T^m = \bigotimes_{p+q=m} F^p \otimes_{\mathcal{R}} G^q,$$

and the differential is induced, up to suitable signs, by the differentials on the two complexes. It is common to write $\mathcal{T}or_i^{\mathcal{R}}(u,v)$ for $\mathcal{H}^{-i}(u \otimes^L v)$.

Remark A.29. Since tensor product commutes with localization, it follows that the Tor functors commute with localization as well.

Remark A.30. If $\mathcal{R}$ is a sheaf of commutative rings such that for every $x \in X$ the ring $\mathcal{R}_x$ is regular, of dimension $\leq d$, then $\mathcal{T}or_i^{\mathcal{R}}(M,N) = 0$ for every $\mathcal{R}_x$-modules M and N and every $i > d$ (since every $\mathcal{R}_x$-module has a projective resolution of length $\leq d$, see [**Mat89**, Theorem 19.2]). This implies that for every $\mathcal{R}$-modules $\mathcal{M}$ and $\mathcal{N}$, we have $\mathcal{T}or_i^{\mathcal{R}}(\mathcal{M},\mathcal{N}) = 0$ for $i > d$. A standard argument using the exact triangles in Example A.19 allows us to deduce that $u \otimes^L v \in \mathcal{D}^b(\mathcal{R})$ if $u, v \in \mathcal{D}^b(\mathcal{R})$.

Suppose now that $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces. In this case we have the right exact functor $f^* = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(-)$ from the Abelian category of $\mathcal{O}_Y$-modules to the Abelian category of $\mathcal{O}_X$-modules. Let us write $\mathcal{D}(\mathcal{O}_Y)$ and $\mathcal{D}(\mathcal{O}_X)$ for the corresponding derived categories. Note that the functor f^{-1} on the category of $\mathcal{O}_Y$-modules is exact. We thus have an exact functor

$$\mathbf{L}f^*\colon \mathcal{D}^-(\mathcal{O}_Y) \to \mathcal{D}^-(\mathcal{O}_X), \quad \mathbf{L}f^*(u) = \mathcal{O}_X \otimes^L_{f^{-1}(\mathcal{O}_Y)} f^{-1}(u).$$

Note that if $u = A^\bullet \in \mathcal{D}^-(\mathcal{O}_Y)$ and $A^\bullet$ has a flat resolution $P^\bullet \to A^\bullet$, then $f^{-1}(A^\bullet)$ has the flat resolution $f^{-1}(P^\bullet)$, and thus $\mathbf{L}f^*(u)$ is given by $f^*(P^\bullet)$, which accounts for the fact that we end up in the derived category of $\mathcal{O}_X$-modules. We write $L^i f^*$ for $\mathcal{H}^i \circ \mathbf{L}f^*$.

Remark A.31. If $\mathcal{O}_{Y,y}$ is a regular ring of dimension $\leq d$ for every $y \in Y$, then it follows from Remark A.30 that $\mathbf{L}f^*$ induces an exact functor $\mathbf{L}f^*\colon \mathcal{D}^b(\mathcal{O}_Y) \to \mathcal{D}^b(\mathcal{O}_X)$.

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