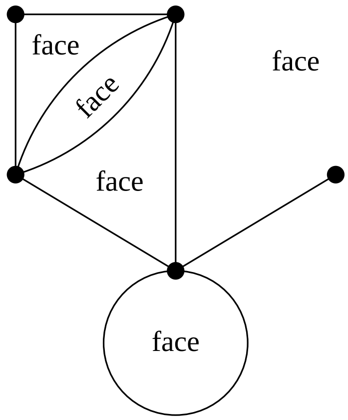
Complex Systems 535/Physics 508: Homework 3

1. **Mean degree of a planar graph:** Consider a connected planar graph with n vertices and m edges. Let f by the number of "faces" of the graph, i.e., areas bounded by edges when the graph is drawn in planar form. The "outside" of the graph, the area extending to infinity on all sides, is also considered a face. The graph can have self-loops and multiedges.



   (i) Write down the values of n, m, and f for a graph with a single vertex and no edges.

   (ii) How do n, m, and f change when we add a single vertex to the graph along with a single edge attaching it to another vertex?

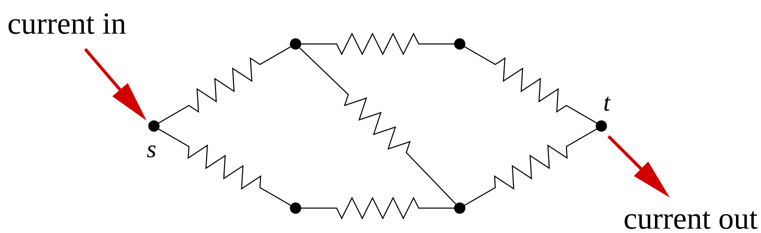
   (iii) How do n, m, and f change when we add a single edge between two extant vertices (or a self-loop attached to just one vertex), in such a way as to maintain the planarity of the graph?

   (iv) Hence by induction prove a general relation between n, m, and f for all connected planar graphs.

   (v) Now suppose our graph is simple (i.e., contains no multiedges) and has no self-loops. Show that the mean degree $z = \langle k \rangle$ of such a graph is strictly less than six.

   The mean degree of intersections in the US interstate road network (which is very nearly planar) is 2.86.

2. **Resistor networks and the graph Laplacian:** Consider a network in which each edge consists of a resistor of resistance 1Ω, with resistors joined together electrically by their leads at the vertices:



   One ampere of current is injected into the network at a source vertex s and one ampere is removed again at the target vertex t. Let V_i be the voltage a vertex i, measured relative to any convenient reference potential, which you can choose however you like.

(i) Kirchhoff's current law says that the current flowing through any vertex in the network is conserved, i.e., that the total amount flowing out of the vertex equals the total amount flowing in at all times. Show that the vector **V** of voltages satisfies

$$\mathbf{L} \cdot \mathbf{V} = \mathbf{s}, \tag{1}$$

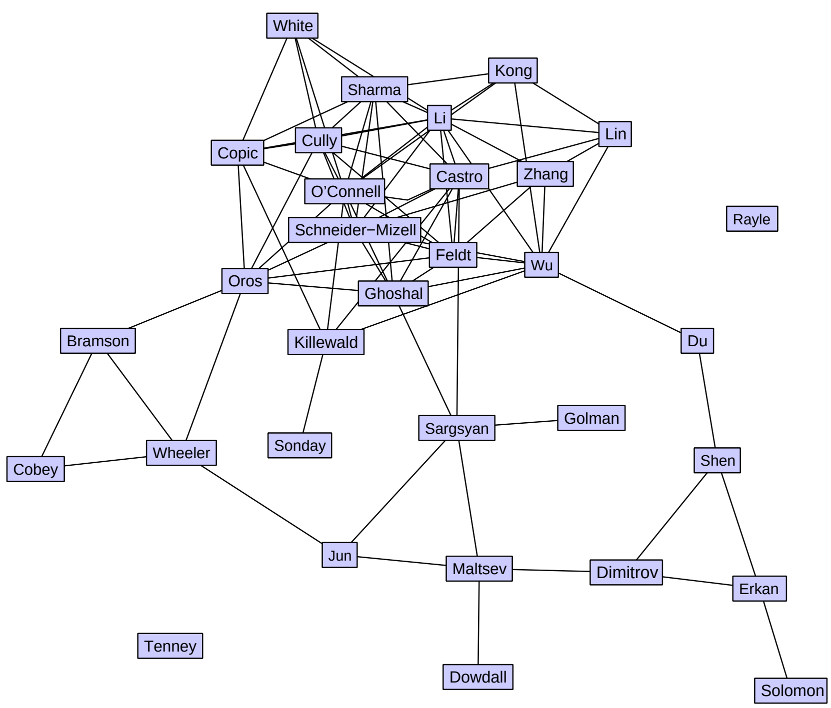
where **L** is the graph Laplacian and **s** is the vector with components

$$s_i = \begin{cases} +1 & \text{for } i = s, \\ -1 & \text{for } i = t, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Equation (1) unfortunately cannot be inverted to solve for the voltages. Why not? Hence suggest a way of modifying the equation so that it *can* be inverted.

(iii) Hence calculate the voltages at the vertices for the network in the figure, measured relative to the target vertex (i.e., giving the target vertex voltage zero). Hint: you'll need a computer for this bit unless you want to invert the matrix by hand, which you don't.

3. **Centrality:** Here's my sorry excuse for a plot of the (undirected) class social network:



(i) Which four vertices have the highest degree? (We did this before.)

(ii) Which four vertices have the highest PageRank for $\alpha = 0.85$? Give a (one sentence) conjecture about the reason for the apparent conflict between the answers to (i) and (ii).

(iii) Recalculate the PageRank for $\alpha = 0.5$. What does the difference from (ii) mean?

(iv) **Extra credit:** Find the four individuals in the class with highest betweenness centrality by any method you like.