Complex Systems 535/Physics 508: Homework 9

1. **The small-world model:** Consider the version of the small-world model in which one adds shortcuts to the one-dimensional lattice but never takes any edges away. Let p be the probably per edge on the underlying lattice of adding a shortcut, so that the expected number of shortcuts is nrp, where r is the maximum range of edges on the underlying lattice. You can assume that $r \ll n$ and that n is very large.

   (i) What is the probability that a vertex has degree k?

   (ii) How many triangles are there in the network?

   (iii) How many two-stars are there in the network, i.e., a vertex connected to an unordered pair of other vertices—count separately every unordered pair for each vertex. (On the midterm exam you showed that this number is $nr(2r-1)$ when $p=0$.) You can assume that p is small enough that each vertex is attached to only zero or one shortcuts.

   (iv) Hence show that the clustering coefficient for this version of the small-world network is

   $$\frac{3(r-1)}{2(2r-1)+8rp}$$

   in this limit. Confirm that this gives the correct result in the limit $p \to 0$.

2. **Generating functions for growing graphs:** Recall the rate equation for Price's model of a citation network in the limit of large n:

   $$p_k = \frac{c}{c+a}[(k-1+a)p_{k-1} - (k+a)p_k] \qquad \text{(for } k > 0\text{)},$$
   $$p_0 = 1 - \frac{c}{c+a} p_0.$$

   (i) Write down the special case of these equations for $c = a = 1$.

   (ii) Show that the degree distribution generating function $g(x) = \sum_{k=0}^{\infty} p_k x^k$ for this case satisfies the differential equation

   $$g(x) = 1 + \tfrac{1}{2}(x-1)[xg'(x) + g(x)].$$

   (iii) Show that the function

   $$h(x) = \frac{x^3 g(x)}{(1-x)^2}$$

   satisfies

   $$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{2x^2}{(1-x)^3}.$$

   (iv) Hence find a closed-form solution for the generating function $g(x)$. Confirm that your solution has the correct limiting values $g(0) = p_0$ and $g(1) = 1$.

   (v) Thus find a value for the mean in-degree of a vertex in Price's model. Is this what you expected?