Midterm Examination : October 28, 1997
MEAM 501 Analytical Methods in Mechanics

1997 Fall

1. (50%) Two vectors

$$\boldsymbol{a} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{Bmatrix} -1 \\ -1 \\ 2 \end{Bmatrix}$$

are given.

(1) Are they orthogonal ? If not orthogonalize the vector ***b*** with respect to ***a***.

$$\boldsymbol{a}^T\boldsymbol{b} = -1 - 1 + 2 = 0 \quad \Rightarrow \quad \boldsymbol{a} \perp \boldsymbol{b}$$

(2) Are they linearly independent ? Check it.

$$\alpha\boldsymbol{a} + \beta\boldsymbol{b} = \begin{Bmatrix} \alpha - \beta \\ \alpha - \beta \\ \alpha + 2\beta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \Rightarrow \quad \alpha - \beta = 0 \text{ and } \alpha + 2\beta = 0 \quad \Rightarrow \quad \alpha = \beta = 0$$

that is, there are independent.

(3) Let S be the linear subspace of $\boldsymbol{R}^3$ spanned by the vectors ***a*** and ***b***. Find the orthogonal compliment $S^\perp$ of S.

$$\boldsymbol{c} \in S^\perp \quad \Leftrightarrow \quad \boldsymbol{c}^T\boldsymbol{a} = 0 \text{ and } \boldsymbol{c}^T\boldsymbol{b} = 0$$

that is

$$c_1 + c_2 + c_3 = 0 \quad \text{and} \quad -c_1 - c_2 + 2c_3 = 0$$

This implies $\boldsymbol{c} = c_1 \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}$, that is, the orthogonal compliment of S is spanned by a vector $\begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}$.

Now let us define a matrix $\boldsymbol{A}$ by the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as the column vectors of the matrix : $\boldsymbol{A} = [\boldsymbol{a}, \boldsymbol{b}]$.

(4) What is the matrix $\boldsymbol{A}$?

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$$

(5) What is the transpose of $\boldsymbol{A}$?

$$\boldsymbol{A}^T = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

(6) What is the null space of $\boldsymbol{A}$? Give the definition.

$$N(\boldsymbol{A}) = \left\{ \boldsymbol{x} \in R^2 : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \right\}$$

Solving $\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$, we have $\boldsymbol{x} = \boldsymbol{0}$, that is, the null space of $\boldsymbol{A}$ is singleton.

(7) What is the rank of $\boldsymbol{A}$? Give the definition as well as the rank of $\boldsymbol{A}$.

Since the rank of $\boldsymbol{A}$ is the number of linearly independent column vectors, and since $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent, the rank of $\boldsymbol{A}$ is 2.

(8) Is the transformation $\boldsymbol{A}$ one-to-one ? After state the definition of a “one-to-one” mapping, explain whether $\boldsymbol{A}$ is one-to-one. Explain why, too.

The transformation $\boldsymbol{A}$ is one-to-one if $\boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{A}\boldsymbol{x}_2 \;\Rightarrow\; \boldsymbol{x}_1 = \boldsymbol{x}_2$ that is the null space of $\boldsymbol{A}$ is just $\boldsymbol{0}$, i.e. singleton. Since the null space of $\boldsymbol{A}$ is zero, $\boldsymbol{A}$ is one-to-one.

(9) Is the transformation $\boldsymbol{A}$ onto ? After state the definition of a "onto" mapping, explain whether $\boldsymbol{A}$ is onto. Explain why, too.

If the range of $\boldsymbol{A}$ is the whole space, then $\boldsymbol{A}$ is onto. Noting that the range of $\boldsymbol{A}$ is a linear space spanned by two linearly independent column vectors a and b of $\boldsymbol{A}$, the range is not the same with the linear space of three component vectors $\boldsymbol{R}^3$. Thus, $\boldsymbol{A}$ is not onto.

(10) Show that the transformation (mapping) $\boldsymbol{A}$ is linear.

$$\boldsymbol{A}(\alpha\boldsymbol{x}+\beta\boldsymbol{y}) = \alpha\boldsymbol{A}\boldsymbol{x}+\beta\boldsymbol{A}\boldsymbol{y} \quad , \quad \forall\alpha,\beta\in\boldsymbol{R}, \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{R}^2$$

(11) Find $\boldsymbol{A}^T\boldsymbol{A}$.

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

while

$$\boldsymbol{A}\boldsymbol{A}^T = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

(12) Compute the determinant of $\boldsymbol{A}^T\boldsymbol{A}$.

$$\det\left(\boldsymbol{A}^T\boldsymbol{A}\right) = \begin{vmatrix} 3 & 0 \\ 0 & 6 \end{vmatrix} = 18 \qquad \text{while} \qquad \det\left(\boldsymbol{A}\boldsymbol{A}^T\right) = 0$$

(13) Give the characteristic polynomials of $\boldsymbol{A}^T\boldsymbol{A}$.

$$\det\left(\boldsymbol{A}^T\boldsymbol{A}-\lambda\boldsymbol{I}\right) = \begin{vmatrix} 3-\lambda & 0 \\ 0 & 6-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda) = 0$$

(14) How to find the roots of the characteristic functions ? Find the roots of $\boldsymbol{A}^T\boldsymbol{A}$.

$$\Rightarrow \quad \lambda = 3 \quad \text{and} \quad 6$$

(15) What is the definition of the eigenvector of $\boldsymbol{A}^T\boldsymbol{A}$? How many eigenvectors we have ?

Non-zero vectors satisfying the equation $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ for the roots of the characteristic polynomial, which are normalized in a specified way, for example $\|\boldsymbol{x}\| = 1$, are called the eigen vectors.

$$\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \quad \Leftrightarrow \quad \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 3\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 6\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Solving these, we have

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Here we have normalize so as to be unit.

2. (30%) Let a matrix $\boldsymbol{A}$ be a m-by-n real rectangular matrix, and let $\boldsymbol{b}$ be a m-component real vector.

(1) State the singular value decomposition theorem.

There exist orthogonal matrices $\boldsymbol{U} \in \boldsymbol{R}^{m\times m}$ and $\boldsymbol{V} \in \boldsymbol{R}^{n\times n}$ and a “diagonal” matrix $\Sigma \in \boldsymbol{R}^{m\times n}$ such that $\boldsymbol{A} = \boldsymbol{U}\Sigma\boldsymbol{V}^T$ where

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & ..... & 0 \\ 0 & \sigma_2 & ..... & 0 \\ : & : & & : \\ 0 & 0 & ..... & \sigma_p \\ 0 & 0 & ..... & 0 \\ : & : & & : \\ 0 & 0 & ..... & 0 \end{bmatrix} \quad \text{or} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & ..... & 0 & 0 & ..... & 0 \\ 0 & \sigma_2 & ..... & 0 & 0 & ..... & 0 \\ : & : & & : & : & & : \\ 0 & 0 & ..... & \sigma_p & 0 & ..... & 0 \end{bmatrix}$$

$$\sigma_1 \ge \sigma_2 \ge ..... \ge \sigma_r > \sigma_{r+1} \ge \sigma_p \ge 0 \quad , \quad p = \min\{m,n\},$$

where r is the rank of the matrix $\boldsymbol{A}$.

(2) Obtain the pseudo-inverse (generalized inverse) $\boldsymbol{A}^+$ by using the singular value decomposition.

$$\boldsymbol{A}^+ = \boldsymbol{V}\Sigma^+\boldsymbol{U}^T$$

where

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & 0 & ..... & 0 & 0 & ..... & 0 \\ 0 & 1/\sigma_2 & ..... & 0 & 0 & ..... & 0 \\ : & : & & : & : & & : \\ 0 & 0 & ..... & 1/\sigma_r & 0 & ..... & 0 \\ 0 & 0 & ..... & 0 & 0 & ..... & 0 \\ : & : & & : & : & & : \\ 0 & 0 & ..... & 0 & 0 & ..... & 0 \\ 0 & 0 & ..... & 0 & 0 & ..... & 0 \\ : & : & & : & : & & : \\ 0 & 0 & ..... & 0 & 0 & ..... & 0 \end{bmatrix} \qquad p = \min\{m,n\} \quad , \quad m > n$$

or

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & 0 & \ldots\ldots & 0 & 0 & \ldots\ldots & 0 & 0 & \ldots\ldots & 0 \\ 0 & 1/\sigma_2 & \ldots\ldots & 0 & 0 & \ldots\ldots & 0 & 0 & \ldots\ldots & 0 \\ : & : & & : & : & & : & : & & : \\ 0 & 0 & \ldots\ldots & 1/\sigma_r & 0 & \ldots\ldots & 0 & 0 & \ldots\ldots & 0 \\ 0 & 0 & \ldots\ldots & 0 & 0 & \ldots\ldots & 0 & 0 & \ldots\ldots & 0 \\ 0 & 0 & \ldots\ldots & : & : & & : & : & & : \\ 0 & 0 & \ldots\ldots & 0 & 0 & \ldots\ldots & 0 & 0 & \ldots\ldots & 0 \end{bmatrix} \quad p = \min\{m,n\} \quad , \quad m \le n$$

that is, the singular values σ_i , $i = 1,\ldots, p$ are taken their inverse only for non-zeros.

(3) Show that $\boldsymbol{x} = \boldsymbol{A}^+ \boldsymbol{b}$ is a solution of the least squares problem of

$$\min_{\boldsymbol{x}} \frac{1}{2} \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|^2$$

where $\|\boldsymbol{g}\|$ is the natural norm of a vector $\boldsymbol{g}$ defined by $\|\boldsymbol{g}\| = \sqrt{\boldsymbol{g}^T \boldsymbol{g}}$.

Noting that

$$\min_{\boldsymbol{x}} \frac{1}{2} \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|^2 \quad \Leftrightarrow \quad \boldsymbol{A}^T \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T \boldsymbol{b}$$

we shall show that $\boldsymbol{x} = \boldsymbol{A}^+ \boldsymbol{b}$ satisfies the equation $\boldsymbol{A}^T \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T \boldsymbol{b}$. Indeed

$$\boldsymbol{A}^T \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T \boldsymbol{A}\boldsymbol{A}^+ \boldsymbol{b} = \left(\boldsymbol{U}\Sigma\boldsymbol{V}^T\right)^T \left(\boldsymbol{U}\Sigma\boldsymbol{V}^T\right)\left(\boldsymbol{V}\Sigma^+\boldsymbol{U}^T\right)\boldsymbol{b}$$

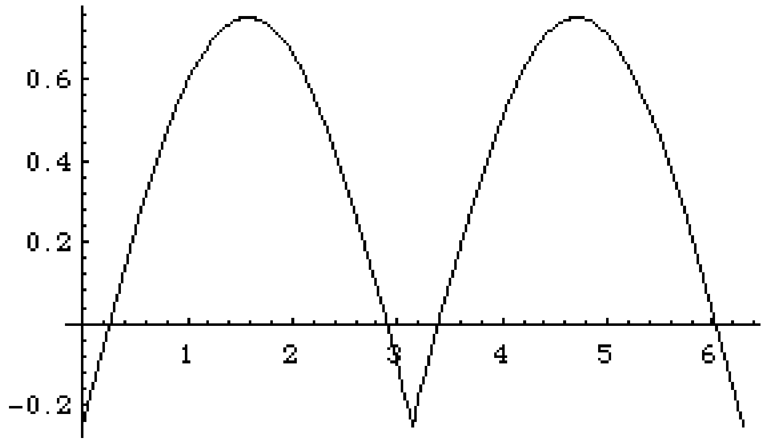
$$= \boldsymbol{V}\Sigma^T\Sigma\Sigma^+\boldsymbol{U}^T\boldsymbol{b} = \boldsymbol{V}\Sigma^T\boldsymbol{U}^T\boldsymbol{b} = \boldsymbol{A}^T\boldsymbol{b}$$

3. (20%) State the Newton method to solve a nonlinear equation

$$f(x) = 0$$

where *f* is a continuously differentiable function in *x*. Similarly, state the bisection method to solve the same nonlinear equation. Describe difference of the two methods according to a function *f*. Now, how can you solve the following nonlinear equation

$$f(x) = |\sin x| - 1/4 = 0 \quad \text{in} \quad (0, 2\pi) \ ?$$



Can we apply the bisection method ? Can we apply the Newton method ?

Using the Taylor series of a differentiable function, we shall obtain the next step approximation so that its linear approximation can satisfy the equality :

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x + \frac{1}{2} f''(x)\Delta x^2 + ...$$

$$\Rightarrow \ f\left(x^{(k)} + \Delta x^{(k)}\right) \approx f\left(x^{(k)}\right) + f'\left(x^{(k)}\right)\Delta x^{(k)} = 0$$

$$\Rightarrow \ \Delta x^{(k)} = -\left[f'\left(x^{(k)}\right)\right]^{-1} f\left(x^{(k)}\right)$$

$$\Rightarrow \ x^{(k+1)} = x^{(k)} + \Delta x^{(k)} = x^{(k)} - \left[f'\left(x^{(k)}\right)\right]^{-1} f\left(x^{(k)}\right)$$

In order to use the Newton iteration, we must have the first derivative at an evaluation point. Therefore, the function in above that is continuous, but not continuously differentiable on the whole interval, may not be good enough to apply the Newton method. In this case, we must apply the Newton method in the subintervals where the function is continuously differentiable. The initial approximation $x^{(0)}$ must also be located at the point where the function *f* is continuously differentiable.

The bisection method is applicable for any continuous functions, and does not require its differentiability. In this sense, the bisection method is much more general than the Newton method. It is based on the fact that if the product of $f(x_1)$ and $f(x_2)$ is negative for $x_1 < x_2$, i.e., $f(x_1)f(x_2) < 0$ for $x_1 < x_2$, then we can expect at least one solution of $f(x) = 0$ exists in the interval (x_1, x_2). Using this fact that define the mid-point of (x_1, x_2) by $x_3 = \frac{1}{2}(x_1 + x_2)$, we examine the sign of $f(x_1)f(x_3)$ and $f(x_3)f(x_2)$, and then we set the new subinterval by dividing the original one into the half size one by using the following algorithm

$$\text{if } f(x_1)f(x_3) < 0, \text{ then } x_1 \leftarrow x_1 \text{ and } x_2 \leftarrow x_3$$
$$\text{if } f(x_3)f(x_2) < 0, \text{ then } x_1 \leftarrow x_3 \text{ and } x_2 \leftarrow x_2$$

Since at each step, the length of the interval is reduced into the half. Continuing this process we can reduce the subinterval length as small as we wish, i.e., up to the tolerance limit specified.