CML Lecture Series

Primer on Homogenization

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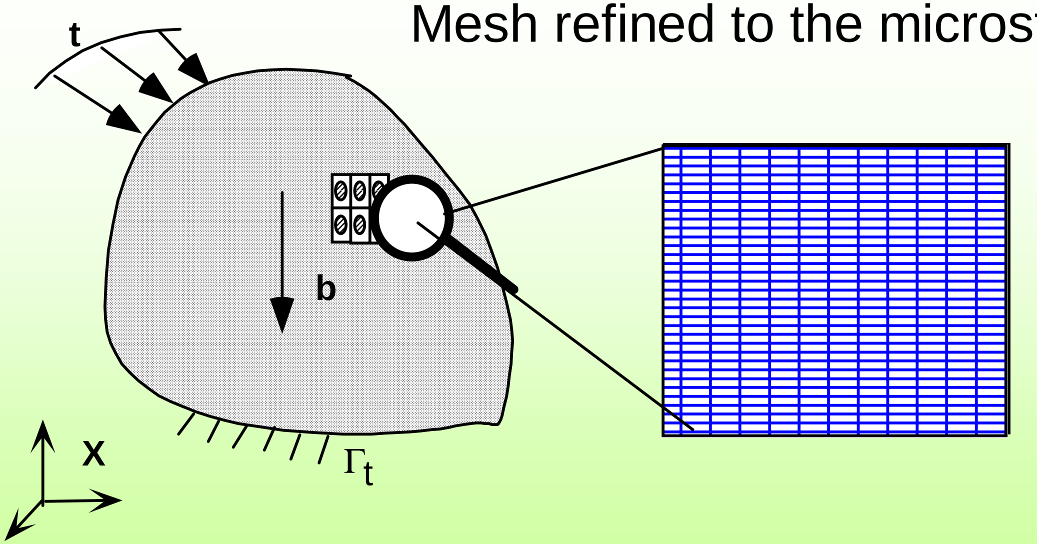
Outline

- Model problem: asymptotic analysis for a uni-directional problem
- Asymptotic analysis for elastic problem
- A more intuitive approach
- Implementation issues: symmetry & periodicity
- A Simply example: layered materials

Heterogeneous Material Problem with highly-oscillating coefficients

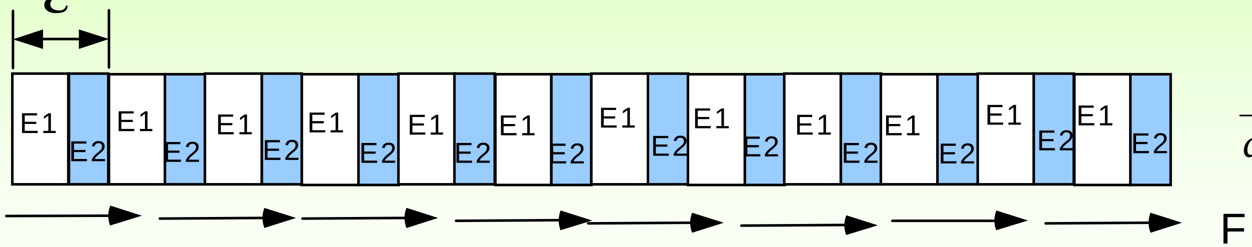
- Need very refined meshes to find the local responses

Mesh refined to the microstructure level

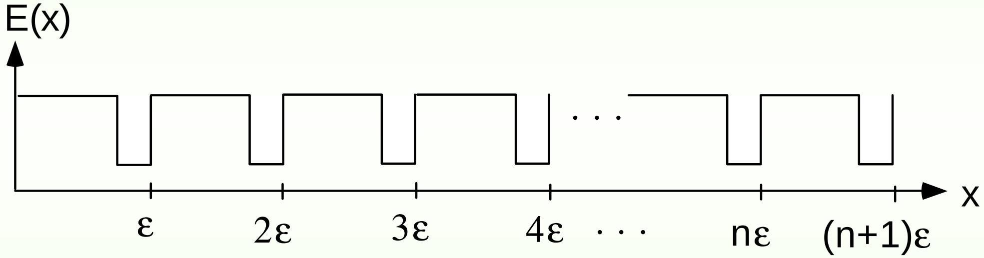


Multiple-scale problem

- Periodic material properties: rapidly varying

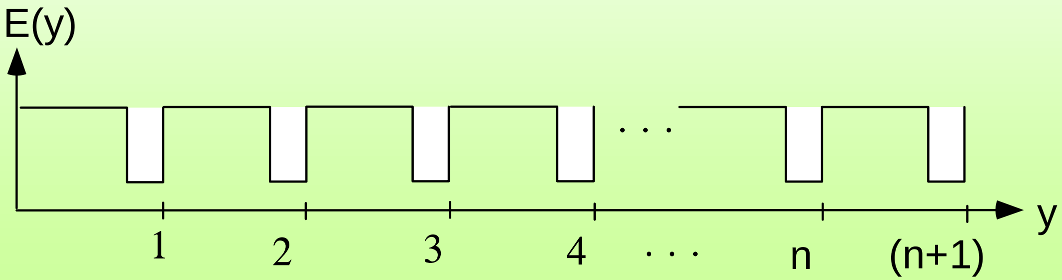


$$\frac{d}{dx}\left(E^{\varepsilon}(x)\frac{d}{dx}u^{\varepsilon}\right)+f(x)=0$$



$$E(x+n\varepsilon)=E(x)$$

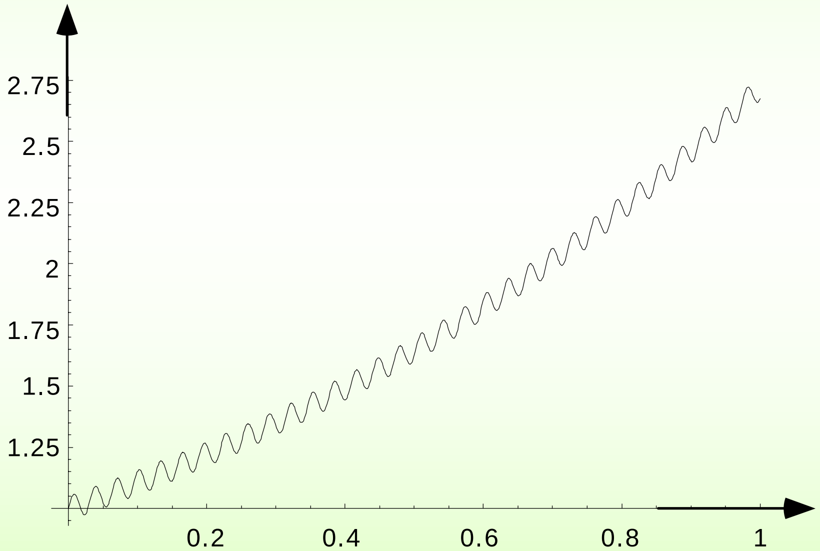
- Introduce another scale y, $y = x/\varepsilon$
  to describe details of oscillation



$$E(y+n)=E(y)$$

Asymptotic Expansion

$$u^{\varepsilon}(x,y)=u^{0}(x,y)+\varepsilon u^{1}(x,y)$$



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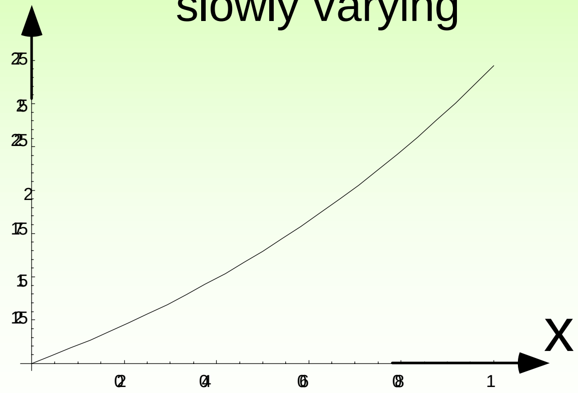
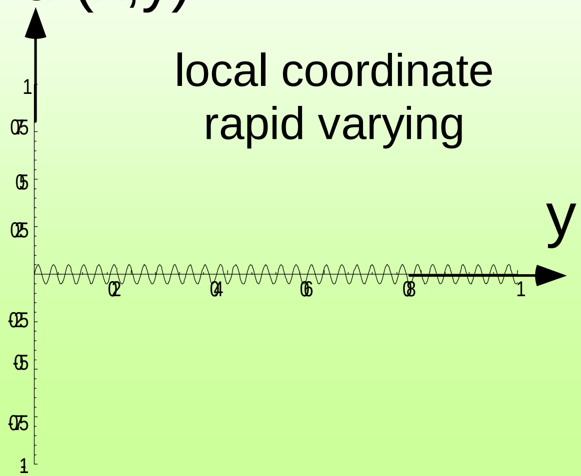
$u^0(x)$ global coordinate slowly varying



+

$u^1(x,y)$

local coordinate rapid varying



$$y = x / \varepsilon$$

Asymptotic Expansion: 1D Problem

- Equilibrium equation $\frac{d}{dx}\left(E^{\varepsilon}(x)\frac{d}{dx}u^{\varepsilon}\right)+f(x)=0$
- Two-scale problem: u_i periodic

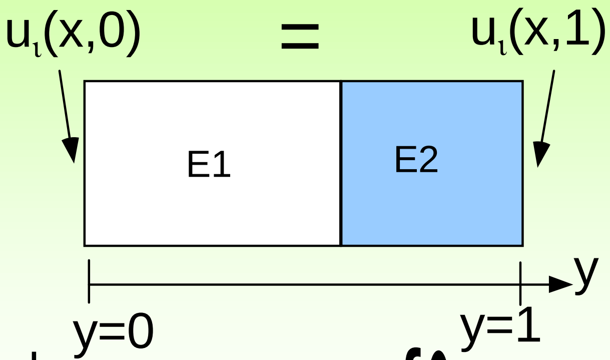
$$u^{\varepsilon}(x,y)=u_0(x,y)+\varepsilon u_1(x,y)+\cdots,\quad y=x/\varepsilon$$

$$\frac{d}{dx}\rightarrow\frac{\partial}{\partial x}+\frac{1}{\varepsilon}\frac{\partial}{\partial y}$$

- Replacing the original differential equation

$$\left(\frac{\partial}{\partial x}+\frac{1}{\varepsilon}\frac{\partial}{\partial y}\right)\left(L^{\varepsilon}(x)\left(\frac{\partial}{\partial x}+\frac{1}{\varepsilon}\frac{\partial}{\partial y}\right)\left\{u_0(x,y)+\varepsilon u_1(x,y)\right\}\right)+f(x)=0$$

Expand the equation: Order analysis



- O(ε[-2])

$$\frac{\partial}{\partial y}\left[E(y)\frac{\partial u_0(x,y)}{\partial y}\right]=0$$

$$E(y)\frac{\partial u_0(x,y)}{\partial y}=c_2(x),\quad \frac{c_2(x)}{E(y)}=\frac{\partial}{\partial y}u_0(x,y)$$

$$c_2(x)\int_0^1\frac{1}{E(x,y)}dY=\int_0^1\frac{\partial}{\partial y}u_0(x,y)dY=u_0(x,1)-u_0(x,0)=0$$

$$\Rightarrow c_2(x)=0,\qquad u_0(x,y)=u_0(x)\ only$$

Order analysis: $O(\varepsilon^{-1})$

- $O(\varepsilon^{-1})$

$$\frac{\partial}{\partial y}\left\{ E(y)\left(\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial y} \right) \right\} = 0$$

$$E(y)\left(\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial y} \right) = c_1(x)$$

$$\int_Y \frac{\partial u_0}{\partial x} dy + \cancel{\int_Y \frac{\partial u_1}{\partial y} dy} = c_1(x) \int_Y \frac{1}{E(y)} dy$$

$$c_1(x) = \left(\frac{\partial u_0}{\partial x} \right) \left\{ \int_Y \frac{1}{E(y)} dy \right\}^{-1}$$

- Microscopic strain proportional to Macroscopic strain

$$\frac{\partial u_1}{\partial y} = \left\{ \frac{1}{E(y)} \left(\int_Y \frac{dy}{E(y)} \right)^{-1} - 1 \right\} \left(\frac{\partial u_0}{\partial x} \right)$$

Order analysis: O(ε⁰)

$$\frac{\partial}{\partial x}\left\{E(y)\left(\frac{\partial u_0}{\partial x}+\frac{\partial u_1}{\partial y}\right)\right\}+\frac{\partial}{\partial y}\left\{E(y)\frac{\partial u_1}{\partial x}\right\}+f(x)=0$$

$$\frac{1}{Y}\int_Y \frac{\partial}{\partial x}\left[c_1(x)\right]dy+\frac{1}{Y}\int_Y \frac{\partial}{\partial y}\left\{E(y)\frac{\partial u_1}{\partial x}\right\}dy+\frac{1}{Y}\int_Y f(x)\,dy=0$$

$$\frac{\partial}{\partial x}\left\{\left(\frac{1}{Y}\int_Y \frac{1}{E(y)}dy\right)^{-1}\left(\frac{\partial u_0}{\partial x}\right)\right\}+f(x)=0$$

$$\frac{\partial}{\partial x}\left\{E^H(x)\left(\frac{\partial u_0}{\partial x}\right)\right\}+f(x)=0$$

- Homogenized coefficient: harmonic means

$$E^H(x)=\frac{1}{Y}\left(\int_Y \frac{1}{E(y)}dy\right)^{-1}$$

Energy consideration

- Minimize the strain energy of the unit cell, subject to a given macroscopic strain $<\varepsilon_0>$

$$\min_{\phi} \frac{1}{2}\frac{1}{Y}\int_Y E(y)\left[\langle\varepsilon_0\rangle + \frac{d\phi}{dy}\right]^2 dy = \frac{1}{2}E^H\langle\varepsilon_0\rangle^2$$

$$\forall\ \phi(0) = \phi(Y)$$

$$\int_Y E(y)\left[\langle\varepsilon_0\rangle + \frac{d\phi}{dy}\right]\frac{d\delta\phi}{dy}\,dy = 0$$

$$-\frac{d}{dy}\left[E(y)\left[\langle\varepsilon_0\rangle + \frac{d\phi}{dy}\right]\right] = 0$$

$$E(y)\left[\langle\varepsilon_0\rangle + \frac{d\phi}{dy}\right] = c_1$$

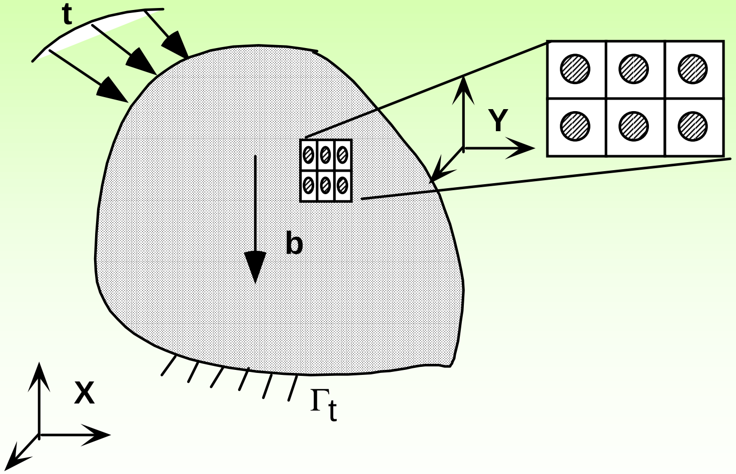
$$c_1 = \langle\varepsilon_0\rangle\left[\int_Y \frac{1}{E(y)}\,dy\right]^{-1}$$

$$\frac{1}{2}\frac{1}{Y}c_1\int_Y\left[\langle\varepsilon_0\rangle + \frac{d\phi}{dy}\right]dy =$$

$$\frac{1}{2}\left[\int_Y \frac{1}{E(y)}\,dy\right]^{-1}\langle\varepsilon_0\rangle^2$$

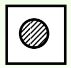
Asymptotic Expansion: General Problem

Global-Local analysis



Local analysis
(PREMAT)

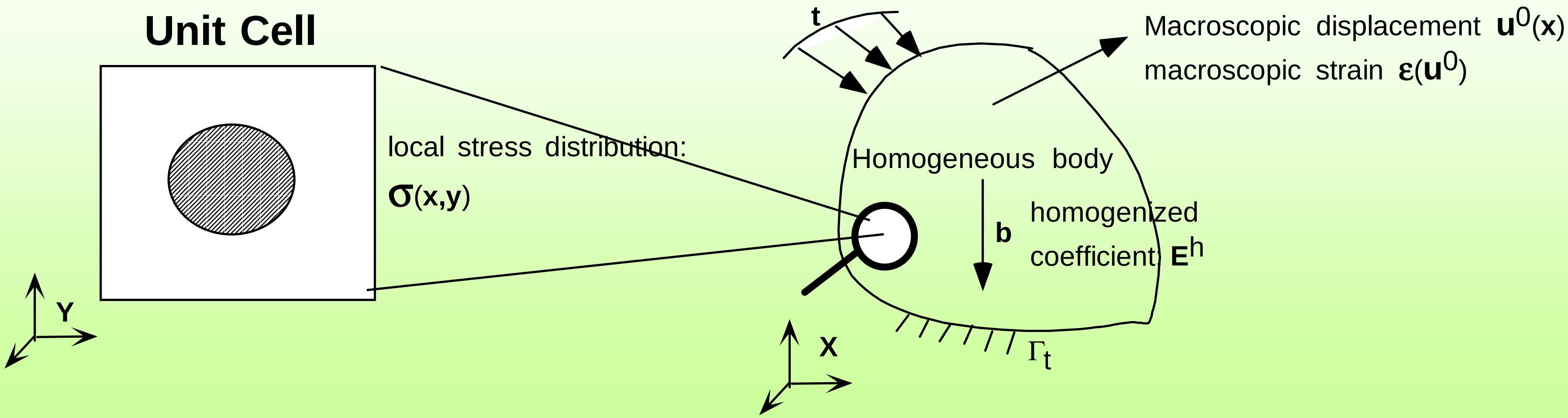
Unit Cell



characteristic displacement: χ
homogenized coefficient: $\mathbf{E}^h$

Global FEM analysis
(NASTRAN, ABAQUS)

Localization Process



Asymptotic Expansion: general problem

- Equilibrium equation $$\frac{\partial}{\partial x_j}\left[L_{ijkl}{}^{\varepsilon}(x)\frac{\partial u_k{}^{\varepsilon}}{\partial x_l}\right] + f_i(x) = 0$$
- Two-scale problem

$$\mathbf{u}^{\varepsilon}(\mathbf{x},\mathbf{y}) = \mathbf{u}^{<0>}(\mathbf{x},\mathbf{y}) + \varepsilon\mathbf{u}^{<1>}(\mathbf{x},\mathbf{y}) + \cdots, \quad \mathbf{y} = \mathbf{x}/\varepsilon$$

$$\frac{\partial}{\partial x_j} \to \frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}$$

- Replacing the original differential equation

$$\left(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\right)\left[L_{ijkl}\left(\frac{\partial}{\partial x_l} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_l}\right)\left\{u_k{}^{<0>}(x,y) + \varepsilon u_k{}^{<1>}(x,y)\right\}\right] + f(x) = 0$$

Summary of the order analysis

- $O(\varepsilon^{-2})$

$$\frac{\partial}{\partial y_j}\left(L_{ijkl}\frac{\partial u_k^{<0>}(x,y)}{\partial y_l}\right)=0$$

$$\mathbf{u}^{<0>}(x,y)=\mathbf{u}^{<0>}(x) \quad \text{only}$$

- $O(\varepsilon^{-1})$

$$\frac{\partial}{\partial y_j}\left(L_{ijkl}\frac{\partial u_k^{<1>}}{\partial y_l}\right)=-\frac{\partial L_{ijkl}}{\partial y_j}\left(\frac{\partial u_k^{<0>}}{\partial x_l}\right)$$

$$\frac{\partial u_k^{<1>}}{\partial y_l}=\frac{\partial \chi_k^{pq}}{\partial y_l}\frac{\partial u_p^{<0>}}{\partial x_q}$$

» Eq. of the characteristic displacement χ^{pq}

$$\frac{\partial}{\partial y_j}\left(L_{ijkl}\frac{\partial \chi_k^{pq}}{\partial y_l}\right)=-\frac{\partial L_{ijpq}}{\partial y_j}$$

- $\chi^{pq}=\chi^{qp}$

Summary of the order analysis (contd)

- $O(\varepsilon^0)$

$$\frac{\partial}{\partial x_j}\left[L_{ijkl}\left(\frac{\partial u_k^{<1>}}{\partial y_l} + \frac{\partial u_k^{<0>}}{\partial y_l} \right) \right] + \cancel{\frac{\partial}{\partial y_j}\left(L_{ijkl} \frac{\partial u_k^{<1>}}{\partial y_l} \right)} + f_i(x) = 0$$

$$\frac{\partial}{\partial x_j}\left[\int_Y L_{ijkl}\left(\frac{\partial u_k^{<1>}}{\partial y_l} + \frac{\partial u_k^{<0>}}{\partial y_l} \right) \right] + f_i(x) = 0$$

$$\frac{\partial}{\partial x_j}\left\{ \boxed{\left[\frac{1}{Y}\int_Y L_{ijkl}\left(\delta_{kp}\delta_{lq} + \frac{\partial \chi_k^{pq}}{\partial y_l} \right) \right]} \frac{\partial u_p^{<0>}}{\partial x_q} \right\} + f_i(x) = 0$$

$$\frac{\partial}{\partial x_j}\left(L_{ijpq}^{H} \frac{\partial u_p^{<0>}}{\partial x_q} \right) + f_i(x) = 0$$

Homogenized coefficients

- Homogenized elastic tensor:

$$L^H_{ijpq} = \frac{1}{Y}\int_Y L_{ijkl}\left(\delta_{kp}\delta_{lq} + \frac{\partial \chi^{pq}_k}{\partial y_l}\right)dy$$

- Microscopic equation for χ^{pq}, in the variational form

$$\int_Y L_{ijkl}\left(\delta_{kp}\delta_{lq} + \frac{\partial \chi^{pq}_k}{\partial y_l}\right)\frac{\partial v_i}{\partial y_j}\,dY = 0 \quad \forall v_i$$

- Symmetric form of the homogenized elastic tensor

$$L^H_{mnpq} = \frac{1}{Y}\int_Y \left(\delta_{im}\delta_{jn}\right)L_{ijkl}\left(\delta_{kp}\delta_{lq} + \frac{\partial \chi^{pq}_k}{\partial y_l}\right)dy$$

$$= \frac{1}{Y}\int_Y \left(\delta_{im}\delta_{jn} + \frac{\partial \chi^{mn}_i}{\partial y_j}\right)L_{ijkl}\left(\delta_{kp}\delta_{lq} + \frac{\partial \chi^{pq}_k}{\partial y_l}\right)dy$$

- Major/Minor symmetries $L^H_{mnpq} = L^H_{mnqp} = L^H_{nmpq} = L^H_{mnpq}$

Localization process

- Given macroscopic strain field, find the microscopic stress field
  - » microscopic strain field

$$\varepsilon_{kl}(\mathbf{x},\mathbf{y}) = \frac{\partial u_p^{<0>}}{\partial x_q} + \frac{\partial u_k^{<1>}}{\partial y_l} = \left(\delta_{pk}\delta_{ql} + \frac{\partial \chi_k^{pq}}{\partial y_l} \right) \frac{\partial u_p^{<0>}}{\partial x_q}$$

  - » microscopic stress field

$$\sigma_{ij} = L_{ijkl}^{\varepsilon}(\mathbf{x},\mathbf{y})\varepsilon_{kl}(\mathbf{x},\mathbf{y}) = L_{ijkl}^{\varepsilon}(\mathbf{x},\mathbf{y})\left(\delta_{pk}\delta_{ql} + \frac{\partial \chi_k^{pq}}{\partial y_l} \right) \frac{\partial u_p^{<0>}}{\partial x_q}$$

  - » Average stress (volume average)

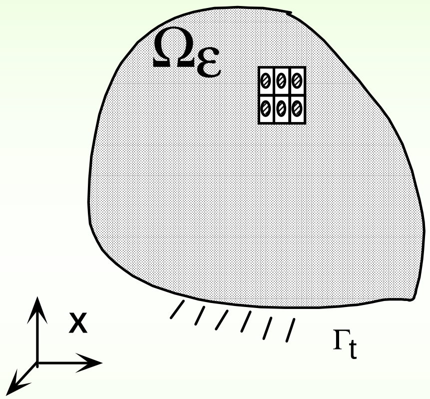
$$\langle \sigma_{ij} \rangle = \frac{1}{Y}\int_Y \sigma_{ij}\, dY = L_{ijkl}^{H}\left(\frac{\partial u_p^{<0>}(\mathbf{x})}{\partial x_q} \right)$$

More Intuitive Approach

Engineering approach

- Two domains

Domain: with heterogeneous material



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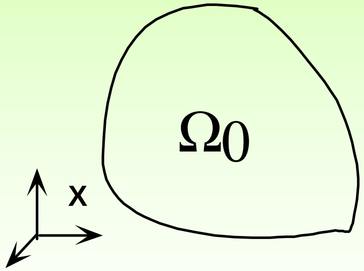
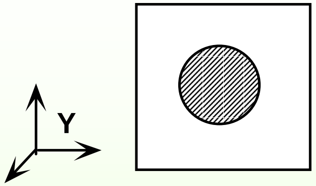
Domain: with homogeneous material



Impose a macroscopc strain: **E**
Homogeneous stress: **Σ**
Macroscopic constitutive law:
Σ= L^h : E

+

Unit Cell Y



Local strain field
ε(u)= E + ε(u¹)

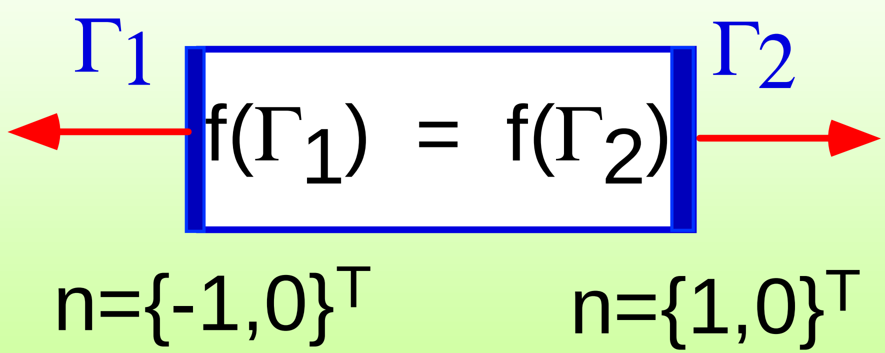
- Local strain ε(**u**) = overall strain (**E**) + local corrector ε(**u**¹)
- small 1st-order variation **u**¹ contributes to zero order effect on derivative

$$\frac{\partial u_i^{\varepsilon}}{\partial x_j} = \left(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\right)\left(u_i^{<0>}(x) + \varepsilon u_i^{<1>}(x,y)\right) \xrightarrow[\varepsilon->0]{} \frac{\partial u_i^{<0>}}{\partial x_j} + \frac{\partial u_i^{<1>}}{\partial y_j}$$

Periodic strain field

- Average operator $$\langle f \rangle = \frac{1}{Y}\int_Y f(\mathbf{y})\,d\mathbf{y}$$
- Average of the gradient operator

$$\langle \nabla f \rangle = \left\langle \frac{\partial f}{\partial y_i} \right\rangle = \frac{1}{V}\int_Y \frac{\partial f}{\partial y_i}\,d\mathbf{y} = \frac{1}{V}\int_{\partial Y} (n_i f)\,dS = 0$$



Periodic strain field (II)

- Average of the strain corrector

$$\left\langle \varepsilon\left(\mathbf{u}^{<1>}\right)\right\rangle = \frac{1}{2}\left\langle \nabla\mathbf{u}^{<1>} + \mathbf{u}^{<1>}\nabla\right\rangle = \frac{1}{Y}\int_{\partial Y}\frac{1}{2}\left(n_i u_j^{<1>} + n_j u_i^{<1>}\right)dS = 0$$

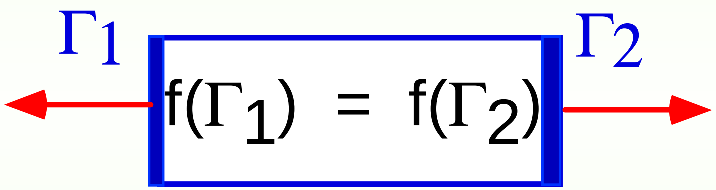
- Average of the microscopic strain

$$\varepsilon\left(\mathbf{u}\right) = \mathbf{E} + \varepsilon\left(\mathbf{u}^{<1>}\right)$$

$$\left\langle \varepsilon\left(\mathbf{u}\right)\right\rangle = \left\langle \mathbf{E}\right\rangle + \left\langle \varepsilon\left(\mathbf{u}^{<1>}\right)\right\rangle = \mathbf{E}$$

Periodic stress field

- Local stress field σ satisfies
  - » in equilibrium $\quad div(\sigma) = 0 \quad \text{in } Y$
  - $\sigma.\mathbf{n}$ are opposite on opposite side of unit cell



$\sigma.\mathbf{n} = \quad -\sigma.\mathbf{n}$

- Overall stress (volume average)

$$\Sigma = \langle \sigma \rangle$$

Effective properties

- Local problem
  - constitutive law: $\boldsymbol{\sigma}(\mathbf{y}) = \mathbf{L}:\left(\mathbf{E} + \varepsilon\left(\mathbf{u}^{<1>}(\mathbf{y})\right)\right) \quad \forall \mathbf{y} \in Y$
  - equilibrium $\quad div(\boldsymbol{\sigma}) = 0 \quad$ in Y , $\boldsymbol{\sigma}.\mathbf{n}$ are opposite

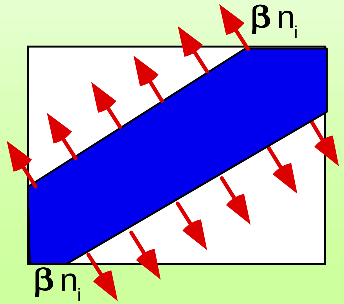
$$div\left(\mathbf{L}:\varepsilon\left(\mathbf{u}^{<1>}(\mathbf{y})\right)\right) = -div(\mathbf{L}:\mathbf{E})$$

$$\int_Y \boldsymbol{\varepsilon}(\delta\mathbf{u}):\mathbf{L}:\boldsymbol{\varepsilon}\left(\mathbf{u}^{<1>}\right) dY = -\int_Y \boldsymbol{\varepsilon}(\delta\mathbf{u}):\mathbf{L}:\mathbf{E}\, dY$$

- linear equation with generalized loading **E**

Thermal-type Loads

$$\int_Y \boldsymbol{\varepsilon}^T(\delta\mathbf{u}):\mathbf{L}:\boldsymbol{\varepsilon}(\mathbf{u})\, dY = \int_Y \boldsymbol{\varepsilon}^T(\delta\mathbf{u}):\mathbf{L}:\boldsymbol{\alpha}\, dY$$



Local problem: Solution by super-position

- Split the strain into fundamental eigen-strain

$$\mathbf{E} = E_{pq}\mathbf{II}^{pq}, \quad II_{ij}^{pq} = \left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}\right)/2$$

- solve each sub-problem

$$\int_Y \boldsymbol{\varepsilon}(\delta\mathbf{u}) : \mathbf{L} : \boldsymbol{\varepsilon}\left(\mathbf{u}^{<1>}\right) dY = -\int_Y \boldsymbol{\varepsilon}(\delta\mathbf{u}) : \mathbf{L} : \mathbf{E}\, dY \quad \forall \delta\mathbf{u}$$

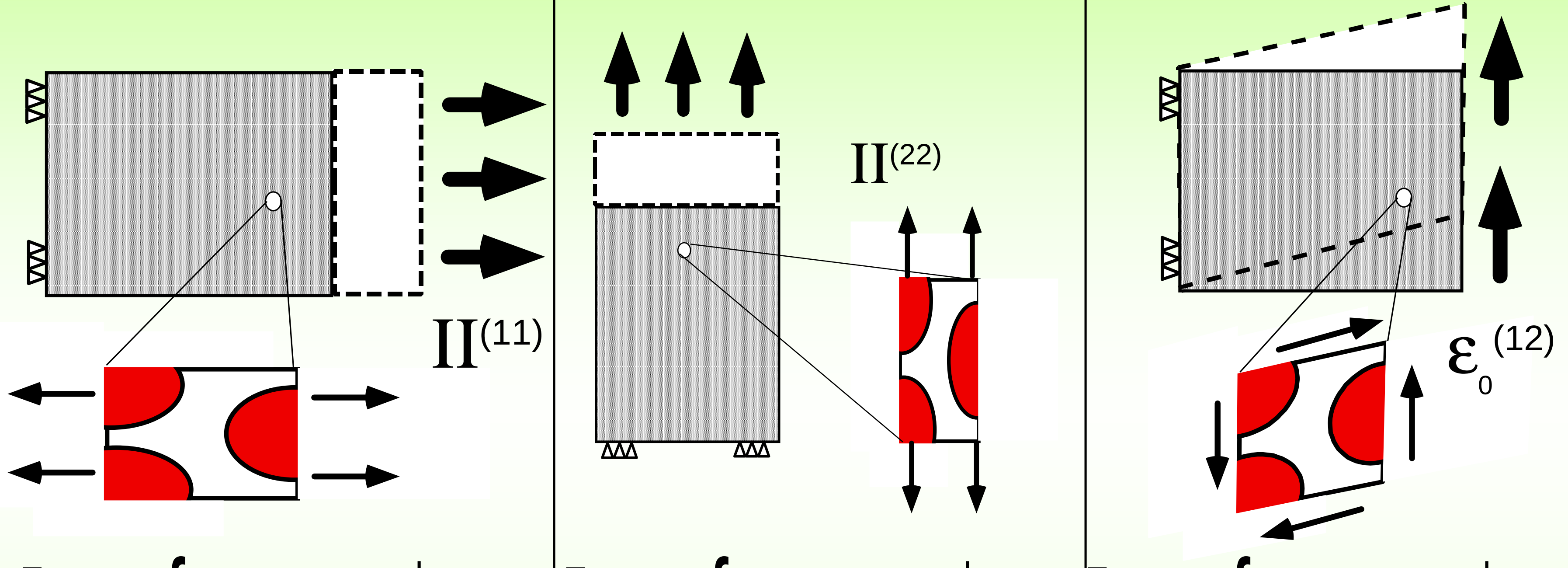
$$\int_Y \boldsymbol{\varepsilon}^T(\delta\mathbf{u})\mathbf{L} : \boldsymbol{\varepsilon}\left(\boldsymbol{\chi}^{pq}\right) dY = \int_Y \boldsymbol{\varepsilon}^T(\delta\mathbf{u})\mathbf{L} : \mathbf{II}^{pq}\, dY \quad \forall \delta\mathbf{u}$$

- super-impose the results

$$\mathbf{u}^{<1>} = E_{pq}\boldsymbol{\chi}^{pq}$$

$$\varepsilon(\mathbf{u}) = \mathbf{E} + \boldsymbol{\varepsilon}\left(\mathbf{u}^{<1>}\right) = E_{pq}\left[\mathbf{II}^{pq} + \boldsymbol{\varepsilon}\left(\boldsymbol{\chi}^{pq}\right)\right]$$

Three Load Cases in the homogenization



$$\int_Y \varepsilon(\delta \mathbf{u}) : \mathbf{L} : \varepsilon(\chi^{(11)})\, dY = \int_Y \varepsilon(\delta \mathbf{u}) : \mathbf{L} : \Pi^{(11)}\, dY$$

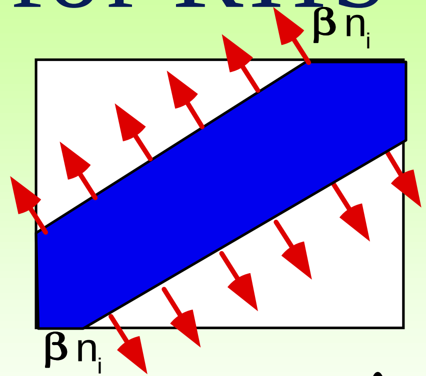
$$\int_Y \varepsilon(\delta \mathbf{u}) : \mathbf{L} : \varepsilon(\chi^{(22)})\, dY = \int_Y \varepsilon(\delta \mathbf{u}) : \mathbf{L} : \Pi^{(22)}\, dY$$

$$\int_Y \varepsilon(\delta \mathbf{u}) : \mathbf{L} : \varepsilon(\chi^{(12)})\, dY = \int_Y \varepsilon(\delta \mathbf{u}) : \mathbf{L} : \Pi^{(12)}\, dY$$

The thermal-load analogy for RHS

Thermal Loads

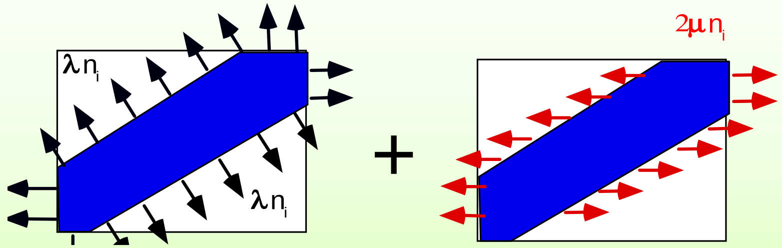
$$\int_Y \boldsymbol{\varepsilon}(\delta\mathbf{u}):\mathbf{L}:\boldsymbol{\varepsilon}(\mathbf{u})\,dY = \int_Y \boldsymbol{\varepsilon}(\delta\mathbf{u}):\mathbf{L}:\boldsymbol{\alpha}\,dY = \int_Y \boldsymbol{\varepsilon}^T(\delta\mathbf{u})\begin{Bmatrix}\beta\\ \beta\\ 0\end{Bmatrix} dY$$



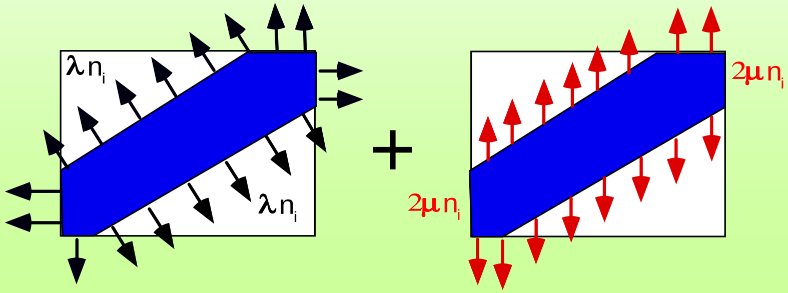
Thermal-type Loads in the cell problem

$$\int_Y \boldsymbol{\varepsilon}(\delta\mathbf{u}):\mathbf{L}:\boldsymbol{\varepsilon}(\boldsymbol{\chi}^{(pq)})\,dY = \int_Y \boldsymbol{\varepsilon}(\delta\mathbf{u}):\mathbf{L}:\mathbf{II}^{(kl)}\,dY = \int_Y \boldsymbol{\varepsilon}(\delta\mathbf{u})\left\{\underbrace{\begin{Bmatrix}\lambda+2\mu\\ \lambda\\ 0\end{Bmatrix}}_{pq=11}\;\underbrace{\begin{Bmatrix}\lambda\\ \lambda+2\mu\\ 0\end{Bmatrix}}_{pq=22}\;\underbrace{\begin{Bmatrix}0\\ 0\\ \mu\end{Bmatrix}}_{pq=12}\right\} dY$$

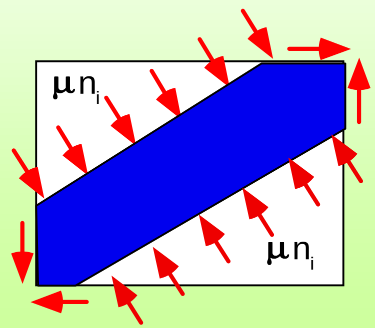
(pq)=11



(pq)=22



(pq)=12



Effective stiffness tensor

$$\mathbf{L}^H : \mathbf{E} = \Sigma = \langle \sigma \rangle \quad = \quad \langle \mathbf{L} : \boldsymbol{\varepsilon}(\mathbf{u}) \rangle = \left\langle \mathbf{L} : \left(\mathbf{E} + \boldsymbol{\varepsilon}\left(\mathbf{u}^{<1>}\right) \right) \right\rangle$$

strain energy of a homogeneous body subject to macroscopic strain

Average strain energy of the unit cell

$$\mathbf{L}^H : E_{pq} \mathbf{II}^{pq} = E_{pq} \left\langle \mathbf{L} : \left(\mathbf{II}^{pq} + \boldsymbol{\varepsilon}\left(\boldsymbol{\chi}^{pq}\right) \right) \right\rangle$$

$$\mathbf{L}^H : \mathbf{II}^{pq} = \left\langle \mathbf{L} : \left(\mathbf{II}^{pq} + \boldsymbol{\varepsilon}\left(\boldsymbol{\chi}^{pq}\right) \right) \right\rangle$$

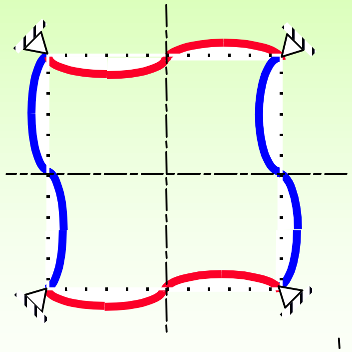
- Effective stiffness tensor

$$L^H_{ijkl} = \left\langle L_{ijpq} : \left(II_{pq}{}^{kl} + \varepsilon_{pq}\left(\chi^{kl}\right) \right) \right\rangle$$

$$L^H_{ijkl} = \left\langle \left(\mathbf{II}^{ij} + \boldsymbol{\varepsilon}\left(\boldsymbol{\chi}^{ij}\right) \right) : \mathbf{L} : \left(\mathbf{II}^{kl} + \boldsymbol{\varepsilon}\left(\boldsymbol{\chi}^{kl}\right) \right) \right\rangle$$

Implementation Issues: symmetry & periodicity

Periodicity conditions

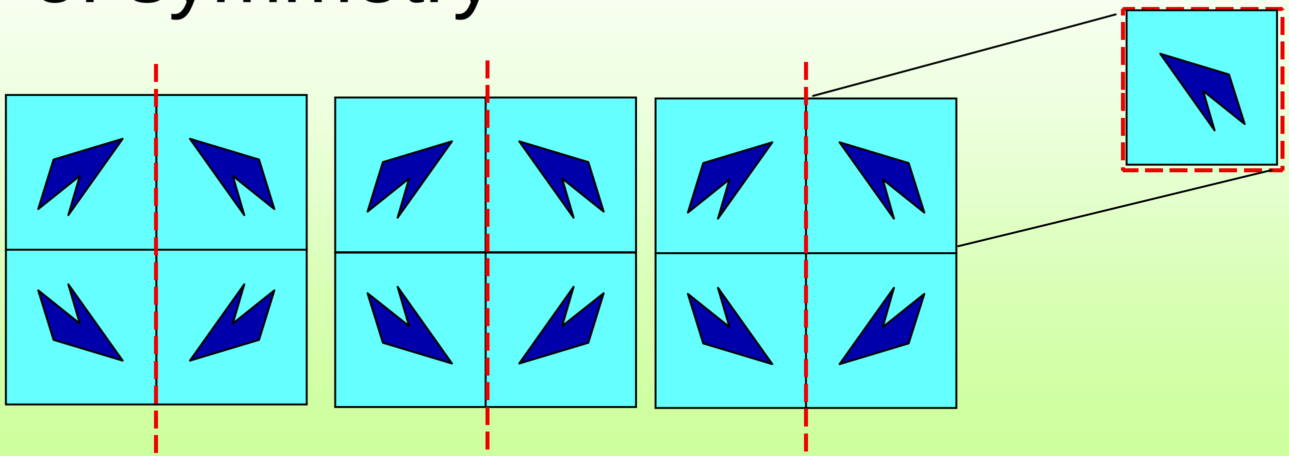


periodicity conditions enforced in the unit cell

- Symmetry conditions
- Elimination
- Lagrange multiplier: (MPC)

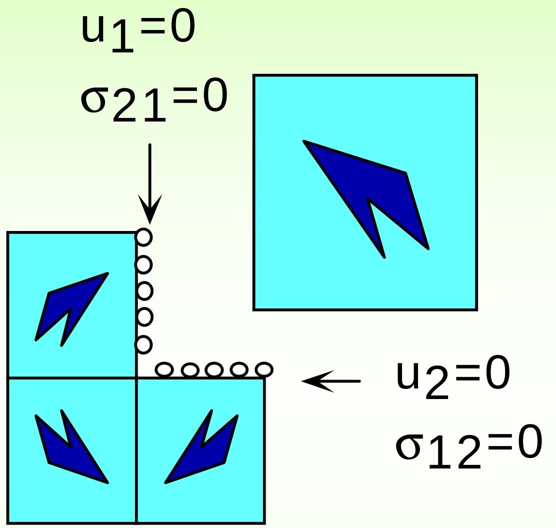
Symmetry conditions

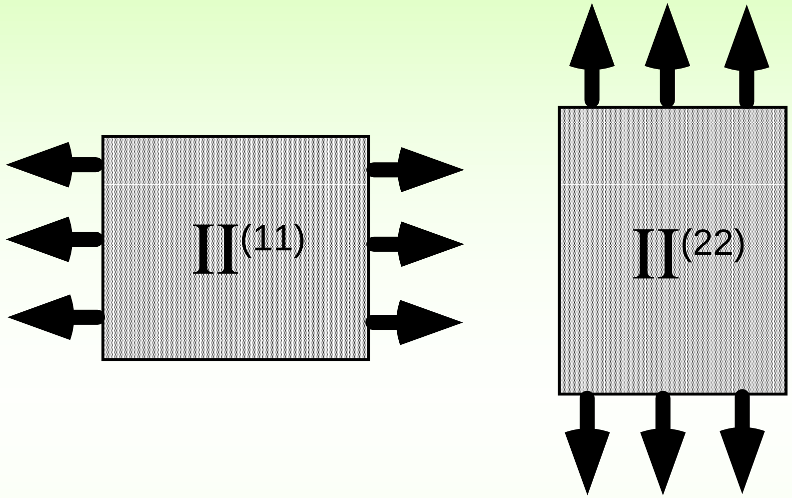
- unit cell has planes of symmetry
  - » material properties
  - » geometry
- local problem can be solved (1/2 or 1/4 model) with standard B.C. on the plane of symmetry



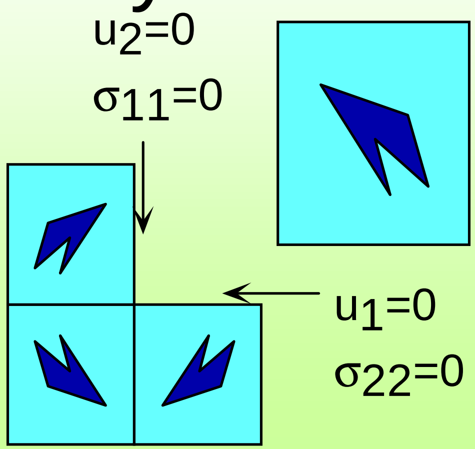
Symmetry conditions (II)

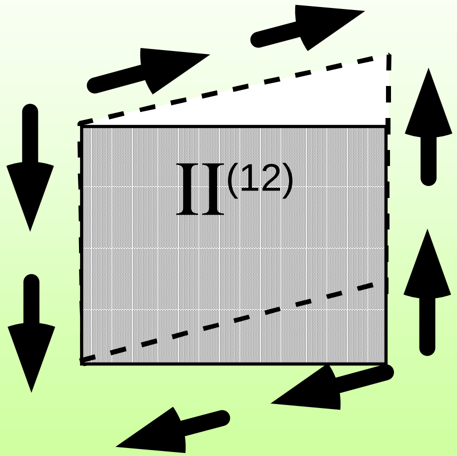
- Symmetric loads (pq=11, 22)





- anti-symmetric load (pq=12)

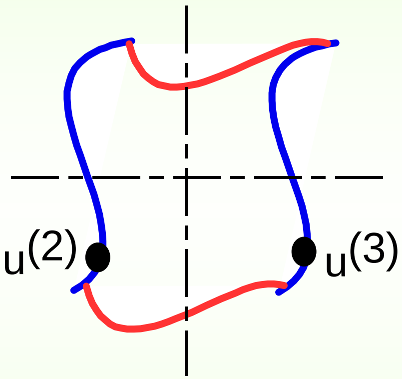




Periodicity condition: elimination

- Reduce DOF by eliminating the DOF on the opposite side of the unit cell

$$\begin{Bmatrix} v_1\\ v_2\\ v_3\end{Bmatrix}^T \begin{bmatrix} K_{11} & K_{12} & K_{13}\\ K_{12}^T & K_{22} & K_{23}\\ K_{13}^T & K_{23}^T & K_{33}\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix} = \begin{Bmatrix} v_1\\ v_2\\ v_3\end{Bmatrix}^T\begin{Bmatrix}f_1\\f_2\\f_3\end{Bmatrix}$$

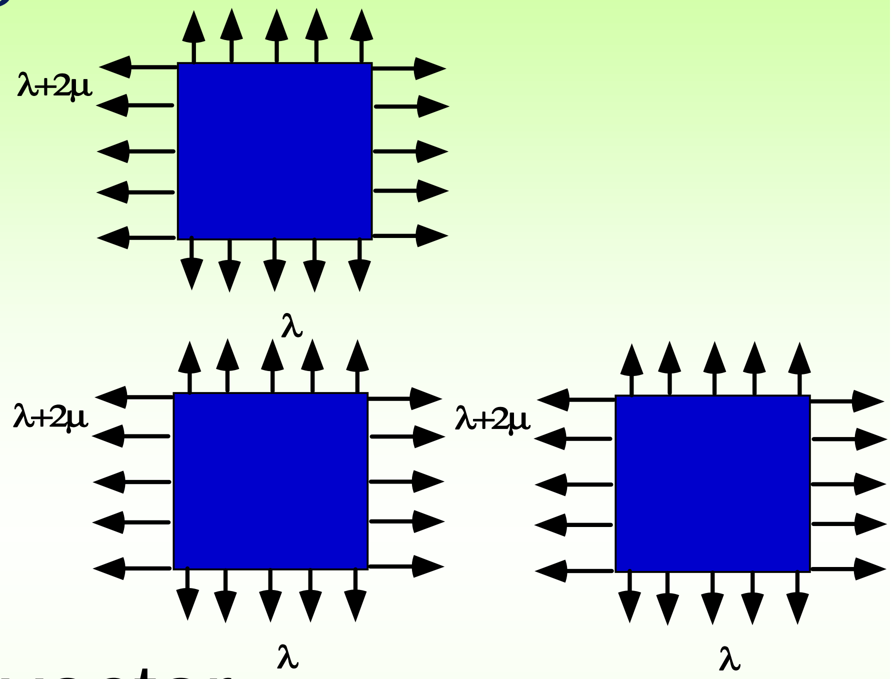


$$\begin{Bmatrix} v_1\\ v_2\end{Bmatrix}^T \begin{bmatrix} K_{11} & K_{12}+K_{13}\\ K_{12}^T+K_{13}^T & K_{22}+K_{33}+K_{23}+K_{23}^T\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \begin{Bmatrix} v_1\\ v_2\end{Bmatrix}^T\begin{Bmatrix}f_1\\f_2+f_3\end{Bmatrix}$$

Special case: homogeneous material

$$\int_Y \boldsymbol{\varepsilon}(\delta\mathbf{u}) : \mathbf{L} : \boldsymbol{\varepsilon}(\boldsymbol{\chi}^{(pq)})\, dY = \int_Y \boldsymbol{\varepsilon}(\delta\mathbf{u}) : \mathbf{L} : \mathbf{II}^{(kl)}\, dY$$

$$= \int_Y \boldsymbol{\varepsilon}(\delta\mathbf{u}) \left\{ \underbrace{\begin{Bmatrix}\lambda+2\mu\\ \lambda\\ 0\end{Bmatrix}}_{pq=11} \quad \underbrace{\begin{Bmatrix}\lambda\\ \lambda+2\mu\\ 0\end{Bmatrix}}_{pq=22} \quad \underbrace{\begin{Bmatrix}0\\ 0\\ \mu\end{Bmatrix}}_{pq=12} \right\} dY$$



- Zero RHS; χ^{pq}: zero vector
- Localization tensor: $(\mathbf{II}^{pq}+\varepsilon(\chi^{pq})) = \mathbf{II}^{pq}$

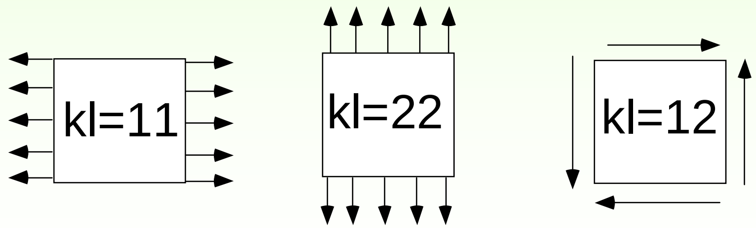
$$E^H_{ijkl} = \left\langle \left(II^{(ij)}_{rs} + \underbrace{\varepsilon_{rs}(\chi^{ij})}_{0} \right) E_{pqrs} \left(II^{(ij)}_{rs} + \underbrace{\varepsilon_{pq}(\chi^{kl})}_{0} \right) \right\rangle = \langle E_{ijkl} \rangle = E_{ijkl}$$

2D Unit Cell Problem, 6 load cases

- **Elastic Problem**

$$\int_Y L_{ijpq} \frac{\partial \chi_p^{(kl)}}{\partial y_q} \frac{\partial \delta u_i}{\partial y_j} = -\int_Y L_{ijkl} \frac{\partial \delta u_i}{\partial y_j} dY \quad \forall \delta u_i$$

Unit strain



- **Thermal-elastic Problem**

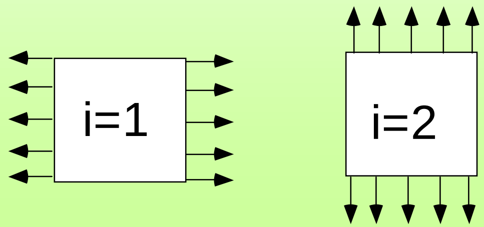
$$\int_Y L_{ijpq} \frac{\partial \phi_p}{\partial y_q} \frac{\partial \delta u_i}{\partial y_j} = -\int_Y \beta_{ij} \frac{\partial \delta u_i}{\partial y_j} dY \quad \forall \delta u_i$$

ΔT

Unit temperature rise

- **Conductivity Problem**

$$\int_Y \lambda_{ij} \frac{\partial \psi^{(k)}}{\partial y_i} \frac{\partial \delta T}{\partial y_j} = -\int_Y \lambda_{ik} \frac{\partial \delta T}{\partial y_i} dY \quad \forall \delta T$$



Unit heat flux

Homogenization of thermoelastic & conductivity

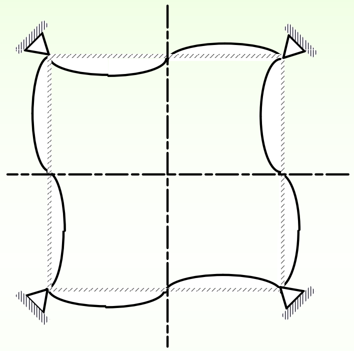
- **Homogenized thermoelastic and conductivity properties**

$$L^H = \frac{1}{Y}\int_Y \left(II + \varepsilon(\chi)\right) : L : \left(II + \varepsilon(\chi)\right) dY$$

$$\beta^H = \frac{1}{Y}\int_Y \left(\beta + \beta : \varepsilon(\phi)\right) dY$$

$$\lambda^H = \frac{1}{Y}\int_Y \left(I + \nabla\psi\right) \cdot \lambda \cdot \left(I + \nabla\psi\right) dY$$

Unit Cell



- **Unit cell problems**

$$\int_Y \left\{\varepsilon(\delta u_1)\right\} : L : \left(I + \varepsilon(\chi)\right) dY = 0 \qquad \forall \delta u_1$$

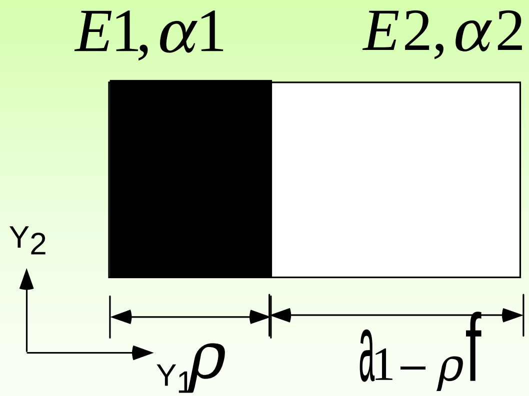
$$\int_Y \left\{\varepsilon(\delta u_1)\right\} : \left(L : \varepsilon(\phi) + \beta \Delta T\right) dY = 0 \qquad \forall \delta u_1$$

$$\int_Y \left(\nabla \delta T\right) \cdot \lambda \cdot \left(I + \nabla\psi\right) dY = 0 \qquad \forall \delta T$$

periodicity conditions enforced in the unit cell

Simply Examples:
Layered Materials

Homogenization of layered material



- Layered along y_1, $\quad \partial(\bullet)/\partial y_2 = 0$

$$\frac{\partial}{\partial y_1}\left\{ L_{i1kl}\left(\delta_{kp}\delta_{lq} + \frac{\partial \chi_k^{pq}}{\partial y_l} \right) \right\} = 0$$

- Layered material is orthotropic

$$L_{1112} = L_{2212} = L_{1211} = L_{1222} = 0$$

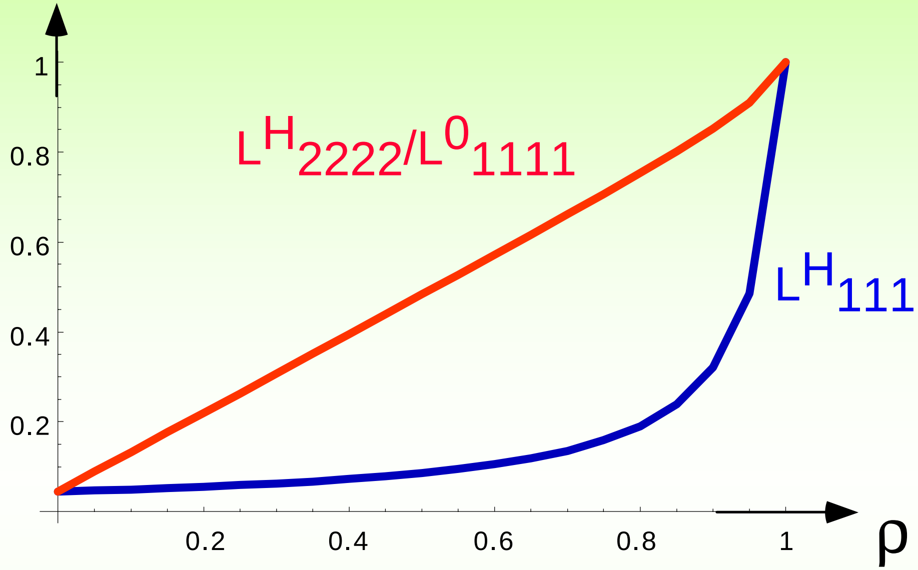
$$\begin{cases} L_{1111}\left(\delta_{1p}\delta_{1q} + \chi_{1,1}^{pq}\right) + L_{1122}\left(\delta_{2p}\delta_{2q}\right) = C_1^{(pq)} \\ L_{1212}\left(\delta_{1p}\delta_{2q}\right) + L_{1212}\left(\chi_{2,1}^{pq} + \delta_{2p}\delta_{1q}\right) = C_2^{(pq)} \end{cases}$$

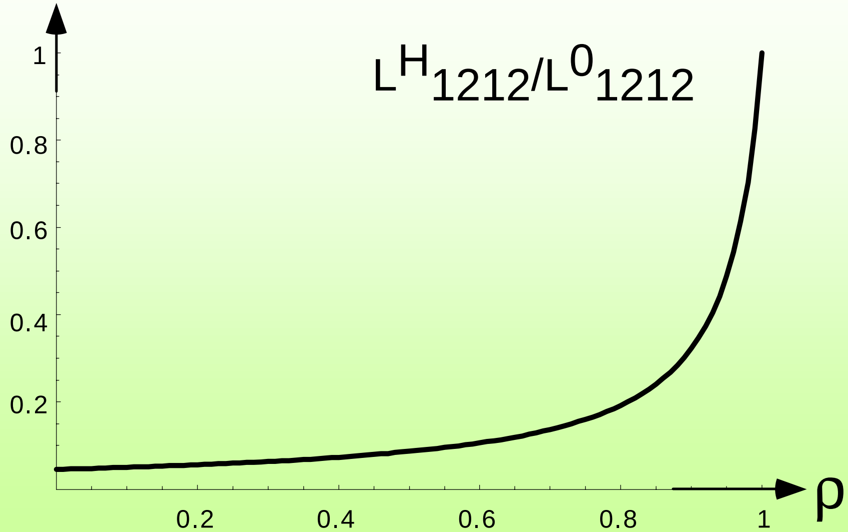
Homogenization Results

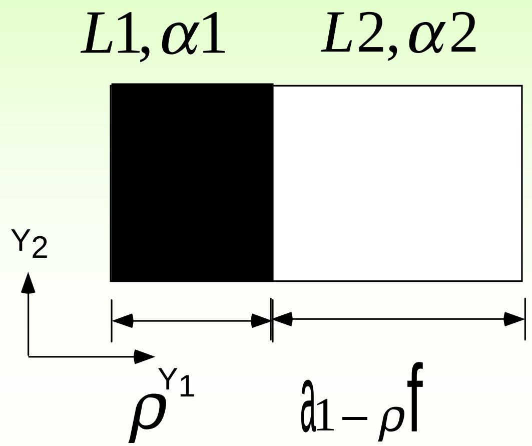
$$
\mathrm{L}^{H} = \begin{pmatrix}
\left\langle \dfrac{1}{\mathrm{L}_{1111}} \right\rangle^{-1} & \left\langle \dfrac{\mathrm{L}_{1122}}{\mathrm{L}_{1111}} \right\rangle \left\langle \dfrac{1}{\mathrm{L}_{1111}} \right\rangle^{-1} & 0 \\
 & \left\langle \mathrm{L}_{2222} \right\rangle - \left\langle \dfrac{\mathrm{L}_{1122}^{2}}{\mathrm{L}_{1111}} \right\rangle + \left\langle \dfrac{\mathrm{L}_{1122}}{\mathrm{L}_{1111}} \right\rangle^{2} \left\langle \dfrac{1}{\mathrm{L}_{1111}} \right\rangle^{-1} & 0 \\
 & & \left\langle \dfrac{1}{\mathrm{L}_{1212}} \right\rangle^{-1}
\end{pmatrix}
$$

$$
\boldsymbol{\beta}^{H} = \begin{pmatrix}
\left\langle \dfrac{\beta_{11}}{\mathrm{L}_{1111}} \right\rangle \left\langle \dfrac{1}{\mathrm{L}_{1111}} \right\rangle^{-1} & 0 \\
 & \left\langle \beta_{22} - \dfrac{\mathrm{L}_{2211}}{\mathrm{L}_{1111}} \left(\beta_{11} - \left\langle \dfrac{\beta_{11}}{\mathrm{L}_{1111}} \right\rangle \left\langle \dfrac{1}{\mathrm{L}_{1111}} \right\rangle^{-1} \right) \right\rangle^{-1}
\end{pmatrix}
$$

Homogenization Results





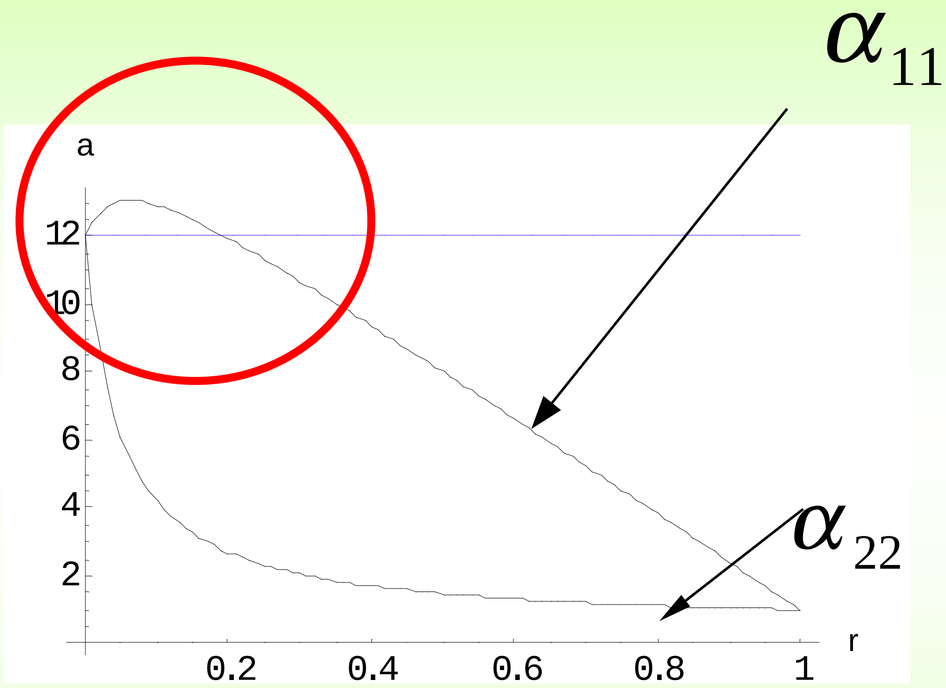


$$\frac{L1}{L2} = \frac{22}{1},$$

$$\frac{\alpha 1}{\alpha 2} = \frac{1}{12}$$

$Red + Yellow \; ?= \; Orange$

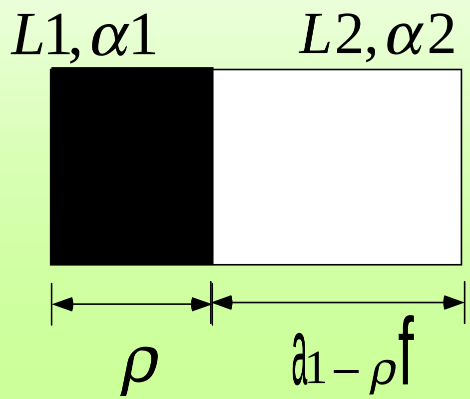
Adding the CTE is *not* like adding paints



- Levin's example, 1967
- The effective CTE dose not form a convex set
- Unusual CTE occurs when $\rho < 0.2$

$$\frac{L1}{L2} = \frac{22}{1},$$

$$\frac{\alpha 1}{\alpha 2} = \frac{1}{12}$$



Summary

- A problem with highly oscillating coefficients can be solved by global-local analysis
- Local analysis: characteristic displacement of the unit cell, find *homogenized coefficient*
- Global analysis: the original domain is treated as if homogeneous to find the stress/strain distribution
- Localization process: given the find the local stress distribution within the unit cell