Lectures on the stable trace formula with emphasis on SL_2

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Contents

Preface

These notes represent an expanded version of the material of five introductory lectures on the theory of endoscopy delivered at the special program "On the Langlands Program: Endoscopy and Beyond" held at the Institute for Mathematical Sciences of the National University of Singapore from 17th December 2018 to 18th January 2019. I wish to thank the organizers of this program, Dihua Jiang, Lei Zhang, Bill Casselman, Pierre-Henri Chaudouard, Wee Teck Gan, and Chengbo Zhu, for their generous hospitality and for the excellent working conditions during the special program. I also thank Ali Altug, Bill Casselman, and Pierre-Henri Chaudouard, for answering various questions related to these notes.

1 Introduction

The theory of endoscopy, initially proposed by Langlands and subsequently developed by many people, centers around the concept of stability of invariant distributions on a connected reductive group, and its Lie algebra, defined over a local or global field. Stability, or rather its failure, in turn arises from the discrepancy between two notions of conjugacy on the group $G(F)$ of F-points of the reductive group G, where F is the local or global ground field: one of them, called "rational conjugacy", being the usual notion of conjugacy for the abstract group $G(F)$, and the other, called "stable conjugacy", being closely related to conjugacy by the group $G(\bar{F})$, where $\bar{F}$ is an algebraic (or separable, but we will work with fields of characteristic zero in these notes) closure of F. The comparison of these two notions of conjugacy involves both the group theory of G and the arithmetic of F.

The problem of stability arises in various places of the Langlands program, all of which are ultimately related to the trace formula. The trace formula gives, in principle, a formula to compute the trace of the operator $r(f)$ by which a test function f acts on the discrete automorphic spectrum of a reductive group G defined over a global field F, which in this introduction we will take to be $\mathbb{Q}$. We briefly recall this notion. Let $\mathbb{A}$ denote the ring of adeles of $\mathbb{Q}$. The group $G(\mathbb{Q})$ embeds into the locally compact group $G(\mathbb{A})$ as a discrete subgroup and one can consider the coset space $G(\mathbb{Q}) \setminus G(\mathbb{A})$. It carries a natural measure and one can consider the Hilbert space $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$. It carries the right regular action of $G(\mathbb{A})$. The discrete automorphic spectrum is a certain subspace, roughly speaking the largest subspace that decomposes as a Hilbert direct sum of irreducible representations of $G(\mathbb{A})$.

The simplest case of the trace formula is when G is anisotropic, i.e. it does not have an embedded copy of the multiplicative group $\mathbb{G}_m$ defined over $\mathbb{Q}$. An example of such a group is given by taking a division algebra D over $\mathbb{Q}$ and letting G be the group SL_1/D. Other examples arise by taking a unitary or special orthogonal group of a non-degenerate anisotropic Hermitian or symmetric bilinear form. When G is anisotropic the quotient $G(\mathbb{Q}) \setminus G(\mathbb{A})$ is compact, the Hilbert space $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ decomposes under the right regular action of $G(\mathbb{A})$ into a Hilbert direct sum of irreducible admissible representations π with finite multiplicities $m(\pi)$, and the trace formula takes the form

$$\sum_\pi m(\pi) \mathrm{tr}\, \pi(f) = \sum_\gamma \mathrm{vol}(G_\gamma(\mathbb{Q}) \setminus G_\gamma(\mathbb{A})) O_\gamma(f),$$

where the right sum runs over conjugacy classes in $G(\mathbb{Q})$ and O_γ is the integral of f over the $G(\mathbb{A})$-conjugacy class of γ.

When the group is isotropic, i.e. when it has an embedded copy of $\mathbb{G}_m$ defined over $\mathbb{Q}$, the situation is vastly more complicated. When such a copy is central in G, it causes a very minor complication which can be easily dealt with by replacing $G(\mathbb{A})$ by a certain natural subgroup $G(\mathbb{A})^1$ that is a group theoretic complement of $A_G(\mathbb{R})^0$ in $G(\mathbb{A})$, where A_G is the maximal $\mathbb{Q}$-split central torus in G. The serious complications are caused by copies of $\mathbb{G}_m$ that are not central in G. Their existence is equivalent to the existence of parabolic $\mathbb{Q}$-subgroups in G. The space $G(\mathbb{Q}) \setminus G(\mathbb{A})^1$ is then no longer compact and the Hilbert space $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A})^1)$ does not decompose as a direct sum of irreducible representations of $G(\mathbb{A})$ any more. It contains the continuous spectrum, which is fairly well understood by the work of Langlands, and the discrete spectrum, which is much less well understood and is the object of main interest. The trace formula gives an expression of the left hand side above, where now π runs over the irreducible constituents of the discrete spectrum, but the right hand side becomes significantly more complicated, both because the geometric terms become more complicated, and because it also involves auxiliary spectral contributions coming from the continuous spectrum. Said slightly differently, the trace formula is an identity

$$J_{\text{geom}}(f) = J_{\text{spec}}(f),$$

of two distributions taking a test function f, one distribution being of "geometric nature" in the sense that it is a sum of contributions from conjugacy classes of $G(\mathbb{Q})$, and one distribution being of "spectral nature", in the sense that it is made of contributions from representations of $G(\mathbb{A})$. Some of the contributions in the geometric side are usual orbital integrals, as was the case for an anisotropic group, but other contributions involve more complicated objects. One of the contributions of the spectral side is the sum of traces of discrete automorphic representations, weighted by their multiplicity, but again there are further more complicated contributions having to do with the continuous spectrum.

There are two principal ways in which the trace formula can be used.

1. Evaluate the geometric side to obtain information about the spectral side side (or vice versa).

2. Compare the geometric side to another formula, for example the trace formula for a different group, to obtain a relation between the other sides of the two formulas.

The first approach appears very difficult due to the complexity of the geometric side of the trace formula. But some versions of it have been carried out. An early example is Langlands' computation of the dimension of the space of automorphic forms [Lan63]. Another classical example is Drinfeld's computation of the number of rank 2 ℓ-adic local systems on a curve over a finite field [Dri81]. More recent examples include Müller's Weyl Law [Mül07], the computation of dimension of level 1 automorphic forms for classical groups by Taïbi [Taï17], or the work of Shin-Templier on asymptotics of families of automorphic forms [ST16].

The second approach was championed by Langlands – he used it to compare the spectra of inner forms, such as GL_2 and quaternion algebra [JL70], or SL_2 and norm-1-elements in a quaternion algebra [LL79]; to compare groups over different fields [Lan80]; to express the Hasse-Weil zeta function of Shimura varieties in terms of automorphic L-functions by comparing the (automorphic) trace formula with the Grothendieck-Lefschetz trace formula – all subjects that he began and that have since flourished in the hands of many people.

For each of these applications the original trace formula needs to be prepared – made more explicit, or invariant under (rational) conjugacy, or even invariant under stable conjugacy. Indeed, the distributions $J_{\text{geom}}(f)$ and $J_{\text{spec}}(f)$ are not invariant under rational conjugacy, let alone stable conjugacy. Since in most applications of the trace formula the test function f is not given directly, but rather only indirectly by specifying its orbital integrals, or stable orbital integrals, a trace formula that is not invariant, or not stably invariant, cannot be applied.

It is the theory of endoscopy, i.e. the study of the problem of stability and its failure, that is involved in the stabilization of the trace formula. This theory gives a different point of view on representation theory and harmonic analysis and is closely intertwined with the local and global Langlands correspondences. It entails a grouping of representations into packets, both locally and globally, and the relationship between packets and Galois representations. One of the central conjectures is an expression of the integer $m(\pi)$ in terms of Galois representations and endoscopic quantities, cf. (6.5.3).

In these notes we will give an introduction to the theory of endoscopy, both from the local and global perspectives. The group SL_2 will serve as our main example. We will review the stabilization of the trace formula for this group – first only of its "regular elliptic term" in a way that is as elementary as possible, and eventually of the full trace formula. In between we will discuss endoscopy for general reductive groups and the stabilization of the regular elliptic term of their trace formula. Part of the discussion will focus on the problem of normalizing transfer factors.

2 Stabilization of the elliptic regular part of the trace formula for $\mathbf{SL}_2$

{sec:ellregsl2}

2.1 The problem of stability

The non-invariant trace formula is an identity of distributions

$$J_{\text{geom}}(f) = J_{\text{spec}}(f).$$

When the reductive group G is anisotropic over $\mathbb{Q}$ the geometric side consists of orbital integrals, while the spectral side consists of traces of automorphic representations. Both sides are invariant distributions – if we replace the test function $f : G(\mathbb{A}) \to \mathbb{C}$ by a conjugate $f^g(\gamma) = f(g^{-1}\gamma g)$ for some $g \in G(\mathbb{A})$, then the values of J_{geom} and J_{spec} are unchanged.

When G is isotropic this is no longer the case. In fact, there is an even stronger condition than invariance and it turns out to be essential for many applications as well. It is called *stable invariance*. Even when G is anisotropic, the invariant distributions J_{geom} and J_{spec} need not be stably invariant.

Let $F/\mathbb{Q}$ be a field and G a connected reductive group G defined over F.

Definition 2.1.1. Let $\gamma_1, \gamma_2 \in G(F)$ be semi-simple elements with connected centralizers. Then γ_1, γ_2 are called *stably conjugate* if they are conjugate in $G(\bar{F})$, where $\bar{F}$ is the algebraic closure of F.

Remark 2.1.2. If the derived subgroup of G is simply connected, which is the case for example with $G = \mathrm{SL}_n$ or $G = \mathrm{GL}_n$, then the condition that the centralizers of γ_i are connected is automatic due to Steinberg's result [Ste68, Corollary 8.5]. The definition of stable conjugacy without assuming connectedness of centralizers is more complicated and will be given in Definition 3.1.1 below.

In order to better distinguish between the usual notion of conjugacy, namely under $G(F)$, and stable conjugacy, it is customary to say that two elements of $G(F)$ are *rationally conjugate*, if they are conjugate in $G(F)$. This terminology is used for any field F, not just $F = \mathbb{Q}$.

Remark 2.1.3. For the groups SL_n and GL_n there is a simple interpretation of stable conjugacy in terms of linear algebra: two semi-simple elements are stably conjugate if and only if the corresponding matrices have the same characteristic polynomial. There is a similar, but more complicated, interpretation for other classical groups.

It follows from the rational canonical form of matrices that for the group $G = \mathrm{GL}_n$ stable and rational conjugacy are the same. This is not true in general.

Example 2.1.4. The elements

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

of $\mathrm{SL}_2(\mathbb{R})$ are conjugate in $\mathrm{SL}_2(\mathbb{C})$, namely by

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

but are not conjugate in $\mathrm{SL}_2(\mathbb{R})$.

Definition 2.1.5. The *stable conjugacy class* of $\gamma \in G(F)$ is the set of all $\gamma' \in G(F)$ that are stably conjugate to γ.

Stable conjugacy is an equivalence relation that is in general looser than rational conjugacy – a stable class may contain multiple rational conjugacy classes.

Defining what it means for a function to be stably invariant is easy:

Definition 2.1.6. A function $f : G(F) \to \mathbb{C}$ is called *stably invariant* if $f(\gamma_1) = f(\gamma_2)$ whenever γ_1, γ_2 are stably conjugate.

However, since stable conjugacy classes are not orbits in $G(F)$ for the action of a group, unlike it is the case with rational conjugacy classes, defining what a stably invariant *distribution* is is a bit more subtle. It is modeled on the deep result of Harish-Chandra about the density of strongly regular semi-simple orbital integrals in the space of invariant distributions. An element $\gamma \in G$ is called *strongly regular semi-simple* if its centralizer T_γ in G is a torus. A *distribution* is a linear functional on the complex vector space $\mathcal{C}_c^\infty(G(F))$ of smooth compactly supported functions $G(F) \to \mathbb{C}$. Here smooth has the usual meaning in terms of the Lie group $G(F)$ when F is archimedean, and one requires that a distribution be continuous with respect to a suitable topology on $\mathcal{C}_c^\infty(G(F))$, cf. [Wal88, 8.A.1]. When F is non-archimedean, one does not put a topology on $\mathcal{C}_c^\infty(G(F))$ and does not require any continuity of distributions. For a function $f \in \mathcal{C}_c^\infty(G(F))$ and a strongly regular semi-simple element $\gamma \in G(F)$ let $O_\gamma(f)$ denote the orbital integral

$$O_\gamma(f) = \int_{T_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)\frac{dg}{dt},$$

where dg and dt are choices of Haar measures on $G(F)$ and $T_\gamma(F)$. This is a distribution on $G(F)$. Harish-Chandra's density result ([HC78], [HC99]) can be stated as follows.

Theorem 2.1.7. *Let F be a local field and let I be a distribution on $G(F)$. The following statements are equivalent*

1. *I is invariant. That is, $I(f^g) = I(f)$ for all $f \in \mathcal{C}_c^\infty(G(F))$ and all $g \in G(F)$.*
2. *If $f \in \mathcal{C}_c^\infty(G(F))$ has the property that $O_\gamma(f) = 0$ for all strongly regular semi-simple $\gamma \in G(F)$, then $I(f) = 0$.*

It is easy to define a stable analog of the orbital integral O_γ:

Definition 2.1.8. Let F be a local field. Let $\gamma \in G(F)$ be strongly regular semi-simple. The *stable orbital integral* at γ is

$$SO_\gamma(f) := \sum_{\gamma'} O_{\gamma'}(f),$$

where the sum is taken over (a set of representatives for) the set of rational conjugacy classes inside the stable conjugacy class of γ.

Remark 2.1.9. The definition of the stable orbital integral for an element $\gamma \in G(F)$ that is not strongly regular semi-simple is more subtle, see [Kot86, §5.2] for the case of semi-simple elements that are not strongly regular.

Definition 2.1.10. Let F be a local field. An invariant distribution I on $G(F)$ is called *stably invariant* if $I(f) = 0$ for all $f \in \mathcal{C}_c^\infty(G(F))$ which satisfy $SO_\gamma(f) = 0$ for all strongly regular semi-simple $\gamma \in G(F)$.

Using the Weyl integration formula the following is easy to see:

Fact 2.1.11. *If the distribution I is represented by a function ϕ, then I is stably invariant if and only if ϕ is stably invariant up to a set of measure zero.*

2.2 Rational and stable conjugacy for $\mathbf{SL}_2$

Before we dive into the discussion of the regular elliptic term of the trace formula for SL_2 and its stabilization, we take a closer look at rational and stable conjugacy for that group.

For the group SL_2, and more generally SL_n, something very special happens – since usual and stable conjugacy in $\mathrm{GL}_2(F)$ are the same, and since two elements of $\mathrm{SL}_2(F)$ are conjugate under $\mathrm{SL}_2(\bar{F})$ if and only if they are conjugate under $\mathrm{GL}_2(\bar{F})$, the stable classes in $\mathrm{SL}_2(F)$ do happen to be orbits under a group action, namely the conjugation action of $\mathrm{GL}_2(F)$. This allows us to compute the set of conjugacy classes inside of a stable conjugacy class easily. Indeed, the stable class of a given γ receives a transitive action of $\mathrm{GL}_2(F)$ by conjugation, so it is enough to compute the stabilizer for this action of the rational class of γ.

Let γ be a regular semi-simple element, i.e. one with distinct eigenvalues. Let T_γ and $\tilde{T}_\gamma$ be its centralizers in $G = \mathrm{SL}_2$ and $\tilde{G} = \mathrm{GL}_2$. These are maximal tori in their respective groups. The stabilizer in $\mathrm{GL}_2(F)$ of the $\mathrm{SL}_2(F)$-orbit of γ is $\mathrm{SL}_2(F) \cdot \tilde{T}_\gamma(F) \subset \mathrm{GL}_2(F)$.

If γ is split, that is to say its eigenvalues lie in $F^\times$, then $\mathrm{SL}_2(F) \cdot \tilde{T}_\gamma(F) = \mathrm{GL}_2(F)$, so the stable class of γ contains a single rational class.

If $\gamma \in \mathrm{SL}_2(F)$ is elliptic, i.e. its eigenvalues do not lie in F, then these eigenvalues generate a quadratic extension E/F and we have $\tilde{T}_\gamma(F) = E^\times$. The restriction of $\det : \mathrm{GL}_2(F) \to F^\times$ to $T_\gamma(F)$ becomes the norm map $N_{E/F} : E^\times \to F^\times$ and so $T_\gamma(F)$ is the kernel of $N_{E/F}$, which we shall denote by E^1. We have the exact sequence

$$1 \to \mathrm{SL}_2(F) \cdot \tilde{T}_\gamma(F) \to \mathrm{GL}_2(F) \to F^\times / N_{E/F}(E^\times) \to 1,$$

whose final term parameterizes the set of conjugacy classes in the stable class of γ. More explicitly, given $g \in \mathrm{GL}_2(F)$, the element $g\gamma g^{-1}$ lies in the stable class of γ; it lies in the rational class of γ if and only if the element $\det g \in F^\times$ is in the image of the norm map $N_{E/F}$.

When F is local the quotient $F^\times / N_{E/F}(F^\times)$ has order 2. When F is global, say $F = \mathbb{Q}$, this quotient is infinite.

Now that we understand the set of rational classes inside a given stable class, let us consider the set of stable classes of regular semi-simple elements. Such a class is uniquely determined by its eigenvalues, i.e. an unordered pair $\{c, c^{-1}\}$ of elements lying either in $F^\times$ or in E^1 for a quadratic extension E/F. In the first case, the stable class is represented by the matrix

$$\begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix}.$$

In the second case we choose $\eta \in E^\times \smallsetminus F^\times$ with $\eta^2 \in F^\times$ and write $c = a + \eta b$ and obtain the following representative

$$\begin{bmatrix} a & b \\ \eta^2 \cdot b & a \end{bmatrix}. \tag{2.2.1}$$ {eq:rep}

This representative depends on the choice of η. For example, if we replace η by $-\eta$ then the representative becomes

$$\begin{bmatrix} a & -b \\ -\eta^2 \cdot b & a \end{bmatrix}$$

and lies in the same rational class as (2.2.1) if and only if $-1 \in F^\times$ lies in $N_{E/F}(E^\times)$. The same statement is true for the representative

$$\begin{bmatrix} a & \eta^2 \cdot b \\ b & a \end{bmatrix}.$$

2.3 The regular elliptic term for SL_2

We consider for $G = \mathrm{SL}_2$ the contributions of the regular elliptic elements to the geometric side of the trace formula (invariant or not, it doesn't make a difference), i.e. the so called regular elliptic term. It has the following form

$$\sum_{\gamma \in G(\mathbb{Q})_{\text{reg.ell}}/G(\mathbb{Q})\text{-conj}} \mathrm{vol}(T_\gamma(\mathbb{Q}) \setminus T_\gamma(\mathbb{A})) \cdot O_\gamma(f),$$

where

$$O_\gamma(f) = \int_{T_\gamma(\mathbb{A})\setminus G(\mathbb{A})} f(g^{-1}\gamma g)\frac{dg}{dt},$$

is the adelic orbital integral of f. We are summing here over the set of $G(\mathbb{Q})$-conjugacy classes of elliptic regular elements and T_γ is the centralizer of such an element – an anisotropic maximal torus in G. We are using the Tamagawa measures on $G(\mathbb{A})$ and $T_\gamma(\mathbb{A})$ and the counting measure on $T_\gamma(\mathbb{Q})$, and f is a smooth compactly supported function on $G(\mathbb{A})$. The definition of the Tamagawa measures in this special case is very simple: one chooses an arbitrary differential form of top degree defined over $\mathbb{Q}$ and takes the measure induced by it and by the standard measure on $\mathbb{A}$; this measure is independent of the chosen top form due to the adelic product formula. For the definition of the Tamagawa measure in general we refer to [Ono66] and [Wei82, Appendix II]. This measure depends only on the isomorphism class of G. In particular, if T_{γ_1} and T_{γ_2} are isomorphic, the corresponding volume terms are equal.

Each individual O_γ is an invariant distribution, but not a stably invariant distribution. This does not a-priori mean that the entire sum is not a stably invariant distribution, so we set out to investigate. We can split the sum into two sums. First we sum over the set of stable classes, and then over the set of rational classes within each stable class. When γ_1, γ_2 are stably conjugate, they are conjugate by some $g \in \mathrm{GL}_2(\mathbb{Q})$ and then $\mathrm{Ad}(g) : T_{\gamma_1} \to T_{\gamma_2}$ is an isomorphism defined over $\mathbb{Q}$, so the volume term depends only on the stable class. We get

$$\sum_{\gamma_0 \in G(\mathbb{Q})_{\text{reg.ell}}/\text{st-conj}} \mathrm{vol}(T_{\gamma_0}(\mathbb{Q}) \setminus T_{\gamma_0}(\mathbb{A})) \sum_{\gamma \sim \gamma_0} O_\gamma(f)$$

where the second sum runs over the set of $\mathrm{SL}_2(\mathbb{Q})$-classes inside the $\mathrm{GL}_2(\mathbb{Q})$-class of γ_0.

At this point it may seem that the inner sum is just $SO_{\gamma_0}(f)$ and we are done. This is not so! The distribution O_γ integrates f over the orbit of γ in

$G(\mathbb{A})$, not $G(\mathbb{Q})$, so the stability we are looking for is in terms of the group $G(\mathbb{A})$ and not the group $G(\mathbb{Q})$. In our special case this means invariance under $\mathrm{GL}_2(\mathbb{A})$, not just $\mathrm{GL}_2(\mathbb{Q})$. In order to obtain a stable distribution we must therefore sum over all $\mathrm{GL}_2(\mathbb{A})$ conjugates of γ. We have the similar exact sequence as above

$$1 \to \mathrm{SL}_2(\mathbb{A}) \cdot \tilde{T}_\gamma(\mathbb{A}) \to \mathrm{GL}_2(\mathbb{A}) \to \mathbb{A}^\times / N_{E/F}(\mathbb{A}_E^\times) \to 1,$$

showing us that the set of $\mathrm{SL}_2(\mathbb{A})$ conjugacy classes in the $\mathrm{GL}_2(\mathbb{A})$-orbit of γ is in bijection with $\mathbb{A}^\times / N_{E/F}(\mathbb{A}_E^\times)$.

Thus, we are summing so far over $F^\times / N_{E/F}(E^\times)$, but we would like to sum instead over $\mathbb{A}^\times / N_{E/F}(\mathbb{A}_E^\times)$. What is the discrepancy? It is measured by the exact sequence

$$1 \to F^\times / N_{E/F}(E^\times) \to \mathbb{A}^\times / N_{E/F}(\mathbb{A}_E^\times) \to (\mathbb{A}^\times / F^\times)/(N_{E/F}(\mathbb{A}^\times / E^\times)) \to 1.$$

The injectivity here requires a small remark – it follows from the Hasse norm theorem stating that an element of a global field is a norm from a cyclic extension if and only if it is everywhere locally so. Global class field theory [CF86, Chap. VII] identifies $(\mathbb{A}^\times / F^\times)/(N_{E/F}(\mathbb{A}^\times / E^\times))$ with the Galois group $\Gamma_{E/F}$ of the quadratic extension E/F – a group of order 2.

We find ourselves in the following situation: We have the function $\gamma \mapsto O_\gamma(f)$ defined for every regular semi-simple $\gamma \in \mathrm{SL}_2(\mathbb{A})$. We are summing its values over the set B_0 of $\mathrm{SL}_2(\mathbb{Q})$-orbits inside of the $\mathrm{GL}_2(\mathbb{Q})$-orbit of $\gamma_0 \in \mathrm{SL}_2(\mathbb{Q})$. We would however like to sum it over the set of $\mathrm{SL}_2(\mathbb{A})$-orbits inside of the $\mathrm{GL}_2(\mathbb{A})$-orbit of γ_0, in order to obtain a stable distribution. The latter set is a disjoint union $B_0 \cup B_1$ corresponding to the two cosets of the subgroup $F^\times / N_{E/F}(E^\times)$ of $\mathbb{A}^\times / N_{E/F}(\mathbb{A}_E^\times)$. Therefore

$$\begin{aligned}\sum_{\gamma \in B_0} O_\gamma(f) &= \frac{1}{2}\left((\sum_{\gamma \in B_0} O_\gamma(f) + \sum_{\gamma \in B_1} O_\gamma(f)) + (\sum_{\gamma \in B_0} O_\gamma(f) - \sum_{\gamma \in B_1} O_\gamma(f)) \right) \\ &= \frac{1}{2}(SO_{\gamma_0}(f) + O^\kappa_{\gamma_0}(f))\end{aligned}$$

The elliptic regular part of the trace formula then becomes

$$\sum_{\gamma_0 \in G(\mathbb{Q})_{\text{reg.ell}}/\text{st-conj}} \mathrm{vol}(T_{\gamma_0}(\mathbb{Q}) \setminus T_{\gamma_0}(\mathbb{A})) \frac{1}{2}(SO_{\gamma_0}(f) + O^\kappa_{\gamma_0}(f)).$$

We write

$$\mathrm{STF}^G_{\text{reg.ell}}(f) = \sum_{\gamma_0 \in G(\mathbb{Q})_{\text{reg.ell}}/\text{st-conj}} \mathrm{vol}(T_{\gamma_0}(\mathbb{Q}) \setminus T_{\gamma_0}(\mathbb{A})) \frac{1}{2} SO_{\gamma_0}(f).$$

This is a stable distribution. In fact, one can compute that $\mathrm{vol}(T_{\gamma_0}(\mathbb{Q}) \backslash T_{\gamma_0}(\mathbb{A})) = 2$, so this distribution is simply

$$\mathrm{STF}^G_{\text{reg.ell}}(f) = \sum_{\gamma_0 \in G(\mathbb{Q})_{\text{reg.ell}}/\text{st-conj}} SO_{\gamma_0}(f).$$

To deal with the remainder, namely the contributions of $O^{\kappa}_{\gamma_0}(f)$, we recall our discussion of the set of regular elliptic stable classes in $\mathrm{SL}_2(\bar{\mathbb{Q}})$ and rewrite this remainder as

$$\frac{1}{4}\sum_{E}\sum_{\gamma\in E^1,\gamma\neq\{\pm 1\}}\mathrm{vol}(E^1\setminus\mathbb{A}^1_E)O^{\kappa}_{\gamma}(f).$$

Here E runs over the set of quadratic field extensions $E/\mathbb{Q}$ contained in $\bar{\mathbb{Q}}$. The factor $1/2$ has become $1/4$ because in the inner sum γ and γ^{-1} give separate contributions, yet the pair $\{\gamma,\gamma^{-1}\}$ accounts for the same stable class. The isomorphism class of the centralizer T of γ depends only on E (it is the unique one-dimensional anisotropic torus defined over F and split over E). The inner sum is most of the trace formula for the torus T evaluated at the function $f^T(\gamma)=O^{\kappa}_{\gamma}(f)$. We say most because it is missing the two summands corresponding to $\gamma=\pm 1$, i.e. to the singular elements of $G(\mathbb{Q})$. Let us denote it by

$$\mathrm{STF}^{\text{G-reg}}_{T}(f^T).$$

Our final form for the elliptic regular part of the trace formula for G is

$$\mathrm{TF}^{G}_{\text{reg.ell}}(f)=\mathrm{STF}^{G}_{\text{reg.ell}}(f)+\frac{1}{4}\sum_{T}\mathrm{STF}^{T}_{\text{G-reg}}(f^T). \tag{2.3.1}$$ {eq:ellregsl2}

2.4 Adelic κ-orbital integrals

But there is a catch! This would only work if we knew that $f^T\in\mathcal{C}^{\infty}_c(T(\mathbb{A}))$. Standard results about orbital integrals tell us that the function f^T is smooth on $T(\mathbb{A})\smallsetminus\{\pm 1\}$. But it needs to be checked that it extends smoothly to $\{\pm 1\}$.

Assume that the test function f is factorizable, i.e. $f=\prod f_v$ with $f_v\in\mathcal{C}^{\infty}_c(G(\mathbb{Q}_v))$ for all $v\leq\infty$ and f_p is the characteristic function of $G(\mathbb{Z}_p)$ for almost all $p<\infty$. A general test function is a sum of factorizable test functions, and since our problem is linear there is no loss in generality. Then the adelic orbital integral becomes the product of local orbital integrals:

$$O_{\gamma}(f)=\prod_{v}O_{\gamma_v}(f_v),\qquad O_{\gamma_v}(f_v)=\int_{T_\gamma(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)}f(g_v^{-1}\gamma_v g_v)dg_v.$$

We claim that the analogous factorization holds for O^{κ}_{γ}. To see this we consider again the surjective map

$$\mathbb{A}^{\times}/N_{E/\mathbb{Q}}(\mathbb{A}^{\times}_E)\to(\mathbb{A}^{\times}/\mathbb{Q}^{\times})/(N_{E/\mathbb{Q}}(\mathbb{A}^{\times}/E^{\times}))$$

and recall that $O^{\kappa}_{\gamma}(f)$ is defined as the sum of the orbital integrals for all orbits in the trivial fiber, minus the sum for all orbits in the non-trivial fiber. At any place v of $\mathbb{Q}$ that is unramified in E the norm map $N_{E_v/\mathbb{Q}_v}:O^{\times}_{E_v}\to\mathbb{Z}^{\times}_v$ is surjective. This leads to the isomorphism

$$\mathbb{A}^{\times}/N_{E/\mathbb{Q}}(\mathbb{A}^{\times}_E)=\bigoplus_{v}\mathbb{Q}^{\times}_v/N_{E_v/\mathbb{Q}_v}(E^{\times}_v).$$

By local class field theory [CF86, Chap. VI] we have

$$\mathbb{Q}_v^\times/N_{E_v/\mathbb{Q}_v}(E_v^\times) = \begin{cases} 0, & v \text{ split} \\ \mathbb{Z}/2\mathbb{Z}, & \text{else} \end{cases}$$

and the above surjective map becomes the summation map

$$\bigoplus_{v \text{ non-split}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}.$$

If $\kappa : (\mathbb{A}^\times/\mathbb{Q}^\times)/(N_{E/\mathbb{Q}}(\mathbb{A}^\times/E^\times)) \to \{\pm 1\}$ is the unique non-trivial character then its pull-back to $\mathbb{Q}_v^\times/N(E_v^\times)$ is the unique non-trivial character κ_v, so

$$\begin{aligned} O_\gamma^\kappa(f) &= \int_{\tilde{T}_\gamma(\mathbb{A})\backslash \tilde{G}(\mathbb{A})} f(g^{-1}\gamma_0 g)\kappa(\det g)dg \\ &= \prod_v \int_{\tilde{T}_\gamma(\mathbb{Q}_v)\backslash \tilde{G}(\mathbb{Q}_v)} f_v(g_v^{-1}\gamma_v g_v)\kappa_v(\det g_v)dg_v \\ &= \prod_v O_{\gamma_v}^{\kappa_v}(f_v), \end{aligned}$$

proving the claim. Recalling that $\mathbb{Q}_v^\times/N(E_v^\times)$ parameterizes the $\mathrm{SL}_2(\mathbb{Q}_v)$-conjugacy classes in the stable class of γ_v we see that

$$O_{\gamma_v}^{\kappa_v}(f_v) = O_{\gamma_v}(f_v) - O_{\gamma_v'}(f_v),$$

where γ_v' is a member of the other $\mathrm{SL}_2(\mathbb{Q}_v)$-conjugacy class in the stable class of γ_v when $E_v/\mathbb{Q}_v$ doesn't split, and otherwise $O_{\gamma_v}^{\kappa_v}(f_v) = O_{\gamma_v}(f_v)$. Letting $f_{v,\text{naive}}^T = O_{\gamma_{0,v}}^{\kappa_v}(f_v)$ we have shown that $f^T = \prod_v f_{v,\text{naive}}^T$ and have thus reduced the problem of studying the global object f^T to the problem of studying its local components $f_{v,\text{naive}}^T$.

In fact, we need to be slightly careful here. While $O_\gamma^\kappa(f)$ depends only on the $\mathrm{GL}_2(\mathbb{Q})$-conjugacy class of γ, its local component $O_{\gamma_v}^{\kappa_v}(f_v)$ depends on the $\mathrm{SL}_2(\mathbb{Q}_v)$-conjugacy class of γ_v, and not just on the $\mathrm{GL}_2(\mathbb{Q}_v)$-conjugacy class. If we change γ by a $\mathrm{GL}_2(\mathbb{Q})$-conjugate each local component might change by a sign. There will be only finitely many changes and their product will be 1. Thus, when analyzing the local components we need to be mindful of the element γ, and not just its $\mathrm{GL}_2(\mathbb{Q})$-conjugacy class.

Therefore we will fix the quadratic extension $E/\mathbb{Q}$ as well as $\eta \in E^\times \smallsetminus \mathbb{Q}^\times$ with $\eta^2 \in \mathbb{Q}^\times$, so that $E = \mathbb{Q}(\eta)$, so that we obtain the particular embedding $E^1 \to \mathrm{SL}_2(\mathbb{Q})$ sending $\gamma = a + \eta \cdot b$ to the matrix (2.2.1).

2.5 Local harmonic analysis

We now consider the local component $f_{v,\text{naive}}^T$ and hope that it extends smoothly to ± 1. This hope is immediately dashed. In fact, this function blows up towards the singular elements. The first step is to renormalize it. The function

$$|\gamma_v - \gamma_v^{-1}|_v \cdot f_{v,\text{naive}}^T(\gamma_v)$$

does not blow up any more as $\gamma_v \to \pm 1$. If E splits at v this is all that is needed – this function extends smoothly to $\gamma = \pm 1$. But if E is inert at v this need not be the case. When $-1 \in \mathbb{Q}_v^\times$ is not a norm from $E_v^\times$ then γ_v and $\gamma_v^{-1} = \bar{\gamma}_v$ are stably conjugate but not rationally conjugate, which shows that $|\gamma_v - \bar{\gamma}_v|_v \cdot f^T_{v,\text{naive}}(\gamma_v)$ changes sign if we replace γ_v by $\bar{\gamma}_v$. For such a function to have a continuous extension to $\gamma_v = 1$ its value there must be zero, but one can show that this need not be the case. It turns out that adding the appropriate sign resolves this problem: for any local field the following function extends smoothly:

$$\kappa_v\left(\frac{\gamma_v - \bar{\gamma}_v}{\eta_v}\right)|\gamma_v - \bar{\gamma}_v|_v f^T_{v,\text{naive}}(\gamma_v).$$

Note here that both $\gamma_v - \bar{\gamma}_v$ and η_v are elements of $E_v^\times$ that change sign under Galois conjugation, so their ratio lies in $\mathbb{Q}_v^\times$. Replacing γ_v by $\bar{\gamma}_v$ introduces the factor $\kappa_v(-1)$, which is -1 precisely when $-1 \in F_v^\times$ is not a norm from $E_v^\times$, i.e. precisely when $f^T_{v,\text{naive}}(\gamma_v)$ also changes sign upon replacing γ_v with $\bar{\gamma}_v$.

In fact, we shall define

$$f_v^T(\gamma_v) = \lambda(E_v/\mathbb{Q}_v, \psi_v)\kappa_v\left(\frac{\gamma_v - \bar{\gamma}_v}{\eta_v}\right)|\gamma_v - \bar{\gamma}_v|_v f^T_{v,\text{naive}}(\gamma_v). \qquad (2.5.1)$$ {eq:loctsl2}

The constant $\lambda(E_v/\mathbb{Q}_v, \psi_v) = \epsilon(\frac{1}{2}, \text{sgn}_{\Gamma_{E_v/\mathbb{Q}_v}}, \psi_v)$ has a spectral meaning. Namely, to each $\theta_v : E_v^1 \to \mathbb{C}^\times$ (except those of order 2, where the situation is a bit more complicated) one can associate two irreducible representations $\pi^+(\theta_v)$ and $\pi^-(\theta_v)$ of $\text{SL}_2(\mathbb{Q}_v)$. For $v = \infty$ and $\theta(z) = z^k$ these would be the two discrete series representations of weight $k+1$ when $k > 0$, and the two limit-of-discrete-series representations when $k = 0$. The set $\Pi(\theta_v) = \{\pi^+(\theta_v), \pi^-(\theta_v)\}$ depends only on θ_v, but the labeling $\pi^\pm(\theta_v)$ of its constituents depends also on ψ_v. Then it turns out that

$$\pi^+(\theta_v)(f_v) - \pi^-(\theta_v)(f_v) = \theta_v(f_v^T) \qquad (2.5.2)$$ {eq:lcisl2}

where $\theta_v(f_v^T)$ is the integral over $T(\mathbb{Q}_v)$ of the product $\theta_v \cdot f_v^T$. This is called an "endoscopic character identity" and is important for the spectral interpretation of the stabilized trace formula, because it accounts for the spectral contribution of the trace formula for T. When ψ_v changes, the change in the labeling $\{\pi^+, \pi^-\}$ of the constituents of $\Pi(\theta_v)$ matches the sign change of $\lambda(E_v/\mathbb{Q}_v, \psi_v)$, hence of f_v^T.

We thus had to decorate $f^T_{v,\text{naive}}$ with a number of auxiliary terms in order to obtain a smooth function. Thankfully, these local auxiliary terms disappear globally: For $\gamma \in E^1$ we have

$$\prod_v \lambda(E_v/F_v, \psi_v)\kappa_v\left(\frac{\gamma_v - \bar{\gamma}_v}{\eta_v}\right)|\gamma_v - \gamma_v^{-1}|_v = 1.$$

Indeed each of the three individual sub-products equals 1, the first being a property of the Langlands λ-constants, the second because κ is trivial on $\mathbb{Q}^\times \subset \mathbb{A}^\times$, and the third by the adelic product formula. Note that by definition the first two factors are trivial at the split places.

3 Endoscopy for quasi-split connected reductive groups

We shall now consider a general connected reductive group G and set the stage for the stabilization of the trace formula. After some preliminary discussion we will assume that G is quasi-split, leaving the more general case for the next lecture. We let F denote an arbitrary field of characteristic zero, $\bar{F}$ an algebraic closure, and $\Gamma = \mathrm{Gal}(\bar{F}/F)$ the absolute Galois group. We will eventually specialize to the case that F is a local field, i.e. a finite extension of $\mathbb{R}$ or $\mathbb{Q}_p$.

3.1 Stable conjugacy

For the group $\mathrm{SL}_n(F)$ we saw that stable conjugacy is an equivalence relation that is effected by the conjugation action of $\mathrm{GL}_n(F)$ on $\mathrm{SL}_n(F)$. For a general group G there is no natural bigger group $\tilde{G}$ so that the orbits of the conjugation action of $\tilde{G}(F)$ are the stable classes in $G(F)$. Instead we work directly with G and use Galois cohomology. For basic definitions and results on Galois cohomology we refer the reader to [Ser97] and [Ser79].

The definition of stable conjugacy in the general case, for elements that may not be semi-simple, and when their centralizers may not be connected, is due to Kottwitz [Kot82].

{dfn:stconjgen}

Definition 3.1.1. Two elements $\delta, \delta' \in G(F)$ are called *stably conjugate* if there exists $g \in G(\bar{F})$ such that

1. $g\delta g^{-1} = \delta'$.
2. For every $\sigma \in \Gamma$ the element $g^{-1}\sigma(g)$ belongs to $G^\circ_{\delta_s}$, the connected component of the centralizer of the semi-simple part of δ.

Remark 3.1.2. Note that we automatically have $g\delta_s g^{-1} = \delta'_s$ and therefore $g^{-1}\sigma(g) \in G_{\delta_s}$. As remarked earlier, when G_{der} is simply connected, then G_{δ_s} is connected by a theorem of Steinberg, and therefore stable conjugacy is the same as $\bar{F}$-conjugacy. More generally, stable conjugacy and $G(\bar{F})$-conjugacy coincide for elements whose semi-simple parts have connected centralizers.

For a moment assume that δ, δ' are strongly regular semi-simple elements with centralizers T and T', maximal tori of G. For any $g \in G(\bar{F})$ such that $g\delta g^{-1} = \delta'$ we have the isomorphism $\mathrm{Ad}(g) : T \to T'$ sending δ to δ'. We have the following facts that are immediate to check.

Fact 3.1.3. {fct:stconj}

1. *The isomorphism $Ad(g) : T \to T'$ depends only on δ and δ' but not on the choice of g. It is defined over F. We shall call it $\varphi_{\delta,\delta'}$.*
2. *$\sigma \mapsto g^{-1}\sigma(g)$ belongs to $Z^1(\Gamma, T)$ and its cohomology class is independent of the choice of g. We shall call it $inv(\delta, \delta')$.*
3. *The map $\delta' \mapsto inv(\delta, \delta')$ sets up a bijection between the set of F-classes inside of the $\bar{F}$-class of δ' and the set $ker(H^1(\Gamma, T) \to H^1(\Gamma, G))$.*

Example 3.1.4. Let $G = \mathrm{SL}_2$, $T \subset G$ a maximal torus corresponding to a quadratic extension E/F, $\tilde{G} = \mathrm{GL}_2$ and $\tilde{T}$ the maximal torus there. The exact sequence

$$1 \to T \to \tilde{T} \to \mathbb{G}_m \to 1$$

given by the determinant map induces the exact sequence of cohomology

$$1 \to E^1 \to E^\times \to F^\times \to H^1(\Gamma, T) \to 1$$

and thus identifies $H^1(\Gamma, T) \cong F^\times / N_{E/F}(E^\times)$. We have used here the vanishing of $H^1(\Gamma, \tilde{T})$, which follows from Shapiro's lemma and the fact that $\tilde{T}(\bar{F})$ is an induced Γ-module. Furthermore $H^1(\Gamma, G) = 1$. Therefore the set the set $\ker(H^1(\Gamma, T) \to H^1(\Gamma, G))$ is simply $F^\times / N_{E/F}(E^\times)$, which is what we saw in the previous Section.

There is a similar statement for general elements. But we need to introduce another definition.

{dfn:inner}

Definition 3.1.5. Let G and H be algebraic groups defined over F. An isomorphism $\xi : G \to H$ defined over $\bar{F}$ is called an *inner twist* if for every $\sigma \in \Gamma$ the automorphism $\xi^{-1}\sigma(\xi) = \xi^{-1} \circ \sigma_H \circ \xi \circ \sigma_G^{-1}$ of the group G is inner. Here σ_G and σ_H are the actions of σ on G and H.

Now take δ, δ' general elements and g as in the definition of stable conjugacy. Define $G_\delta^* = G_{\delta_s}^\circ \cap G_\delta$. This group may be disconnected.

Fact 3.1.6.

1. *The isomorphism $Ad(g)$ is an inner twist $G_\delta^* \to G_{\delta'}^*$. It depends on g only up to an F-rational point of G_δ / G_δ^*.*

2. *The element $g^{-1}\sigma(g)$ lies in $Z^1(\Gamma, G_\delta^*)$ and the image in $H^1(\Gamma, G_\delta)$ of its class is independent of the choice of g.*

3. *This sets up a bijection between the set of F-classes inside of the stable class and the image of*

$$\mathit{ker}(H^1(\Gamma, G_\delta^*) \to H^1(\Gamma, G)) \to \mathit{ker}(H^1(\Gamma, G_\delta) \to H^1(\Gamma, G)).$$

For more discussion and results about stable conjugacy we refer the reader to [Kot82].

3.2 The Langlands dual group

We assume from now on that G is quasi-split and fix a Borel pair (T_0, B_0), i.e. a Borel subgroup $B_0 \subset G$ and a maximal torus $T_0 \subset B_0$, both defined over F. Let $X = X^*(T_0)$ and $Y = X_*(T_0)$ be the character and co-character modules of T_0. Let $R \subset X$ and $R^\vee \subset Y$ be the root system and its dual, and let $\Delta \subset R$ and $\Delta^\vee \subset R^\vee$ be the sets of simple roots and simple coroots. The tuple

$(X,\Delta,Y,\Delta^\vee)$ is called the *based root datum* of (G,B_0,T_0). There is an action of Γ on X and Y leaving invariant $\Delta \subset R$ and $\Delta^\vee \subset R^\vee$ and the canonical pairing between X and Y.

The *connected Langlands dual* group $\widehat{G}$ is defined to be the unique triple $(\widehat{G},\widehat{B}_0,\widehat{T}_0)$, taken say over $\mathbb{C}$, whose based root datum is $(Y,\Delta^\vee,X,\Delta)$. We augment the Borel pair $(\widehat{T}_0,\widehat{B}_0)$ to a pinning $(\widehat{T}_0,\widehat{B}_0,\{\widehat{X}_{\alpha^\vee}\})$, which means that for each simple root $\alpha^\vee \in \Delta^\vee$ we have fixed a non-zero vector $\widehat{X}_{\alpha^\vee}$ in the 1-dimensional subspace of the Lie algebra of $\widehat{G}$ corresponding to $\alpha^\vee$. The action of each $\sigma \in \Gamma$ on $(Y,\Delta^\vee,X,\Delta)$ lifts uniquely to an automorphism of $\widehat{G}$ that preserves the pinning. The *Galois form of the L-group* is then ${}^LG = \widehat{G} \rtimes \Gamma$. This construction works over any ground field F. When F is a local or global field, there is a variation that is often useful, called the *Weil form of the L-group*, which is ${}^LG = \widehat{G} \rtimes W_F$, where W_F is the Weil group of the field F. Unfortunately there seems to be no distinction made in the literature between the two forms, so one needs to infer from the context which one is being used; in many cases it does not matter.

Remark 3.2.1. The auxiliary choices of (T_0,B_0) and $(\widehat{T}_0,\widehat{B}_0,\{\widehat{X}_{\alpha^\vee}\})$ are of no importance. We can forget them once the construction is done. All objects involving the relationship between G and ${}^LG$ will then be well-defined up to conjugacy. One can make this formal by taking the limit over all Borel pairs. In practice what is important to note is the following: Given any two Γ-invariant Borel pairs (T_0,B_0) of G and $(\widehat{T}_0,\widehat{B}_0)$ of $\widehat{G}$ there is a canonical identification $X_*(T_0)=X^*(\widehat{T}_0)$; it sends the set of B_0-simple coroots to the set of $\widehat{B}_0$-simple roots.

Fact 3.2.2. *Any Γ-invariant Borel pair of $\widehat{G}$ can be extended to a Γ-invariant pinning. Equivalently, any two Γ-invariant Borel pairs of $\widehat{G}$ are conjugate under $\widehat{G}^\Gamma$.*

Proof. The equivalence is implied by [Kot84b, Corollary 1.7] and the proof is in fact a minor modification of that of [Kot84b, Lemma 1.6]. Let $(\widehat{T}_0,\widehat{B}_0)$ be Γ-invariant and let $g \in \widehat{G}$ be such that $\mathrm{Ad}(g)(\widehat{T}_0,\widehat{B}_0)$ is also Γ-invariant. Then $g^{-1}\sigma(g) \in \widehat{T}_0$ for all $\sigma \in \Gamma$. This implies that σ fixes the $\widehat{B}_0$-double coset containing g, and hence there exists $n \in \widehat{G}^\Gamma$ normalizing $\widehat{T}_0$ such that $g \in \widehat{B}_0 \cdot n\widehat{B}_0$ by [Bor79, Lemma 2]. Write uniquely $g = unvt$ with $u \in \widehat{U}_0 \cap n\widehat{U}_0^- n^{-1}$, $v \in \widehat{U}_0$, $t \in \widehat{T}_0$, where $\widehat{U}_0$ is the unipotent radical of $\widehat{B}_0$ and $\widehat{U}_0^-$ is the unipotent radical of the $\widehat{T}_0$-opposite of $\widehat{B}_0$. The relation $\sigma(g)=gt_\sigma$ with $t_\sigma \in \widehat{T}_0$ shows that $u,n,v \in \widehat{G}^\Gamma$. But $\mathrm{Ad}(unv)(\widehat{T}_0,\widehat{B}_0)=\mathrm{Ad}(g)(\widehat{T}_0,\widehat{B}_0)$. □

{ex:dualsl2}

Example 3.2.3. When $G=\mathrm{SL}_2$ we can take T_0 to be the diagonal matrices and B_0 the upper triangular matrices. We have the canonical isomorphism

$$\mathbb{G}_m \to T_0, \qquad x \mapsto \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}$$

which gives the isomorphisms $X = \mathbb{Z}$ and $Y = \mathbb{Z}$; namely $1 \in Y$ is the above isomorphism and $1 \in X$ its inverse. Then $R = \{2, -2\}$, $\Delta = \{2\}$, $R^\vee = \{1, -1\}$, $\Delta^\vee = \{1\}$.

Dually $\widehat{G} = \mathrm{PGL}_2(\mathbb{C})$ with $\widehat{T}_0$ again the diagonal torus, and $\widehat{B}_0$ again the upper triangular matrices. Here we have $\mathbb{Z} = Y = X^*(\widehat{T}_0)$ and $\mathbb{Z} = X = X_*(\widehat{T}_0)$ where now $1 \in X$ and $1 \in Y$ correspond to the isomorphism

$$\mathbb{G}_m \to \widehat{T}_0, \qquad x \mapsto \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}$$

and its inverse, respectively. The Γ-action is trivial, so ${}^L G = \widehat{G} \times \Gamma$.

{cns:te}

Construction 3.2.4. Let $T \subset G$ be a maximal torus. There is a canonical Γ-invariant $\widehat{G}$-conjugacy class of embeddings of complex algebraic groups $\widehat{T} \to \widehat{G}$. It is obtained as follows. Choose $g \in G(\bar{F})$ such that $gT_0g^{-1} = T$ and obtain an isomorphism $\mathrm{Ad}(g) : T_0 \to T$ defined over $\bar{F}$. Compose its dual with the canonical identification of the dual of T_0 with $\widehat{T}_0$. This gives an embedding $\widehat{T} \to \widehat{G}$, whose $\widehat{G}$ conjugacy class is independent of the choice of g, and in particular Γ-invariant. Note that a particular embedding $\widehat{T} \to \widehat{G}$ in this $\widehat{G}$-conjugacy class will usually not be Γ-invariant.

3.3 Endoscopic data

In the case of SL_2 we rewrote the elliptic regular part of the trace formula in terms of trace formulas for elliptic maximal tori. These are the endoscopic groups for SL_2. For a general group, the endoscopic groups are neither tori nor subgroups. The relationship is again one of "sub", but goes via the dual group.

There are various equivalent ways to package the necessary data. In [Kot84b, §7] they are introduced as "endoscopic data" and "endoscopic triples" and their equivalence is discussed. In [LS87, §1.2] the authors introduce a concept they call "endoscopic datum" that is optically different from that of [Kot84b, §7], so it has now become customary to call the former "endoscopic pair".

Definition 3.3.1. An *endoscopic triple* (H, s, η) consists of

1. a quasi-split connected reductive group H;
2. an embedding $\eta : \widehat{H} \to \widehat{G}$ of complex algebraic groups;
3. an element $s \in [Z(\widehat{H})/Z(\widehat{G})]^\Gamma$ (we have used η to identify $Z(\widehat{G})$ with a subgroup of $Z(\widehat{H})$);

subject to the conditions

1. η identifies $\widehat{H}$ with $\mathrm{Cent}(\eta(s), \widehat{G})^\circ$;
2. the $\widehat{G}$-conjugacy class of η is stable under the action of Γ that is defined by $\sigma\eta = \sigma_{\widehat{G}} \circ \eta \circ \sigma_{\widehat{H}}^{-1}$;

3. • If F is a local field, s lifts to $Z(\widehat{H})^\Gamma$
   • If F is a global field, s lifts to $Z(\widehat{H})^{\Gamma_v}$ for each place v.

Definition 3.3.2. An isomorphism of endoscopic triples $(H_1, s_1, \eta_1) \to (H_2, s_2, \eta_2)$ is an isomorphism of algebraic groups $f : H_1 \to H_2$ defined over F subject to the conditions

1. $\eta_1 \circ \widehat{f}$ and η_2 are $\widehat{G}$-conjugate.
2. The images of $\widehat{f}(s_2)$ and s_1 in $\pi_0([Z(\widehat{H}_2)/Z(\widehat{G})]^\Gamma)$ coincide.

Among all endoscopic triples the most essential ones are the elliptic.

Definition 3.3.3. An endoscopic triple (H, s, η) is called *elliptic* if $Z(\widehat{H})^{\Gamma,\circ} = Z(\widehat{G})^{\Gamma,\circ}$.

{exa:ellendosl2}

Example 3.3.4. For $G = \mathrm{SL}_2$ the elliptic endoscopic triples, up to isomorphism, are the following: (T, s, η), where T is an anisotropic torus of dimension 1, $\eta : \mathbb{C}^\times \to \widehat{G}$ is the identification of $\mathbb{C}^\times$ with the diagonal torus in $\widehat{G}$ as in Example 3.2.3, and $s = -1$. In addition, we have the trivial elliptic endoscopic triple $(G, 1, \mathrm{id})$. Finally, we have the non-elliptic endoscopic triple (T, s, η), where T is the split one-dimensional torus, η is as before, and s is any non-trivial element.

3.4 Admissible isomorphisms

In Fact 3.1.3 we saw that when two strongly regular semi-simple $\delta, \delta' \in G(F)$ are stably conjugate, there exists a unique isomorphism $\varphi_{\delta,\delta'} : T_\delta \to T_{\delta'}$ between their centralizers that sends δ to δ'. This is an example of an admissible isomorphism. We shall now extend this notion to the setting where (H, s, η) is an endoscopic datum, $\gamma \in H(F)$, and $\delta \in G(F)$.

Definition 3.4.1. Let $T^H \subset H$ and $T \subset G$ be maximal tori. An *admissible isomorphism* $T^H \to T$ is one whose dual is the composition of

1. An embedding $\widehat{T}^H \to \widehat{H}$ in the canonical $\widehat{H}$-conjugacy class of Construction 3.2.4;
2. The embedding $\eta : \widehat{H} \to \widehat{G}$;
3. The inverse of an embedding $\widehat{T} \to \widehat{G}$ in the canonical $\widehat{G}$-conjugacy class of Construction 3.2.4.

Definition 3.4.2. An *admissible embedding* $T^H \to G$ is the composition of an admissible isomorphism $T^H \to T$ with the inclusion $T \to G$.

Definition 3.4.3. Two semi-simple elements $\gamma \in H(F)$ and $\delta \in G(F)$ are called *related* if there exists an admissible isomorphism $\varphi : T^H \to T$ of tori such that $\varphi(\gamma) = \delta$.

Fact 3.4.4. *If δ is strongly regular then φ is unique if it exists and will be called $\varphi_{\gamma,\delta}$.*

Theorem 3.4.5. *[Kot82, §2] For every maximal torus $T^H \subset H$ there exists an admissible embedding $T^H \to G$ defined over F.*

3.5 Transfer factors

In the case of $G = \mathrm{SL}_2$, starting from a test function on f on $G(\mathbb{A})$ we produced a test function f^T on $T(\mathbb{A})$. There was an issue in proving that f^T is smooth at singular points. We had to decorate f^T_v with auxiliary terms in order to achieve this, and these auxiliary terms disappeared in the global product.

The issues for general G are significantly more severe, and involve decade-long work of Langlands, Shelstad, Kottwitz, Waldspurger, Hales, Arthur, and others, and culminate with the work of Laumon and Ngô.

We shall describe here the auxiliary local terms that ensure the smoothness of the transfer function. These are called *transfer factors*.

Let F be a local field.

Definition 3.5.1. A *Whittaker datum* $\mathfrak{w}$ for G is a $G(F)$-conjugacy class of pairs (B, ψ), where $B \subset G$ is a Borel subgroup defined over F and $\psi : B_u(F) \to \mathbb{C}^\times$ is a generic character, i.e. one whose restriction to each relative simple root subgroup is non-trivial.

Definition 3.5.2. An *extended endoscopic triple* $(H, s, {}^L\eta)$ consists of an endoscopic triple (H, s, η) and an extension of $\eta : \widehat{H} \to \widehat{G}$ to an embedding of L-groups ${}^L\eta : {}^LH \to {}^LG$.

Remark 3.5.3. An extension ${}^L\eta$ does not always exist. It does if G has simply connected derived subgroup, by a result of Langlands [Lan79]. We shall ignore this technicality, because it has a simple work-around (the concept of a z-pair of [KS99, §2]) that only complicates notation. We only remark that it is this technical issue that is responsible for the need of the more complicated notion of endoscopic datum introduced in [LS87].

Given $\mathfrak{e} = (H, s, {}^L\eta)$ and $\mathfrak{w}$ there is a function

$$\Delta : H_{\mathrm{sr}}(F) \times G_{\mathrm{sr}}(F) \to \mathbb{C}$$

called the Langlands-Shelstad transfer factor (G_{sr} denotes the set of strongly regular semi-simple elements in G). We will now review its construction following [LS87], but incorporating conventions from [KS]. Let $\gamma \in H_{\mathrm{sr}}(F)$ and $\delta \in G_{\mathrm{sr}}(F)$. If they are not related, then $\Delta(\gamma, \delta) = 0$, so we assume from now on that they are related. Recall this means that there exists an admissible isomorphism $\varphi_{\gamma,\delta} : T_\gamma \to T_\delta$ between the respective centralizers that maps γ to δ.

The complex number $\Delta(\gamma, \delta)$ is a product[1]

$$\epsilon \cdot \Delta_I^{-1} \cdot \Delta_{II} \cdot \Delta_{III_2} \cdot \Delta_{IV}.$$

[1]See Appendix for more information about normalization.

Some of the pieces in this product depend on auxiliary data, but the product does not. Readers familiar with [LS87] will note the absence of Δ_{III_1}. This is because in our construction we will arrange the auxiliary data so that it becomes trivial.

We begin by defining the auxiliary data. For this, let $R(T_\delta, G)$ be the absolute root system. It is a finite set with a Γ-action.

Definition 3.5.4. Let $\alpha \in R(T_\delta, G)$. Define $\Gamma_\alpha = \mathrm{Stab}(\alpha, \Gamma)$, $\Gamma_{\pm\alpha} = \mathrm{Stab}(\{\pm\alpha\}, \Gamma)$, $F_\alpha = \bar{F}^{\Gamma_\alpha}$, $F_{\pm\alpha} = \bar{F}^{\Gamma_{\pm\alpha}}$. Then $[F_\alpha : F_{\pm\alpha}] \leq 2$. Call α *symmetric* if $[F_\alpha : F_{\pm\alpha}] = 2$ and *asymmetric* otherwise.

Definition 3.5.5. A set of *a-data* for $R(T_\delta, G)$ is a set $\{a_\alpha \in \bar{F}^\times | \alpha \in R(T_\delta, G)\}$ subject to

1. $a_{\sigma\alpha} = \sigma(a_\alpha)$ for all $\sigma \in \Gamma$;
2. $a_{-\alpha} = -a_\alpha$.

Definition 3.5.6. A set of *χ-data* for $R(T_\delta, G)$ is a set $\{\chi_\alpha | \alpha \in R(T_\delta, G)\}$ subject to

1. $\chi_\alpha : F_\alpha^\times \to \mathbb{C}^\times$ is a continuous character;
2. $\chi_{\sigma\alpha} = \chi_\alpha \circ \sigma^{-1}$;
3. $\chi_{-\alpha} = \chi_\alpha^{-1}$;
4. If α is symmetric, then $\chi_\alpha|_{F_{\pm\alpha}^\times}$ is the quadratic character of $F_{\pm\alpha}^\times \to \{\pm 1\}$ associated to the quadratic field extension $F_\alpha / F_{\pm\alpha}$ by local class field theory.

Lemma 3.5.7. *Sets of a-data and χ-data exist.*

Proof. One can choose arbitrarily an element α in a given Γ-orbit in $R(T_\delta, G)$ and choose $a_\alpha \in F_\alpha^\times$ such that, when α is symmetric, $\mathrm{tr}_{F_\alpha/F_{\pm\alpha}}(a_\alpha) = 0$. Then $a_{\sigma\alpha} = \sigma(a_\alpha)$ is well-defined for any $\sigma \in \Gamma$. Repeating this for each Γ-orbit one obtains a set of a-data. A similar procedure produces a set of χ-data. $\square$

We fix such choices. We further fix a pinning $(T_0, B_0, \{X_\alpha\})$ of G and a nontrivial additive character $\Lambda : F \to \mathbb{C}^\times$. The pinning gives the second map in the following chain of algebraic groups:

$$B_u \to (B_u/[B_u, B_u]) \to \prod_{\alpha \in \Delta} \mathbb{G}_a \to \mathbb{G}_a,$$

for indeed the quotient $B_u/[B_u, B_u]$ is the direct product of the absolute simple root subgroups, and the pinning identifies each of them with $\mathbb{G}_a$. Composing this homomorphism with Λ we obtain a generic character $B_u(F) \to \mathbb{C}^\times$. We require of our choices that this generic character belongs to the fixed Whittaker datum $\mathfrak{w}$.

Next we recall the *Tits section*: The choice of pinning specifies a (usually non-multiplicative) map $\Omega(T_0) \to N(T_0, G_{\mathrm{sc}})$ from the absolute Weyl group to the normalizer of the maximal torus: it sends a simple reflection s_α to the element $n_\alpha = \exp(X_\alpha)\exp(-X_{-\alpha})\exp(X_\alpha)$, where $X_{-\alpha}$ is the unique element in the $-\alpha$ eigenspace of the Lie algebra such that $[X_\alpha, X_{-\alpha}]$ is the coroot H_α. A general $w \in \Omega(T_0)$ is written as a shortest product of simple reflection $s_1 \dots s_k$ and then sent to $\dot{w} = n_1 \dots n_k$; the element $\dot{w}$ turns out to be independent of the choice of shortest expression for w.

Then we can define the pieces of the transfer factor.

1. $\epsilon = \epsilon(\frac{1}{2}, X_*(T_0)_{\mathbb{C}} - X_*(T_0^H)_{\mathbb{C}}, \Lambda)$, where T_0^H is a maximally split maximal torus in H.

2. $\Delta_I = \langle \lambda, \widehat{\varphi}_{\gamma,\delta}^{-1}(s)\rangle$. Here $\lambda \in H^1(\Gamma, T_{\delta,\mathrm{sc}})$ is the Langlands-Shelstad splitting invariant. It is constructed as follows. Choose $g \in G_{\mathrm{sc}}(\bar{F})$ such that $gT_0g^{-1} = T_\delta$. Then $g^{-1}\sigma(g)$ is an element of $Z^1(F, N(T_0, G_{\mathrm{sc}}))$. Let w_σ be its image in $Z^1(\Gamma, \Omega(T_0))$, where $\Omega(T_0)$ is the Weyl group. Let $\dot{w}_\sigma \in N(T_0, G_{\mathrm{sc}})$ be the Tits lift of w_σ associated to the fixed pinning. Since the Tits section is in general not multiplicative, $\sigma \mapsto \dot{w}_\sigma$ is generally not a 1-cocycle. To deal with this, Langlands and Shelstad define

$$x_\sigma := \prod_{\substack{\alpha>0 \\ \sigma^{-1}\alpha<0}} \alpha^\vee(a_\alpha) \in T_{\delta,\mathrm{sc}}(\bar{F}),$$

   where $\alpha > 0$ means $\alpha \in R(T_\delta, gB_0g^{-1})$. Then $g^{-1}x_\sigma g \cdot \dot{w}_\sigma$ is another element of $Z^1(F, N(T_0, G_{\mathrm{sc}}))$, whose image in $Z^1(F, \Omega(T_0))$ coincides with that of $g^{-1}\sigma(g)$. Therefore the product $(g^{-1}x_\sigma g \cdot \dot{w}_\sigma) \cdot (g^{-1}\sigma(g))^{-1}$ takes values in $T_{0,\mathrm{sc}}(\bar{F})$ and moreover its image under $\mathrm{Ad}(g) : T_{0,\mathrm{sc}} \to T_{\delta,\mathrm{sc}}$, i.e. the element $x_\sigma g\dot{w}_\sigma\sigma(g)^{-1}$, belongs to $Z^1(F, T_{\delta,\mathrm{sc}})$. Its class is independent of g and will be denoted by λ. On the other hand, $\widehat{\varphi}_{\gamma,\delta}^{-1}(s) \in [\widehat{T}_\delta/Z(\widehat{G})]^\Gamma$. The pairing $\langle -, - \rangle$ is given by Tate-Nakayama duality, see Remark 3.5.8.

3. $$\Delta_{II} = \prod_\alpha \chi_\alpha\left(\frac{\alpha(\delta) - 1}{a_\alpha}\right),$$

   where the product is taken over the Γ-orbits in $R(T_\delta, G) \setminus R(T_\gamma, H)$. Here we are using the isomorphism $\varphi_{\gamma,\delta} : T_\gamma \to T_\delta$ to identify $R(T_\gamma, H)$ with a subset of $R(T_\delta, G)$.

4. $\Delta_{III} = \theta(\delta)$, where $\theta : T_\delta(F) \to \mathbb{C}^\times$ is a character constructed as follows. The choice of χ-data for $R(T_\delta, G)$ leads to a $\widehat{G}$-conjugacy class of L-embeddings ${}^LT_\delta \to {}^LG$; it can be constructed explicitly, but the construction is rather technical, and we refer the reader to [LS87, §2.6]. We can use $\varphi_{\gamma,\delta}$ to transport this choice to $R(T_\gamma, H)$ obtaining an $\widehat{H}$-conjugacy

class of L-embeddings ${}^LT_\gamma \to {}^LH$. These are the horizontal arrows in the diagram

$$\begin{array}{ccc} {}^LT_\gamma & \longrightarrow & {}^LH \\ \downarrow & & \downarrow \\ {}^LT_\delta & \longrightarrow & {}^LG \end{array}$$

in which the left vertical arrow comes from $\varphi_{\gamma,\delta}$ and the right from ${}^L\eta$. In general this diagram fails to commute and the failure is measured by an element of $H^1(W_F, \widehat{T}_\delta)$, which by the local correspondence for tori gives a character $\theta : T_\delta(F) \to \mathbb{C}^\times$.

5. $\Delta_{IV} = \prod_\alpha |\alpha(\delta) - 1|^{\frac{1}{2}}$, where the product is over $R(T_\delta, G) \smallsetminus R(T_\gamma, H)$.

{rem:tnd}

Remark 3.5.8. We recall Langlands' reinterpretation of Tate-Nakayama duality. The original statement due to Tate (cf. [Sha72, Theorem 45]) is that for any torus T defined over the local field F the pairing

$$H^1(\Gamma, T) \otimes H^1(\Gamma, X^*(T)) \to H^2(\Gamma, \mathbb{G}_m),$$

when composed with the invariant map $H^2(\Gamma, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$ of local class field theory, becomes perfect. Tensoring with $X^*(T)$ the exponential sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 1$$

and using the defining property $X^*(T) = X_*(\widehat{T})$ of the dual torus $\widehat{T}$ one obtains the exact sequence

$$0 \to X_*(\widehat{T}) \to \mathrm{Lie}(\widehat{T}) \to \widehat{T} \to 1.$$

Taking Γ-invariants this leads to the isomorphism

$$\pi_0(\widehat{T}^\Gamma) = \mathrm{cok}(\mathrm{Lie}(\widehat{T})^\Gamma \to \widehat{T}^\Gamma) \to H^1(\Gamma, X^*(T)).$$

We use again the exponential map to obtain the embedding $\mathbb{Q}/\mathbb{Z} \to \mathbb{C}^\times$ and obtain the perfect pairing

$$H^1(\Gamma, T) \otimes \pi_0(\widehat{T}^\Gamma) \to \mathbb{C}^\times. \qquad (3.5.1)$$ {eq:tnd}

The following is a fundamental property of the transfer factor, which we may call its *κ-behavior*:

{pro:tfinv}

Proposition 3.5.9. *If δ, δ' are stably conjugate, then*

$$\Delta(\gamma, \delta') = \Delta(\gamma, \delta) \cdot \langle inv(\delta, \delta'), \widehat{\varphi}_{\gamma,\delta}^{-1}(s) \rangle,$$

where $inv(\delta, \delta') \in H^1(\Gamma, T_\delta)$ is as in Fact 3.1.3.

We need to explain the second factor on the right. It will again be given by the Tate-Nakayama pairing (3.5.1), but there seems to be a mismatch: $\widehat{\varphi}_{\gamma,\delta}^{-1}(s)$ lies in $[\widehat{T}_\delta/Z(\widehat{G})]^\Gamma$ and $\widehat{T}_\delta/Z(\widehat{G})$ is the torus dual to $T_{\delta,\mathrm{sc}}$. On the other hand, $\mathrm{inv}(\delta,\delta') \in H^1(\Gamma, T_\delta)$. What one needs to note is that this latter element lies in the image of the natural map

$$H^1(\Gamma, T_{\delta,\mathrm{sc}}) \to H^1(\Gamma, T_\delta).$$

Indeed, it was represented by a cocycle $\sigma \mapsto g^{-1}\sigma(g)$ for any $g \in G(\bar{F})$ such that $\mathrm{Ad}(g)\delta = \delta'$, but such a g already exists in $G_{\mathrm{sc}}(\bar{F})$. Note that this map need not be injective, and that in general there is no canonical lift of $\mathrm{inv}(\delta,\delta')$ to $H^1(\Gamma, T_{\delta,\mathrm{sc}})$. But this is where we use the property of the endoscopic element $s \in [Z(\widehat{H})/Z(\widehat{G})]^\Gamma$ that it is liftable to $Z(\widehat{H})^\Gamma$. This implies that $\widehat{\varphi}_{\gamma,\delta}^{-1}(s) \in [\widehat{T}/Z(\widehat{G})]^\Gamma$ is liftable to $\widehat{T}^\Gamma$. Thus we want to pair the element $\mathrm{inv}(\delta,\delta') \in \mathrm{im}(H^1(\Gamma, T_{\delta,\mathrm{sc}}) \to H^1(\Gamma, T_\delta))$ with the element $s \in \mathrm{im}(\widehat{T}_\delta^\Gamma \to [\widehat{T}_\delta/Z(\widehat{G})]^\Gamma)$.

We find ourselves in the following abstract situation: We have a homomorphism $f : A \to B$ of finite abelian groups and its Pontryagin dual $f^* : A^* \to B^*$. Then the images of f and f^* are in duality, namely via the canonical isomorphisms

$$\mathrm{im}(f)^* = (\mathrm{cok}(\ker(f)))^* = \ker(\mathrm{cok}(f^*)) = \mathrm{im}(f^*).$$

3.6 Local transfer

We continue to assume that F is a local field.

{dfn:matching}

Definition 3.6.1. The functions $f \in \mathcal{C}_c^\infty(G(F))$ and $f^H \in \mathcal{C}_c^\infty(H(F))$ are said to be Δ-*matching* if for all strongly G-regular semi-simple elements $\gamma \in H(F)$ we have

$$SO_\gamma(f^H) = \sum_\delta \Delta(\gamma,\delta) O_\delta(f),$$

where the sum runs over the set of (rational) conjugacy classes of strongly regular semi-simple elements.

Remark 3.6.2. A strongly G-regular element γ is one that is related to a strongly regular $\delta \in G(F)$. For a fixed γ the set of such δ forms a single stable class. Therefore, upon fixing one δ_0 the above sum can be rewritten using Proposition 3.5.9 as

$$\Delta(\gamma,\delta_0) \sum \kappa(\mathrm{inv}(\delta_0,\delta)) O_\delta(f),$$

where the sum now runs over the set of rational classes in the stable class of δ_0 and $\kappa : H^1(\Gamma, T_{\delta_0}) \to \mathbb{C}^\times$ is the character given by $\widehat{\varphi}_{\gamma,\delta_0}^{-1}(s)$ and Tate-Nakayama duality. Note that this sum is finite – it runs over $\ker(H^1(\Gamma, T) \to H^1(\Gamma, G))$ and $H^1(\Gamma, T)$ is a finite abelian group, being dual to the finite abelian group $\pi_0(\widehat{T}^\Gamma)$.

{exa:sl2-et}

Example 3.6.3. In the example of $G = \mathrm{SL}_2$ we have $H = T$ an elliptic maximal torus and we can take $\delta_0 = \gamma$. We have $T_\delta = T$ and moreover $H^1(\Gamma, T_\delta) = F^\times / N_{E/F}(E^\times) = \mathbb{Z}/2\mathbb{Z}$, so the above equation becomes

$$f^T(\gamma) = \Delta(\gamma, \delta_0)[O_{\delta_0}(f) - O_{\delta_1}(f)].$$

This gave a construction of f^T on the G-regular elements of H and we only needed to show that it extends smoothly to the singular elements.

This is far from what happens in general. The following is one of the hardest theorems proven in the subject so far.

Theorem 3.6.4. {thm:loctrans}

1. *For every f there exists a matching f^H.*
2. *If G and $(H, s, {}^L\eta)$ are unramified and f is the characteristic function of a hyperspecial maximal compact subgroup, then f^H can be taken as the characteristic function of a hyperspecial maximal compact subgroup.*

The proof of this theorem spans many papers. We only give a selection: [LS87], [LS90], [Wal95], [Wal97], [Kot99], [Wal00], [Wal09], [Ngô10]. The first part of the theorem is known as *endoscopic transfer of functions*, and the second as *the Fundamental Lemma*. Initially it was believed that the two parts, while conceptually related, are independent statements, but Waldspurger proved that the second statement actually implies the first, using a global argument based on the trace formula. For the second statement, we recall that G being unramified means that G is quasi-split and splits over an unramified extension of the non-archimedean local field F. In this case the topological group $G(F)$ has a very special kind of a compact open subgroup, called *hyperspecial*, which arises as $\mathscr{G}(O_F)$, where $\mathscr{G}$ is a reductive group scheme defined over the ring of integers O_F with generic fiber G. This is part of Bruhat-Tits theory, cf. [Tit79] for a summary and [BT72], [BT84] for the main development. The second statement was reduced from the case of groups to the case of Lie algebras via a Harish-Chandra descent argument. The Lie algebra version was proved by Ngô for local fields of positive characteristic, and then transfered to characteristic zero.

{rem:loctrans}

Remark 3.6.5. The first statement of Theorem 3.6.4 is valid even without the assumption that G is quasi-split. In that setting, the transfer factor Δ can still be defined, but it is ambiguous up to multiplication by a complex scalar of absolute value 1. The function f^H of course depends on the choice of Δ and hence inherits this ambiguity. This is the reason for including the notation Δ in Definition 3.6.1. We will discuss this issue in the next section.

We can now state the refined local Langlands correspondence, Conjecture 3.6.7 below. It is the spectral analog of Theorem 3.6.4. We will see in Section 5 that Theorem 3.6.4 is used in the stabilization of the geometric side of the trace formula. Once the geometric side is stabilized, the spectral side is automatically stabilized as well, but its meaning as a spectral distribution is not clear any more. Conjecture 3.6.7 is used to interpret the meaning of the stabilized spectral side.

Recall that a *Langlands parameter* is a continuous homomorphism $L_F \to {}^L G$, where $L_F = W_F$ for $F/\mathbb{R}$ and $L_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ for $F/\mathbb{Q}_p$ that satisfies the following conditions: the image of every element of W_F is semi-simple; its restriction to $\mathrm{SL}_2(\mathbb{C})$ is algebraic; it commutes with the obvious maps to Γ on its source and target. The parameter is called *tempered* if the image of W_F has relatively compact projection to $\widehat{G}$.

{dfn:stabqs}

Definition 3.6.6. Let $\varphi : L_F \to {}^L G$ be a Langlands parameter.

1. Let $S_\varphi = \mathrm{Cent}(\varphi, \widehat{G})$, $\bar{S}_\varphi = S_\varphi / Z(\widehat{G})^\Gamma$, $\mathcal{S}_\varphi = \pi_0(\bar{S}_\varphi)$.
2. For any semi-simple $s \in S_\varphi$ the virtual character
$$\Theta^s_\varphi := \sum_{\pi \in \Pi_\varphi(G)} \mathrm{tr}\rho_\pi(s) \cdot \Theta_\pi$$
is called the *s-stable character* associated to φ, where $\Pi_\varphi(G)$ is as in Conjecture 3.6.7 below.
3. In the special case $s = 1$ we call
$$S\Theta_\varphi = \Theta^1_\varphi = \sum_{\pi \in \Pi_\varphi(G)} \dim \rho_\pi \cdot \Theta_\pi$$
the *stable character* associated to φ.

Conjecture 3.6.7. {cnj:lciqs}

1. *Let $\varphi : L_F \to {}^L G$ be a Langlands parameter. There exists an associated finite set $\Pi_\varphi(G)$ of irreducible representations of $G(F)$ equipped with a map $\rho : \Pi_\varphi(G) \to Irr(\mathcal{S}_\varphi)$, which is injective for $F = \mathbb{R}$ and bijective otherwise.*
2. *When φ is tempered there is a unique $\mathfrak{w}$-generic constituent of $\Pi_\varphi(G)$, and it is mapped to the trivial representation of $\mathcal{S}_\varphi$.*
3. *When φ is tempered the distribution $S\Theta_\varphi$ is stable.*
4. *Let $\varphi^H : L_F \to {}^L H$ be a tempered Langlands parameter and let $\varphi = {}^L\eta \circ \varphi^H$. If f and f^H are Δ-matching, then*
$$\Theta^s_\varphi(f) = S\Theta_{\varphi^H}(f^H).$$

Remark 3.6.8. Various instances of Conjecture 3.6.7 have been proved (see e.g. [She82], [She08] for a complete treatment of $F = \mathbb{R}$, [Art13] for symplectic and orthogonal groups, and [Kal19] for toral supercuspidal representations of fairly general p-adic groups), but the full statement remains open. Most parts of this conjecture were proposed, in slightly weaker form, by Langlands in [Lan83]. The second statement is due to Shahidi [Sha90] and is known as the *generic packet conjecture*. The fourth statement is known as the *endoscopic character identity*. It expresses the s-stable character associated to φ in terms of the stable character associated to a factorization of φ through the endoscopic group H. It is this statement that drives the spectral interpretation of the stable trace formula, namely the stable multiplicity formula (6.5.2) that we will discuss in the setting of SL_2. Note that when $s = 1$, then $H = G$ and the statement is trivial.

Example 3.6.9. We return to the case of $G = \mathrm{SL}_2$. In Example 3.6.3 we had the identity {exa:sl2-lci}

$$f^T(\gamma) = \Delta(\gamma, \delta_0)[O_{\delta_0}(f) - O_{\delta_1}(f)]$$

and we now want to relate it to (2.5.1). For this we need to be careful when identifying the maximal torus T and the endoscopic group H.

Fix a quadratic extension E/F. Let T be the unique one-dimensional anisotropic torus defined over F and split over E. We have $T(F) = E^1$. We have the elliptic endoscopic datum (T, s, η) as in Example 3.3.4. In order to obtain a transfer factor we need to extend $\eta : \widehat{T} \to \widehat{G}$ to an L-embedding ${}^L\eta : {}^LT \to {}^LG$. Here ${}^LT = \mathbb{C}^\times \rtimes \Gamma_{E/F}$, with $\Gamma_{E/F}$ acting by inversion on $\mathbb{C}^\times$, and ${}^LG = \mathrm{PGL}_2(\mathbb{C}) \times \Gamma$. We can forget the direct factor Γ in ${}^LG$, since it acts trivially on $\widehat{G}$. The L-embedding ${}^L\eta$ we will use sends the non-trivial element $\sigma \in \Gamma_{E/F}$ to the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We now need to realize T as a maximal torus of G. Fix $\eta \in E^\times \smallsetminus F^\times$ with $\eta^2 \in F^\times$. We have the embedding $T \to G$ sending $c = a + \eta b \in E^1$ to (2.2.1).

With this we can now compute $\Delta(c, c)$. The group G comes with the standard pinning consisting of the diagonal torus T_0, the upper triangular Borel subgroup B_0, and the positive root vector

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since $X^*(T_0) = \mathbb{Z}$ with trivial $\Gamma_{E/F}$-action and $X^*(T) = \mathbb{Z}$ with the unique non-trivial action we see that $\epsilon(X^*(T_0)_\mathbb{C} - X^*(T)_\mathbb{C}, \psi) = \lambda(E/F, \psi)^{-1}$, which in turn equals $\lambda(E/F, \psi)\kappa_{E/F}(-1)$, where $\kappa_{E/F} : F^\times/N_{E/F}(E^\times) \to \{\pm 1\}$ is the non-trivial character. We furthermore have $\Delta_{IV} = |c - \bar{c}|_F$.

To compute the other contributions we need to fix a-data and χ-data. Consider the matrix

$$g = \begin{bmatrix} 1 & -(2\sqrt{d})^{-1} \\ \sqrt{d} & 1/2 \end{bmatrix} \in \mathrm{SL}_2(E).$$

This matrix conjugates T_0 to the image of our chosen embedding of T into G and hence identifies $R(T, G)$ with $\{2, -2\} \subset \mathbb{Z} = X^*(T_0)$. Take $2\eta = \eta - \bar{\eta}$ as a-data for $2 \in R(T, G)$, and choose an arbitrary extension $\chi : E^\times \to \mathbb{C}^\times$ of $\kappa_{E/F}$ as χ-data for 2. The Langlands-Shelstad splitting invariant relative to this data is the element of $Z^1(\Gamma_{E/F}, T(E))$ that sends $\sigma \in \Gamma_{E/F}$ to the image under $\mathrm{Ad}(g)$ of

$$\begin{bmatrix}\eta & 0\\ 0 & \eta^{-1}\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}(g^{-1}\sigma(g))^{-1} = \begin{bmatrix}4\eta^2 & 0\\ 0 & (4\eta^2)^{-1}\end{bmatrix}.$$

That is, the splitting invariant is the element $4\eta^2 \in F^\times/N_{E/F}(E^\times) = H^1(\Gamma, T)$. This element is equal to the element $-1 \in F^\times/N_{E/F}(E^\times)$. Therefore $\Delta_I = \kappa_{E/F}(-1)$.

For Δ_{II} we obtain

$$\chi\left(\frac{c/\bar{c}-1}{2\eta}\right) = \kappa_{E/F}\left(\frac{c-\bar{c}}{2\eta}\right)\chi(\bar{c})^{-1} = \kappa_{E/F}\left(\frac{c-\bar{c}}{2\eta}\right)\chi(c).$$

For Δ_{III_2} we compute the restriction to $E^\times \subset W_{E/F}$ of the L-embedding ${}^L\xi_\chi : {}^LT \to {}^LG$ obtained from the chosen χ-data to send $e \in E^\times$ to the diagonal element with entries $\chi(e)$ and 1. On the other hand we have ${}^L\eta(e) = 1$. Therefore the element $a \in Z^1(W_{E/F}, \widehat{T})$ with ${}^L\xi_\chi \cdot a = {}^L\eta$ corresponds to the character χ^{-1}. Putting everything together we obtain

$$\Delta(c, c) = \lambda(E/F, \psi) \cdot \kappa_{E/F}\left(\frac{c-\bar{c}}{2\eta}\right)|c-\bar{c}|_F$$

and recover (2.5.1) up to the harmless factor of 2 in the denominator. We have used above the relative Weil group $W_{E/F}$, which is a canonical extension

$$1 \to E^\times \to W_{E/F} \to \Gamma_{E/F} \to 1.$$

At the same time any character $\theta : E^1 \to \mathbb{C}^\times$ has a corresponding Langlands parameter $\varphi_T : W_{E/F} \to {}^LT$. Indeed the restriction map $H^1(W_{E/F}, \widehat{T}) \to \mathrm{Hom}(E^\times, \widehat{T})$ is injective and φ_T corresponds to the homomorphism $E^\times \to \mathbb{C}^\times$ sending e to $\theta(e/\bar{e})$. The L-parameter $\varphi = {}^L\eta \circ \varphi_T$ is a tempered parameter for G, in fact a discrete series parameter unless $\theta = 1$. There is a corresponding L-packet Π_φ of tempered representations of $G(F)$. Unless θ is of order 2, it consists of two members $\pi^+(\theta)$ and $\pi^-(\theta)$, whose labeling depends on the choice of ψ. The local character identity conjectured above specializes to (2.5.2). For more details on this example we refer to [She79] and [LL79].

4 Endoscopy for general reductive groups

{sec:endogen}

We shall now extend our discussion of endoscopy from the case of quasi-split groups to the case of general groups.

4.1 Inner forms

Recall from Definition 3.1.5 that given algebraic groups G and H defined over F an isomorphism $\xi : G \to H$ defined over $\bar{F}$ is called an inner twist if the automorphism $f^{-1}\sigma(f)$ of G is inner for all $\sigma \in \Gamma$. One says that G and H are inner forms of each other. The relation of being inner forms is an equivalence relation, and the equivalence classes are called inner classes. It was the fundamental idea of Adams-Barbasch-Vogan [ABV92], [Vog93], that entire inner classes should be treated simultaneously for the refined local Langlands correspondence.

Thus, let G be a connected reductive group defined over F. Rather than treating G alone, we should simultaneously treat all inner forms of G, even if we are only interested in G for a particular application.

Example 4.1.1. Consider the real reductive group $U(p,q)$ – the unitary group of signature p,q and dimension $n = p+q$. Recall that the refined local Langlands correspondence predicts an injection

$$\Pi_\varphi \to \mathrm{Irr}(\mathcal{S}_\varphi).$$

How far is it from being a bijection? When φ is discrete it turns out that $|\Pi_\varphi| = \binom{p}{q}$, while $\mathcal{S}_\varphi = \bar{S}_\varphi$ is an abelian group of order 2^{n-1}. So the orders seem off. However, one can observe that S_φ is of order $2^n = \sum_{p,q}\binom{p}{q}$. This suggests the relation

$$\mathrm{Irr}(S_\varphi) = \bigcup_{p+q=n} \Pi_\varphi(U(p,q)).$$

We note that the groups $U(p,q)$, $p+q=n$ constitute an inner class. For more details on this example, see [Ada11, §9].

What does it mean to treat *all* inner forms? That is, when do we consider two inner forms truly distinct? For example, $\mathrm{id} : G \to G$ is trivially an inner twist, but we don't want to consider its source and target as different inner forms. At the same time, $U(p,q)$ and $U(q,p)$ are isomorphic as reductive groups over $\mathbb{R}$, but the above example seems to want us to consider them as distinct. So we need to introduce an appropriate equivalence relation.

More precisely, we need to explain what an isomorphism is between two inner forms G_1 and G_2 of a connected reductive group G, and then work with isomorphism classes of inner forms. Since we are interested in invariant harmonic analysis, our notion of isomorphism has to satisfy the following criterion:

{cnd:auto}

Condition 4.1.2. *An automorphism of the inner form G' preserves each conjugacy class of $G'(F)$ and each representation of $G'(F)$.*

The most naive notion is simply isomorphism of algebraic groups. Fix G and consider an inner form G'. That is, there exists an inner twist $\xi : G \to G'$. Then $\sigma \mapsto \xi^{-1}\sigma(x)$ is an element of $Z^1(F, \mathrm{Inn}(G))$, where $\mathrm{Inn}(G)$ is the group of inner automorphisms of G; this group is the same as the quotient G_{ad} of G by its center. The class in $H^1(\Gamma, \mathrm{Inn}(G))$ of that element *does* depend on ξ. Its image in $H^1(\Gamma, \mathrm{Aut}(G))$ is independent of ξ, it only depends on G'.

Exercise 4.1.3.

1. *The assignment $G' \mapsto [\xi^{-1}\sigma(x)]$ is a bijection between the set of isomorphism classes of connected reductive groups G' that are inner forms of G and the image of $H^1(\Gamma, Inn(G)) \to H^1(\Gamma, Aut(G))$.*

2. *The group of automorphisms of G' is $H^0(F, Aut(G')) = Aut(G')(F)$.*

However, this notion does not meet Condition 4.1.2. If we apply it to the trivial inner twist $\mathrm{id} : G \to G$ of $G = \mathrm{GL}_n$, the outer automorphism of $G = \mathrm{GL}_n$ given by transpose inverse sends each representation to its contragredient, and swaps different conjugacy classes.

The next definition takes into account not just G' but also ξ.

Definition 4.1.4. An *isomorphism of inner twists* $\xi_1 : G \to G_1$ and $\xi_2 : G \to G_2$ is an isomorphism $f : G_1 \to G_2$ of algebraic groups defined over F for which $\xi_2^{-1} \circ f \circ \xi_1$ is an inner automorphism of G.

Exercise 4.1.5.

1. *The assignment $(\xi, G') \mapsto [\xi^{-1}\sigma(x)]$ is a bijection between the set of isomorphism classes of inner twists of G and $H^1(\Gamma, Inn(G))$.*

2. *The group of automorphisms of (ξ, G') is $H^0(F, Inn(G')) = G'_{ad}(F)$, acting on $G'(F)$ by conjugation.*

Unfortunately, it turns out that this notion of isomorphism is still not good enough.

Example 4.1.6. Consider the trivial inner twist $\mathrm{id} : G \to G$ of $G = \mathrm{SL}_2$ over $F = \mathbb{R}$. Then $G_{\mathrm{ad}} = \mathrm{PGL}_2$. Consider the matrices

$$g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then $g \in \mathrm{PGL}_2(\mathbb{R})$ conjugates x to y, but x and y but are not conjugate in $\mathrm{SL}_2(\mathbb{R})$. Furthermore, conjugation by g swaps the holomorphic and antiholomorphic discrete series of a given weight.

The following partial solution to this problem was proposed by Vogan in [Vog93].

Definition 4.1.7. A *pure inner twist* is a pair (ξ, z) of an inner twist $\xi : G \to G'$ and $z \in Z^1(F, G)$ such that $\xi^{-1}\sigma(\xi) = \mathrm{Ad}(z_\sigma)$. An *isomorphism of pure inner twists* $(\xi_1, z_1) : G \to G_1$ and $(\xi_2, z_2) : G \to G_2$ is a pair (f, g) where $f : G_1 \to G_2$ is an isomorphism over F and $g \in G(\bar{F})$ is subject to $\xi_2^{-1} \circ f \circ \xi_1 = \mathrm{Ad}(g)$ and $z_2(\sigma) = g z_1(\sigma) \sigma(g^{-1})$.

Exercise 4.1.8.

1. *The assignment $(\xi, z, G') \mapsto [z]$ is a bijection between the set of isomorphism classes of pure inner twists and $H^1(\Gamma, G)$.*

2. *The group of automorphisms of* (ξ, z, G') *is* $H^0(F, G') = G'(F)$*, acting on* $G'(F)$ *by conjugation.*

Finally this enriched notion of inner twist has the right automorphism group. It also has a very natural interpretation for classical groups – an isomorphism of pure inner twists of unitary groups is the same as an isomorphism of the underlying hermitian spaces, and the analogous statement holds for orthogonal groups. Unfortunately, pure inner twists have a major disadvantage: since the map $H^1(\Gamma, G) \to H^1(\Gamma, G_{\mathrm{ad}})$ is rarely surjective, there will be inner twists that cannot be enriched to pure inner twists.

Example 4.1.9. The group $G = \mathrm{SL}_2$ satisfies $H^1(\Gamma, G) = \{1\}$ while $H^1(\Gamma, G_{\mathrm{ad}}) = \mathbb{Z}/2\mathbb{Z}$. Therefore the unique non-trivial inner form of G, i.e. the unit quaternion group, cannot be made into a pure inner twist.

More generally, a theorem due to Kneser ([PR94, Theorem 6.4]) states that $H^1(\Gamma, G) = 1$ for any simply connected group when F is non-archimedean, while $H^1(\Gamma, G_{\mathrm{ad}})$ is very often non-trivial.

Remark 4.1.10. While in these lectures we have introduced the dual group only for a quasi-split group, one can introduce it in general. It is then easy to see that the datum of an inner twist $\xi : G \to G'$ induces a Γ-invariant isomorphism of dual groups $\widehat{G} \to \widehat{G}'$, hence also an isomorphism of L-groups ${}^L G \to {}^L G'$. We will use this isomorphism to define the dual group, and the L-group, of a non-quasi-split group G' equipped with an inner twist $\xi : G \to G'$ with G quasi-split.

4.2 Pure inner twists

Despite the serious drawback of pure inner twists we shall describe how endoscopy works in that setting. This will shed light on what we need to do to overcome their limitation, and also introduce in a more familiar setting the general framework of endoscopy for non-quasi-split groups.

Definition 4.2.1.

1. An *element of a pure inner twist* of G is a tuple (G', ξ, z, γ), where $(\xi, z) : G \to G'$ is a pure inner twist and $\gamma \in G'(F)$.

2. Two elements $(G_1, \xi_1, z_1, \gamma_1)$ and $(G_2, \xi_2, z_2, \gamma_2)$ are called *rationally conjugate* if there exists an isomorphism $(g, f) : (\xi_1, z_1) \to (\xi_2, z_2)$ such that $f(\gamma_1) = \gamma_2$.

3. A *representation of a pure inner twist* of G is a tuple (G', ξ, z, π), where $(\xi, z) : G \to G'$ is a pure inner twist and π is a representation of $G'(F)$.

4. Two representations (G_1, ξ_1, z_1, π_1) and (G_2, ξ_2, z_2, π_2) are called *equivalent* if there exists an isomorphism $(f, g) : (\xi_1, z_1) \to (\xi_2, z_2)$ such that $\pi_1 = \pi_2 \circ f$.

Exercise 4.2.2. *Two elements* (ξ, z, G', γ_1) *and* (ξ, z, G', γ_2) *are conjugate in the sense of the above definition if and only if* γ_1, γ_2 *are* $G'(F)$*-conjugate. Two representations* (ξ, z, G', π_1) *and* (ξ, z, G', π_2) *are equivalent if and only if* π_1 *and* π_2 *are equivalent in the usual sense as representations of* $G'(F)$*.*

We can now develop the notions of local endoscopy. First we need the concept of stable conjugacy and the cohomological classification of rational classes inside a stable class.

Definition 4.2.3.

1. Two strongly regular semi-simple $(G_1, \xi_1, z_1, \gamma_1)$ and $(G_2, \xi_2, z_2, \gamma_2)$ are called *stably conjugate* if $\xi_1^{-1}(\gamma_1)$ and $\xi_2^{-1}(\gamma_2)$ are $G(\bar{F})$-conjugate.

2. Let $\gamma \in G(F)$ be strongly regular semi-simple with centralizer T. If (G', ξ, z, γ') is stably conjugate to $(G, \mathrm{id}, 1, \gamma)$, choose $g \in G(\bar{F})$ such that $\gamma' = \xi(g\gamma g^{-1})$ and let $\mathrm{inv}(\gamma, (G', \xi, z, \gamma'))$ be the class of $g^{-1}z_\sigma\sigma(g)$ in $H^1(\Gamma, T)$. We will also write $\mathrm{inv}(\gamma, \gamma')$ if there is no danger of confusion.

Exercise 4.2.4. *The map* $(G', \xi, z, \gamma') \mapsto$ *inv*$(\gamma, (G', \xi, z, \gamma'))$ *is a bijection between the set of rational classes inside of the stable class of* $(G, id, 1, \gamma)$ *and the finite group* $H^1(\Gamma, T)$*.*

Remark 4.2.5. The difference between the above statement and Fact 3.1.3 is that we are using all of $H^1(\Gamma, T)$ instead of just $\ker(H^1(\Gamma, T) \to H^1(\Gamma, G))$. Each fiber of the map $H^1(\Gamma, T) \to H^1(\Gamma, G)$, say over some element $[z] \in H^1(\Gamma, G)$, accounts for the set of rational classes inside of the stable class of $(G, \mathrm{id}, 1, \gamma)$ that are "hosted" by the pure inner twist whose isomorphism class is given by $[z]$.

We now come to transfer factors. The structure theory of reductive groups says that in each inner class there is a unique quasi-split group. In the example of unitary groups it is the group $U(p, q)$ for which $|p - q|$ is minimal, i.e. $p = q$ when n is even and $p = q + 1$ or $p = q - 1$ when n is odd. Note that $U(p, q)$ and $U(q, p)$ are isomorphic as inner twists, but not as pure inner twists.

We fix the quasi-split form G and will consider pure inner twists of G. We also fix a Whittaker datum $\mathfrak{w}$ for G and an extended endoscopic triple $\mathfrak{e} = (H, s, {}^L\eta)$. Recall here that $s \in [Z(\widehat{H})/Z(\widehat{G})]^\Gamma$ and that it was required that there exists a lift $\dot{s} \in Z(\widehat{H})^\Gamma$. In the quasi-split case the existence of this lift was enough, but the data of the lift was not required, as was discussed after Proposition 3.5.9. Now in the general case this data is required, and leads to the following definition.

Definition 4.2.6. A *pure refined endoscopic triple* is a triple $\dot{\mathfrak{e}} = (H, \dot{s}, \eta)$, where (H, s, η) is an endoscopic triple and $\dot{s} \in Z(\widehat{H})^\Gamma$ is a lift of s. An *isomorphism* of pure refined endoscopic triples $(H_1, \dot{s}_1, \eta_1) \to (H_2, \dot{s}_2, \eta_2)$ is an isomorphism of algebraic groups $f : H_1 \to H_2$ defined over F subject to the conditions

1. $\eta_1 \circ \widehat{f}$ and η_2 are $\widehat{G}$-conjugate.

2. The images of $\widehat{f}(\dot{s}_2)$ and $\dot{s}_1$ in $\pi_0(Z(\widehat{H}_2)^\Gamma)$ coincide.

Note that every endoscopic triple can be made into a pure refined endoscopic triple, simply by choosing a lift $\dot{s}$ of s (this is a feature of local endoscopy that is generally not true for global endoscopy, as we will discuss later). Note also that two such refinements need not be isomorphic.

For any strongly regular semi-simple element (G', ξ, z, δ') there exists $\delta \in G(F)$ that is stably conjugate. This follows from a theorem of Steinberg, cf. [Kot82, §2].

{dfn:tfnqsp}

Definition 4.2.7. Given a pure refined extended endoscopic triple $\dot{\mathfrak{e}} = (H, \dot{s}, {}^L\eta)$, $\gamma \in H(F)$, and $\delta' \in G'(F)$ related to γ, define

$$\Delta[\mathfrak{w}, \dot{\mathfrak{e}}](\gamma, (G', \xi, z, \delta')) = \Delta[\mathfrak{w}, \mathfrak{e}](\gamma, \delta) \cdot \langle \mathrm{inv}(\delta, (G', \xi, z, \delta')), \widehat{\varphi}_{\gamma,\delta}^{-1}(\dot{s}) \rangle.$$

The pairing used in this definition is the Tate-Nakayama pairing between $H^1(\Gamma, T)$ and $\pi_0(\widehat{T}^\Gamma)$.

Theorem 4.2.8. *For every $f \in \mathcal{C}_c^\infty(G(F))$ there exists a $\Delta[\mathfrak{w}, \dot{\mathfrak{e}}]$-matching $f^{\dot{\mathfrak{e}}} \in \mathcal{C}_c^\infty(H(F))$.*

This theorem is in fact the first statement of Theorem 3.6.4, which holds without the assumption that G is quasi-split, as discussed in Remark 3.6.5. The only point that needs verification is that Definition 4.2.7 produces a valid transfer factor, in the sense of [LS87]; this is verified in [Kal11, Proposition 2.2.2].

We can now state the refined local Langlands conjecture for pure inner twists, which is the analog of Conjecture 4.2.9.

Conjecture 4.2.9. {cnj:lcinqsp}

1. *Let $\varphi : L_F \to {}^LG$ be a Langlands parameter. There exists an associated finite set Π_φ of irreducible representations of pure inner twists of G equipped with a* ***bijection*** *$\rho : \Pi_\varphi \to Irr(\pi_0(S_\varphi))$, where $S_\varphi = Cent(\varphi, \widehat{G})$, fitting into the commutative diagram*

$$\begin{array}{ccc} Irr(\pi_0(S_\varphi)) & \longrightarrow & \Pi_\varphi \\ \downarrow & & \downarrow \\ \pi_0(Z(\widehat{G})^\Gamma)^* & \longleftarrow & H^1(\Gamma, G) \end{array}$$

   Write $\Pi_\varphi((G', \xi))$ for the set $\{\pi | (G', \xi, z, \pi') \in \Pi_\varphi\}$.

2. *Let $\varphi^H : L_F \to {}^LH$ be a tempered Langlands parameter and let $\varphi = {}^L\eta \circ \varphi^H$. Then*
$$e(G') \sum_{\pi \in \Pi_\varphi((G', \xi))} tr\rho_\pi(\dot{s})\Theta_\pi(f) = S\Theta_{\varphi^H}(f^{\dot{\mathfrak{e}}}).$$

The left vertical map in the above diagram assigns to $\rho \in \mathrm{Irr}(\pi_0(S_\varphi))$ the character by which the group $\pi_0(Z(\widehat{G})^\Gamma)$ operates on the representation ρ via the map $\pi_0(Z(\widehat{G})^\Gamma) \to \pi_0(S_\varphi)$ induced by the inclusion $Z(\widehat{G})^\Gamma \subset S_\varphi$. The right vertical map sends a tuple (G', ξ, z, π) to the class of z. The bottom horizontal map is known as the Kottwitz map, cf [Kot86, §1]. It is bijective when F is non-archimedean, and has well understood kernel and cokernel when $F = \mathbb{R}$. The quantity $e(G)$ is either $+1$ or -1. It is known as the Kottwitz sign of the reductive group G and is defined in [Kot83]. An alternative interpretation of the Kottwitz map and the Kottwitz sign is given in [Lab99, §1.7]. As the notation suggests, the set $\Pi_\varphi((G', \xi))$ will not depend on z, but the map to $\mathrm{Irr}(\pi_0(S_\varphi))$ will.

{rem:relevant}

Remark 4.2.10. The natural map $\pi_0(Z(\widehat{G})^\Gamma) \to \pi_0(S_\varphi)$ need not be injective. This means that for certain combinations of $\zeta : \pi_0(Z(\widehat{G})^\Gamma) \to \mathbb{C}^\times$ and $\varphi : L_F \to {}^LG$ the fiber of the left vertical map is forced to be empty. It turns out that this is the case precisely when φ is not relevant for the inner twist (or in the real case, inner twists) that correspond(s) to ζ under the Kottwitz map. Here *relevant* is a technical notion: the minimal Γ-stable Levi subgroup of $\widehat{G}$ through which φ factors is dual to a Levi subgroup of G that transfers to an inner twist corresponding to ζ. For more details we refer to [Kal16, §5.5].

{rem:s=1}

Remark 4.2.11. Unlike in the setting of Conjecture 3.6.7, the case $s = 1$ in Conjecture 4.2.9 is not trivial, because in that case $H = G$, which will in general be different from G'. This case contains the assertion that the left hand side in the character identity is a stable distribution, and in addition the assertion that it matches the stable distribution $S\Theta_\varphi$ on the quasi-split form G.

The following generalizes Definition 3.6.6 to the setting of pure inner twists.

{dfn:stabpure}

Definition 4.2.12.

1. For any semi-simple $\dot{s} \in S_\varphi$ the virtual character

$$\Theta^{\dot{s}}_{\varphi,(G',\xi,z)} := e(G') \sum_{\pi \in \Pi_\varphi((G',\xi))} \mathrm{tr}\rho_\pi(\dot{s}) \cdot \Theta_\pi$$

is called the *$\dot{s}$-stable character* on (G', ξ) associated to φ.

2. In the special case $\dot{s} = 1$ we call

$$S\Theta_{\varphi,(G',\xi,z)} = \Theta^{1}_{\varphi,(G',\xi,z)} = e(G') \sum_{\pi \in \Pi_\varphi((G',\xi))} \dim \rho_\pi \cdot \Theta_\pi$$

the *stable character* on (G', ξ) associated to φ.

Example 4.2.13. Let us consider here the example of unitary groups. So let F be a local field, E/F a quadratic extension, and $G = U_{E/F}(N)$ the quasi-split unitary group associated to this quadratic extension. The set $H^1(\Gamma, G)$ classifies Hermitian spaces of rank N. When $F = \mathbb{R}$ these are classified by their signature (p, q) with $p, q \geq 0$ and $p + q = N$; when F is non-archimedean these are classified by their discriminant, an element of $F^\times / N_{E/F}(E^\times) \cong \mathbb{Z}/2\mathbb{Z}$.

We have the commutative diagram

$$\begin{array}{ccc} H^1(\Gamma, G) & \longrightarrow & H^1(\Gamma, G_{\text{ad}}) \\ \downarrow & & \downarrow \\ [Z(\widehat{G})^\Gamma]^* & \longrightarrow & [Z(\widehat{G}_{\text{sc}})^\Gamma]^* \\ \| & & \| \\ \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/\delta\mathbb{Z} \end{array}$$

where $\delta = 2$ if N is even and $\delta = 1$ if N is odd, and the bottom horizontal map is the unique surjective group homomorphism. The top horizontal map is also surjective.

First let F be non-archimedean. Then the two vertical maps are bijective. Thus when N is even there exists a unique non-trivial inner form, which can be made pure in exactly one way; when N is odd the quasi-split unitary group does not have a non-trivial inner form, but there are two non-equivalent ways to view the quasi-split unitary group as a pure inner form of itself.

Let φ be a discrete series parameter. The compound L-packet Π_φ is always the disjoint union of two L-packets $\Pi_\varphi(G) \cup \Pi_\varphi(G')$. The packet $\Pi_\varphi(G)$ is in bijection with those irreducible representations of $\pi_0(S_\varphi)$ on which the central $\{\pm 1\}$ operates trivially, while the packet $\Pi_\varphi(G')$ is in bijection with those irreducible representations of $\pi_0(S_\varphi)$ on which the central $\{\pm 1\}$ operates non-trivially. When N is even G' is the unique non-trivial inner form. When N is odd $G' = G$ and both L-packets contain the same representations, but are indexed by different members of $\text{Irr}(\pi_0(S_\varphi))$.

Now let $F = \mathbb{R}$. The vertical maps are surjective, but not injective. We have

$$H^1(\Gamma, G) = \{(p, q) | p \geq 0, q \geq 0, p + q = N\}.$$

The map $H^1(\Gamma, G) \to H^1(\Gamma, G_{\text{ad}})$ is surjective, with fibers given by $\{(p, q), (q, p)\}$. Thus the fibers all have size 2 when N is odd, while when $N = 2n$ is even all fibers have size 2 except for the fiber $\{(n, n)\}$. The left vertical map maps (p, q) to $\lfloor N/2 \rfloor + q$ mod 2. Note that the occurrence of fibers of different size implies that there is no structure of abelian group on the sets $H^1(\Gamma, G)$ and $H^1(\Gamma, G_{\text{ad}})$ that makes the natural map $H^1(\Gamma, G) \to H^1(\Gamma, G_{\text{ad}})$ a group homomorphism. The groups $U(p, q)$ and $U(q, p)$ are considered different as pure inner forms, but not different as inner forms.

When $N = 2n$ is even the quasi-split unitary group is $G = U(n, n)$. The two fibers of the left vertical map are the subsets $\{(2n - 2k, 2k) | 0 \leq k \leq n\}$ and $\{(2n - 2k - 1, 2k + 1), 0 \leq k < n\}$. The pure inner forms comprising each subset constitute what is known as a K-group (cf. [Art99, §1], [She08, §4], [Kal16, §5.1]). That is, if $(\xi_i, z_i) : G \to G'_i$ are two pure inner twists in the same K-group then the composed pure inner twist $(\xi_2 \circ \xi_1^{-1}, \xi_1(z_2 \cdot z_1^{-1})) : G'_1 \to G'_2$

has the property that $\xi_1(z_2 \cdot z_1^{-1}) \in H^1(\Gamma, G'_1)$ lifts to $H^1(\Gamma, (G'_1)_{\text{sc}})$. In the case at hand, an inner form belongs to exactly one of the two K-groups, and occurs twice in that K-group.

When $N = 2n+1$ is odd the quasi-split unitary group is $U(n, n+1) = U(n+1, n)$. The two fibers of the left vertical map are $\{(2n+1-2k, 2k) | 0 \le k \le n\}$ and $\{(2n-2k, 2k+1) | 0 \le k \le n\}$. Each inner form occurs in both K-groups exactly once.

Again let φ be a discrete series parameter. The compound L-packet Π_φ is again the disjoint union of two subsets $\Pi_\varphi({}^K G) \cup \Pi_\varphi({}^K G')$, each again being in bijection with those irreducible representations of $\pi_0(S_\varphi)$ on which the central $\{\pm 1\}$ operates trivially resp. non-trivially. Here ${}^K G$ and ${}^K G'$ are the two K-groups. The packet $\Pi_\varphi({}^K G)$ decomposes further as the disjoint union of the L-packets corresponding to the individual pure inner forms of G that belong to the K-group ${}^K G$, and the analogous statement holds for ${}^K G'$.

4.3 Rigid inner twists

We now come to the problem that not every inner form is part of a pure inner twist. It came from the possible failure of surjectivity of $H^1(\Gamma, G) \to H^1(\Gamma, G_{\text{ad}})$ for arbitrary connected reductive groups, particularly for quasi-split groups.

The solution, introduced in [Kal16], is to replace the Galois cohomology set $H^1(\Gamma, G)$ with something else, which for a moment we'll call $H^1(?, G)$. This replacement has to have the following properties:

1. An injection $H^1(\Gamma, G) \to H^1(?, G)$.
2. A surjection $H^1(?, G) \to H^1(\Gamma, G_{\text{ad}})$.
3. A description of $H^1(?, G)$ in terms of $\widehat{G}$.
4. Work uniformly for all local fields.
5. Have a version for all global fields equipped with localization maps.

Here is the solution to this problem.

Definition 4.3.1. Consider the pro-finite algebraic group

$$P_F^{\text{rig}} = \varprojlim_{N, E/F} \text{Res}_{E/F} \mu_N / \mu_N,$$

where the limit is taken over all natural numbers N and all finite Galois extensions E/F.

Theorem 4.3.2. *Let F be a local field of characteristic zero. Then*

$$H^1(\Gamma, P_F^{rig}) = 0, \qquad H^2(\Gamma, P_F^{rig}) = \begin{cases} \widehat{\mathbb{Z}}, & F/\mathbb{Q}_p \\ \mathbb{Z}/2\mathbb{Z}, & F = \mathbb{R} \\ 0, & F = \mathbb{C} \end{cases}$$

Note that the above equality signifies a canonical isomorphism, so we can speak of the element $-1 \in H^2(\Gamma, P_F^{\mathrm{rig}})$.

Definition 4.3.3. Let
$$1 \to P_F^{\mathrm{rig}} \to \mathcal{E}_F^{\mathrm{rig}} \to \Gamma_F \to 1$$
be the extension associated to $-1 \in H^2(\Gamma, P_F^{\mathrm{rig}})$.

Definition 4.3.4. Let $H^1(\mathcal{E}_F^{\mathrm{rig}}, G)$ be the cohomology of the pro-finite group $\mathcal{E}_F^{\mathrm{rig}}$ acting on the discrete module $G(\bar{F})$. Let $H^1_{\mathrm{bas}}(\mathcal{E}_F^{\mathrm{rig}}, G)$ be the subset of $H^1(\mathcal{E}_F^{\mathrm{rig}}, G)$ consisting of those classes whose restriction to P_F^{rig} is central.

Theorem 4.3.5. {thm:gerbe}

1. *There is an inflation restriction sequence*
$$1 \to H^1(\Gamma, G) \to H^1_{bas}(\mathcal{E}_F^{rig}, G) \to Hom_F(P_F^{rig}, Z(G))$$

2. *The map $Hom_F(P_F^{rig}, Z(G)) \to H^2(\Gamma, Z(G))$ given by evaluation at the canonical element of $H^2(\Gamma, P_F^{rig})$ is surjective. Consequently the map $H^1_{bas}(\mathcal{E}_F^{rig}, G) \to H^1(\Gamma, G_{ad})$ is surjective.*

3. *There is a functorial homomorphism*
$$H^1_{bas}(\mathcal{E}_F^{rig}, G) \to \pi_0(Z(\widehat{\bar{G}})^+)^*,$$
which is an isomorphism when F is non-archimedean. Here $\widehat{\bar{G}}$ is the universal cover of the complex Lie group $\widehat{G}$, and $Z(\widehat{\bar{G}})^+$ is the preimage of $Z(\widehat{G})^\Gamma$.

{rem:cover}

Remark 4.3.6. If G is semi-simple, then so is $\widehat{G}$, and its universal cover is again a semi-simple algebraic group, namely the simply connected cover $\widehat{G}_{\mathrm{sc}}$ of $\widehat{G}$. If G is a torus, then so is $\widehat{G}$, and its universal cover is a pro-algebraic group, i.e. the projective limit of all isogenies with target $\widehat{G}$.

Instead of working with the universal cover, one can fix a finite central subgroup $Z \subset G$ defined over F and work with the dual group $\widehat{\bar{G}}$ of the quotient $\bar{G} = G/Z$. Then one replaces $H^1_{\mathrm{bas}}(\mathcal{E}_F^{\mathrm{rig}}, G)$ with the subset $H^1(P_F^{\mathrm{rig}} \to \mathcal{E}_F^{\mathrm{rig}}, Z \to G)$ consisting of those cohomology classes in $H^1(\mathcal{E}_F^{\mathrm{rig}}, G)$ whose restriction to P_F^{rig} is not just central, but valued in the chosen subgroup Z.

Fact 4.3.7. *The composition of the homomorphism of part 3 of the above theorem with the inflation map $H^1(\Gamma, G) \to H^1_{bas}(\mathcal{E}_F^{rig}, G)$ takes values in the subgroup $\pi_0(Z(\widehat{G})^\Gamma)^* \subset \pi_0(Z(\widehat{\bar{G}})^+)^*$ and recovers Kottwitz's map.*

Definition 4.3.8. A *rigid inner twist* is a pair (ξ, z) of an inner twist $\xi : G \to G'$ and $z \in Z^1_{\mathrm{bas}}(\mathcal{E}_F^{\mathrm{rig}}, G)$ such that $\xi^{-1}\sigma(\xi) = \mathrm{Ad}(\bar{z}_\sigma)$, where $\bar{z} \in Z^1(F, G_{\mathrm{ad}})$ is the image of z.

An *isomorphism of rigid inner twists* $(\xi_1, z_1) : G \to G_1$ and $(\xi_2, z_2) : G \to G_2$ is a pair (f, g) where $f : G_1 \to G_2$ is an isomorphism over F and $g \in G(\bar{F})$ is subject to $\xi_2^{-1} \circ f \circ \xi_1 = \mathrm{Ad}(g)$ and $z_2(e) = g z_1(\sigma)\sigma_e(g^{-1})$, where $\sigma_e \in \Gamma$ is the image of $e \in \mathcal{E}_F^{\mathrm{rig}}$.

Exercise 4.3.9.

1. *The assignment* $(\xi, z, G') \mapsto [z]$ *is a bijection between the set of isomorphism classes of rigid inner twists and* $H^1_{bas}(\mathcal{E}_F^{rig}, G)$.

2. *The group of automorphisms of* (ξ, z, G') *is* $H^0(F, G') = G'(F)$.

We can now repeat all definitions involving pure inner twists with rigid inner twists, while replacing $H^1(\Gamma, -)$ with $H^1_{\mathrm{bas}}(\mathcal{E}_F^{\mathrm{rig}}, -)$. We leave this replacement to the reader, and refer to [Kal16, §5.1]. In the definition of the transfer factor we have to modify the notion of a refined endoscopic triple.

Definition 4.3.10. A *rigid refined endoscopic triple* is a triple $\dot{\mathfrak{e}} = (H, \dot{s}, \eta)$, where (H, s, η) is an endoscopic triple and $\dot{s} \in Z(\widehat{\widehat{H}})^+$ is a lift of s, where $Z(\widehat{\widehat{H}})^+$ is the pull-back of the diagram $Z(\widehat{H})^\Gamma \xrightarrow{\xi} \widehat{G} \leftarrow \widehat{\widehat{G}}$. An *isomorphism* of rigid refined endoscopic triples $(H_1, \dot{s}_1, \eta_1) \to (H_2, \dot{s}_2, \eta_2)$ is an isomorphism of algebraic groups $f : H_1 \to H_2$ defined over F subject to the conditions

1. $\eta_1 \circ \widehat{f}$ and η_2 are $\widehat{G}$-conjugate.

2. The images of $\widehat{f}(\dot{s}_2)$ and $\dot{s}_1$ in $\pi_0(Z(\widehat{\widehat{H_2}})^+)$ coincide.

Definition 4.3.11. Given a rigid refined extended endoscopic triple $\dot{\mathfrak{e}} = (H, \dot{s}, {}^L\eta)$, {dfn:tfnqsr}
$\gamma \in H(F)$, and $\delta' \in G'(F)$ related to γm define

$$\Delta[\mathfrak{w}, \dot{\mathfrak{e}}](\gamma, (G, \xi, z, \delta')) = \Delta[\mathfrak{w}, \mathfrak{e}](\gamma, \delta) \cdot \langle \mathrm{inv}(\delta, (G, \xi, z, \delta')), \widehat{\varphi}_{\gamma,\delta}^{-1}(\dot{s}) \rangle.$$

The pairing used in this definition is the pairing between $H^1_{\mathrm{bas}}(\mathcal{E}_F^{\mathrm{rig}}, T)$ and $\pi_0([\widehat{\widehat{T}}]^+)$ of Theorem 4.3.5 applied to the group T.

Theorem 4.3.12. *For every* $f \in \mathcal{C}_c^\infty(G(F))$ *there exists a* $\Delta[\mathfrak{w}, \dot{\mathfrak{e}}]$*-matching* $f^{\dot{\mathfrak{e}}} \in$ {thm:transrig}
$\mathcal{C}_c^\infty(H(F))$.

As in the case of pure inner twists, this theorem follows from the first part of Theorem 3.6.4, once it has been verified that Definition 4.3.13 produces a valid transfer factor in the sense of [LS87], which has been done in [Kal16, Proposition 5.6]. We now come to the analogs of Definition 3.6.6 and Conjecture 3.6.7 in the general setting of rigid inner twists.

{dfn:stabnqsr}

Definition 4.3.13. Let $\varphi : L_F \to {}^L G$ be a Langlands parameter.

1. Let $S_\varphi^+ \subset \widehat{\widehat{G}}$ be the preimage of $S_\varphi = \mathrm{Cent}(\varphi, \widehat{G})$.

2. For any semi-simple $\dot{s} \in S_\varphi^+$ the virtual character

$$\Theta_\varphi^{\dot{s}}(G', \xi, z) := e(G') \sum_{\pi \in \Pi_\varphi((G,\xi))} \mathrm{tr}(\rho_\pi(\dot{s})) \cdot \Theta_\pi$$

is called the *$\dot{s}$-stable character* associated to φ, where $\Pi_\varphi((G, \xi))$ is as in Conjecture 4.3.14 below.

3. In the special case $\dot{s} = 1$ we call

$$S\Theta_\varphi = \Theta_\varphi^1 = e(G') \sum_{\pi \in \Pi_\varphi((G,\xi))} \dim(\rho_\pi) \cdot \Theta_\pi$$

the *stable character* associated to φ.

Conjecture 4.3.14. {cnj:lcinqsr}

1. *Let $\varphi : L_F \to {}^LG$ be a Langlands parameter. There exists an associated finite set Π_φ of irreducible representations of rigid inner twists of G equipped with a* ***bijection*** *$\rho : \Pi_\varphi \to Irr(\pi_0(S_\varphi^+))$ that fits into the commutative diagram*

$$\begin{array}{ccc} Irr(\pi_0(S_\varphi^+)) & \longrightarrow & \Pi_\varphi \\ \downarrow & & \downarrow \\ \pi_0(Z(\widehat{\overline{G}})^+)^* & \longleftarrow & H^1_{bas}(\mathcal{E}_F^{rig}, G) \end{array}$$

Write $\Pi_\varphi((G', \xi))$ for the set $\{\pi | (G', \xi, z, \pi) \in \Pi_\varphi\}$.

2. *Let $\varphi^H : L_F \to {}^LH$ be a tempered Langlands parameter and let $\varphi = {}^L\eta \circ \varphi^H$. Then*

$$\Theta_\varphi^{\dot{s}}(f) = S\Theta_{\varphi^H}(f^{\dot{\mathfrak{e}}}).$$

Remark 4.3.15. As in the quasi-split case, the endoscopic character identity expresses the $\dot{s}$-stable character associated to φ in terms of the stable character associated to the factorization of φ through the endoscopic group H. As already remarked for pure inner twists, the case $\dot{s} = 1$ is not trivial any more, because then $H = G$ is in general not equal to G'. As for pure inner twists, the set $\Pi_\varphi((G', \xi))$ will not depend on z, but the map to $\mathrm{Irr}(\pi_0(S_\varphi^+))$ will.

Remark 4.3.16. When $F = \mathbb{R}$ the set $H^1_{\mathrm{bas}}(\mathcal{E}_F^{\mathrm{rig}}, G)$ coincides with the set of *strong real forms* of G in the sense of [ABV92], and the above commutative diagram has been constructed in loc. cit. The endoscopic character identity follows from the work of Shelstad [She82], [She08], as explained in [Kal16, §5.6].

{exa:rigsl2}

Example 4.3.17. We examine here the case of $G = \mathrm{SL}_2$. We have $H^1(\Gamma, G_{\mathrm{ad}}) = \mathbb{Z}/2\mathbb{Z}$ so there is a unique non-trivial inner form G', namely $G'(F) = D^1$, where

D/F is the unique quaternion algebra and D^1 is the subgroup of elements whose reduced norm is equal to 1. The restriction map

$$H^1_{\text{bas}}(\mathcal{E}^{\text{rig}}_F, G) \to \text{Hom}_F(P^{\text{rig}}_F, Z(G))$$

is surjective and the evaluation map

$$\text{Hom}_F(P^{\text{rig}}_F, Z(G)) \to H^2(\Gamma, Z(G)) = \mathbb{Z}/2\mathbb{Z}$$

is an isomorphism. The fiber of the restriction map over the trivial element of $\mathbb{Z}/2\mathbb{Z}$ is $H^1(\Gamma, G)$, which is trivial. The fiber of that map over the non-trivial element is $H^1(\Gamma, G')$, which we will look at more closely.

We have $\widehat{G} = \text{PGL}_2(\mathbb{C})$ and $\widehat{\bar{G}} = \text{SL}_2(\mathbb{C})$. Thus $Z(\widehat{\bar{G}})^+ = Z(\text{SL}_2(\mathbb{C})) = \{\pm 1\}$.

First consider the case $F = \mathbb{R}$. Then $H^1(\Gamma, G') = \mathbb{Z}/2\mathbb{Z}$ according to [PR94, Theorem 6.17]. So again we see that $H^1_{\text{bas}}(\mathcal{E}^{\text{rig}}_F, G)$ has no natural group structure. There are two K-groups, namely the two fibers of the map $H^1(\mathcal{E}^{\text{rig}}_F, G) \to \mathbb{Z}/2\mathbb{Z}$. The trivial fiber is a singleton and the corresponding K-group consists of G alone. The non-trivial fiber contains two points, both of which correspond to the non-trivial inner form G'. In fact, the two points in that non-trivial fiber are a torsor under $H^1(\Gamma, Z(G')) = \mathbb{Z}/2\mathbb{Z}$.

Let φ be a discrete parameter. Then $S_\varphi = \mathbb{Z}/2\mathbb{Z}$ and $S^+_\varphi = \mathbb{Z}/4\mathbb{Z}$. The map $S^+_\varphi \to \pi_0(Z(\widehat{\bar{G}})^+)^* = \mathbb{Z}/2\mathbb{Z}$ is the natural projection map. The trivial fiber of this map indexes the L-packet for the K-group consisting of G alone. This L-packet has two members – the holomorphic and antiholomorphic discrete series representation of a given weight. The non-trivial fiber indexes the L-packet for the K-group consisting of two copies of G'. This L-packet breaks up as a disjoint union of two singleton L-packets, both containing the same representation of G'.

Now consider the case of F non-archimedean. Then $H^1(\Gamma, G') = 0$. The restriction map $H^1_{\text{bas}}(\mathcal{E}^{\text{rig}}_F, G) \to \text{Hom}_F(P^{\text{rig}}_F, Z(G)) = \mathbb{Z}/2\mathbb{Z}$ has trivial fibers and is an isomorphism, giving its source the structure of an abelian group.

Let φ be a discrete parameter. The compound L-packet Π_φ is the disjoint union of two L-packets $\Pi_\varphi(G)$ and $\Pi_\varphi(G')$. There are three possibilities. If φ is the Steinberg parameter then S_φ is trivial and $S^+_\varphi = \mathbb{Z}/2\mathbb{Z}$. Both $\Pi_\varphi(G)$ and $\Pi_\varphi(G')$ are singleton, containing the Steinberg representation of $G(F)$ and the trivial representation of $G'(F)$ respectively. If φ is a regular supercuspidal parameter then $S_\varphi = \mathbb{Z}/2\mathbb{Z}$ and $S^+_\varphi = \mathbb{Z}/4\mathbb{Z}$. Each of $\Pi_\varphi(G)$ and $\Pi_\varphi(G')$ contains two regular supercuspidal representations. Finally if φ is the unique non-regular supercuspidal parameter then $S_\varphi = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and S^+_φ is the quaternion group Q, a non-abelian group of order 8. The L-packet $\Pi_\varphi(G)$ contains the four "exceptional" supercuspidal representations and these corresponds to the four 1-dimensional representations of Q, while $\Pi_\varphi(G')$ contains a single supercuspidal (in fact, finite-dimensional) representation and it corresponds to the unique two-dimensional representation of Q. For more details we refer to [She79] and the discussion in [Kal16, §5.4].

4.4 A slight variation

From the local point of view Conjecture 4.3.14 provides a satisfactory picture. It turns out that from the global point of view a minor shift of that picture is more convenient. This shift aligns well with expectations of Arthur, so we briefly introduce it here following [Kal18, §4.6]. It will also be used in the next section on the stabilization of the elliptic regular part of the trace formula.

We are again interested in an inner twist $\xi : G \to G'$. Instead of an element $z \in Z^1(\mathcal{E}_F^{\text{rig}}, G)$ with $\xi^{-1}\sigma(\xi) = \text{Ad}(\bar{z}_\sigma)$ we now consider an element $z_{\text{sc}} \in Z^1(\mathcal{E}_F^{\text{rig}}, G_{\text{sc}})$ that satisfies $\xi_{\text{sc}}^{-1}\sigma(\xi_{\text{sc}}) = \text{Ad}(\bar{z}_{\text{sc},\sigma})$, where $\xi_{\text{sc}} : G_{\text{sc}} \to G'_{\text{sc}}$ is the inner twist between the simply connected covers. Since the image of z_{sc} under the natural map $G_{\text{sc}} \to G$ will give an element $z \in Z^1(\mathcal{E}_F^{\text{rig}}, G)$, we see that z_{sc} carries more information than z.

We consider an extended endoscopic triple $\mathfrak{e} = (H, s, {}^L\eta)$. Instead of a lift $\dot{s} \in Z(\hat{\bar{H}})^+$ of s, we now consider a lift $s_{\text{sc}} \in Z(\widehat{H}_{G_{\text{sc}}})$, where $\widehat{H}_{G_{\text{sc}}}$ is the algebraic cover of $\widehat{H}_{\text{der}}$ obtained as the fiber product of $\widehat{H} \xrightarrow{\xi} \widehat{G} \leftarrow \widehat{G}_{\text{sc}}$.

We will now use z_{sc} and s_{sc} in place of z and $\dot{s}$ to obtain a transfer factor. This transfer factor will be the product of the one from Definition 4.3.11 with an explicit constant. Following the discussion of Remark 4.3.6 we fix the finite central subgroup $Z(G_{\text{der}}) \subset Z(G)$ and work with the finite quotient $\bar{G} = G/Z(G_{\text{der}}) = G_{\text{ad}} \times Z(G)/Z(G_{\text{der}})$. Dually we have $\hat{\bar{G}} = \widehat{G}_{\text{sc}} \times Z(\widehat{G})^\circ$. Now $\hat{\bar{H}} = \widehat{H}_{G_{\text{sc}}} \times Z(\widehat{G})^\circ$.

Let $s_{\text{der}} \in Z(\widehat{H})$ be the image of s_{sc} under the natural map $\widehat{H}_{G_{\text{sc}}} \to \widehat{H}$. There exists $y \in Z(\widehat{G})$ such that $y \cdot s_{\text{der}} \in Z(\widehat{H})^\Gamma$. Choose $y' \in Z(\widehat{G}_{\text{der}})$ and $y'' \in Z(\widehat{G})^\circ$ such that $y = y' \cdot y''$ and choose a lift $y'_{\text{sc}} \in Z(\widehat{G}_{\text{sc}})$ of y'. Then $\dot{s} = (y'_{\text{sc}} s_{\text{sc}}, y'') \in Z(\widehat{H}_{G_{\text{sc}}}) \times Z(\widehat{G})^\circ$ is an element belonging to $Z(\hat{\bar{H}})^+$. Therefore we have the rigid refined extended endoscopic triple $\dot{\mathfrak{e}} = (H, \dot{s}, {}^L\eta)$ and the transfer factor $\Delta[\mathfrak{w}, \dot{\mathfrak{e}}]$ of Definition 4.3.11.

{dfn:tfnqsa}

Definition 4.4.1.

$$\Delta[\mathfrak{w}] = \langle [z_{\text{sc}}], y'_{\text{sc}} \rangle^{-1} \cdot \Delta[\mathfrak{w}, \dot{\mathfrak{e}}].$$

The pairing in the first factor is between $H^1_{\text{bas}}(\mathcal{E}_F^{\text{rig}}, G_{\text{sc}})$ and $Z(\widehat{G}_{\text{sc}})$ and comes from Theorem 4.3.5 applied the the group G_{sc}.

Proposition 4.4.2. *This factor depends only on s_{sc} and $[z_{sc}]$, but not on the choices of y'_{sc} and y''.*

Now let $\varphi : L_F \to {}^LG$ be a tempered Langlands parameter. Let $S_\varphi^{\text{sc}} \subset \widehat{G}_{\text{sc}}$ be the preimage of $\bar{S}_\varphi = S_\varphi / Z(\widehat{G})^\Gamma \subset \widehat{G}_{\text{ad}}$. We can write elements of $S_\varphi^+ \subset \hat{\bar{G}} = \widehat{G}_{\text{sc}} \times Z(\widehat{G})^\circ$ as tuples (s, z). The map $Z(\hat{\bar{G}}) \to Z(\widehat{G}_{\text{sc}})$ given by projection onto the first factor allows us to form the push-out $S_\varphi^+ \oplus_{Z(\hat{\bar{G}})^+} Z(\widehat{G}_{\text{sc}})$ of the diagram $S_\varphi^+ \leftarrow Z(\hat{\bar{G}})^+ \to Z(\widehat{G}_{\text{sc}})$.

Proposition 4.4.3. *The map*

$$S_\varphi^+ \oplus_{Z(\widehat{G})^+} Z(\widehat{G}_{sc}) \to S_\varphi^{sc}, \qquad ((s,z),x) \mapsto sx$$

is an isomorphism of groups and gives a bijection $Irr(\pi_0(S_\varphi^+),[z]) \to Irr(\pi_0(S_\varphi^{sc}),[z_{sc}])$.

Let $\rho \in \mathrm{Irr}(\pi_0(S_\varphi^+),[z])$ be sent by this bijection to $\rho^{\mathrm{sc}} \in \mathrm{Irr}(\pi_0(S_\varphi^{\mathrm{sc}}),[z_{\mathrm{sc}}])$. The following is immediate.

Fact 4.4.4. *If* $f^{\dot{\mathfrak{e}}} \in \mathcal{C}_c^\infty(H(F))$ *is the transfer of* $f \in \mathcal{C}_c^\infty(G(F))$ *with respect to* $\Delta[\mathfrak{w},\dot{\mathfrak{e}}]$*, then* $f_{sc}^{\dot{\mathfrak{e}}} = \langle [z_{sc}], y'_{sc}\rangle^{-1} \cdot f^{\dot{\mathfrak{e}}}$ *is the transfer of* f *with respect to* $\Delta[\mathfrak{w}]$ *the character identity*

$$e(G') \sum_{\pi \in \Pi_\varphi((G',\xi))} tr\rho_\pi^{sc}(s_{sc})\Theta_\pi(f) = \sum_{\pi^H \in \Pi_{\varphi^H}} dim(\rho_{\pi^H})\Theta_{\pi^H}(f_{sc}^{\dot{\mathfrak{e}}})$$

holds.

5 Stabilization of the elliptic regular part of the trace formula for general reductive groups

{sec:stabgen}

In this section we will review the stabilization of the elliptic regular part of the geometric side of the trace formula. This was done in [Lan83], but we will adopt the approach and the notation of the generalization to non-regular elliptic elements of [Kot86], and will also include a discussion of the use of normalized transfer factors following [Kal18].

5.1 The beginning

Let G be a connected reductive group over the global field F of characteristic zero. The reader may assume that $F = \mathbb{Q}$ if desired. The elliptic (strongly) regular part of the trace formula is

$$\mathrm{TF}_{\mathrm{sr.ell}}(f) = \sum_{\delta \in G(F)_{\mathrm{sr.ell}}/\sim} \tau(T_\delta) O_\delta(f). \qquad (5.1.1)$$

{eq:regell1}

Here the sum runs over the set of (rational) conjugacy classes of strongly regular elliptic elements of $G(F)$, i.e. those whose centralizer T_δ is an elliptic (anisotropic modulo the center of G) maximal torus, $\tau(T_\delta)$ is the Tamagawa number of T_δ, and O_δ is the adelic orbital integral of f. We remind ourselves that the Tamagawa number $\tau(G)$ of a reductive group G is the volume of $G(\mathbb{A})^1/G(F)$ with respect to the Tamagawa measure [Wei82, Appendix 2]. This number also has a cohomological description that we will discuss below.

Mimicking our approach from the case of $G = \mathrm{SL}_2$ we first collect the F-classes inside of a $\bar{F}$-class together and use the fact that if $\delta, \delta' \in G(F)$ are strongly regular semi-simple and stably conjugate there exists (in fact a canonical) isomorphism $T_\delta \to T_{\delta'}$ defined over F. Thus we get

$$\sum_{\delta_0 \in G(F)_{\text{sr.ell}}/\text{st.conj}} \tau(T_{\delta_0}) \sum_{\substack{\delta \in G(F)/\sim \\ \delta \text{ st.conj. } \delta_0}} O_\delta(f).$$

The next step is to replace the inner sum, which runs over the F-classes inside of a $\bar{F}$-class, to a sum running over the $\mathbb{A}$-classes inside of a $\bar{\mathbb{A}}$-class. Here $\bar{\mathbb{A}}$ is the direct limit of $\mathbb{A}_E = \mathbb{A} \otimes_F E$ over all finite extensions E/F. We recall from our discussion of stable conjugacy that the first and second sets are in bijection with the source and target respectively of the map

$$\ker(H^1(F, T_{\delta_0}) \to H^1(F, G)) \xrightarrow{\alpha} \ker(H^1(\mathbb{A}, T_{\delta_0}) \to H^1(\mathbb{A}, G)),$$

where we have used the following short-hand notation: $H^1(F, G) = H^1(\Gamma, G(\bar{F}))$ and $H^1(\mathbb{A}, G) = H^1(\Gamma, G(\bar{\mathbb{A}}))$. We will soon also need the notation $H^1(\mathbb{A}/F, T) = H^1(\Gamma, T(\bar{\mathbb{A}})/T(\bar{F}))$ for a torus T.

In the example of SL_2 we had a number of simplifying properties. First, $H^1(F, G) = 0 = H^1(\mathbb{A}, G)$, so the map was actually just $H^1(F, T_{\delta_0}) \to H^1(\mathbb{A}, T_{\delta_0})$. And second, this map was injective, due to the Hasse norm theorem. Both of these statements are false in general. For any connected reductive group H defined over F we consider the set

$$\ker^1(F, H) = \ker(H^1(F, H) \to H^1(\mathbb{A}, H)).$$

With this notation we have

$$\ker(\alpha) = \ker(\ker^1(F, T_{\delta_0}) \to \ker^1(F, G)).$$

This kernel parameterizes the different F-classes inside of the $\bar{F}$-class of δ_0 that melt together to a single $\mathbb{A}$-class. The corresponding summands O_δ are then equal, so we can simply count and see that (5.1.1) becomes

$$\text{TF}_{\text{sr.ell}}(f) = \sum_{\delta_0 \in G(F)_{\text{sr.ell}}/\text{st.conj}} \tau(T_{\delta_0}) |\ker(\alpha)| \sum_{a \in \text{im}(\alpha)} O_{a\delta_0}(f). \qquad (5.1.2)$$ {eq:regell2}

The notation $a\delta_0$ here means the unique $\mathbb{A}$-conjugacy class of elements $\delta \in G(\mathbb{A})$ satisfying $\text{inv}(\delta_0, \delta) = a$.

5.2 Pre-stabilization 1: with the Hasse principle

Following the approach for SL_2 our next step would be to extend the sum from $\text{im}(\alpha)$ to all of $\ker(H^1(\mathbb{A}, T_{\delta_0}) \to H^1(\mathbb{A}, G))$ so as to obtain stable adelic orbital integrals. For a moment we return to the case $G = \text{SL}_2$ and reinterpret slightly what we did there. We consider the exact sequence

$$1 \to H^1(F, T_{\delta_0}) \to H^1(\mathbb{A}, T_{\delta_0}) \to H^1(\mathbb{A}/F, T_{\delta_0}) \to 1.$$

The injectivity of the first map is due to the Hasse norm principle [PR94, Corollary to Theorem 6.11]. The surjectivity of the second map can be seen either via Kneser's theorem [PR94, Proposition 6.12], or via local and global Tate-Nakayama duality. We have a function $y(a) = O_{a\delta_0}(f)$ defined for all $a \in H^1(\mathbb{A}, T_{\delta_0})$, we are summing it over the subgroup $H^1(F, T_{\delta_0})$, but would like to sum it over all of $H^1(\mathbb{A}, T_{\delta_0})$. We did this by adding and then subtracting an error term. The general case is done by using what Drinfeld once jokingly referred to as a "more scientific terminology for addition and subtraction", namely Fourier inversion.

Exercise 5.2.1. *Let $0 \to A \to B \to C$ be an exact sequence of abelian groups, and assume that C is finite. Let $y : B \to \mathbb{C}$ be a finitely supported function. Then*

$$\sum_{a \in A} y(a) = |C|^{-1} \sum_{\kappa \in C^*} \sum_{b \in B} \kappa(b) y(b).$$

More generally, B be a set, $A \subset B$ a subset, C a finite abelian group, and $B \to C$ a map with the property that the preimage of the identity of C equals A. If $y : B \to \mathbb{C}$ is a finitely supported function, then

$$\sum_{a \in A} y(a) = |C|^{-1} \sum_{\kappa \in C^*} \sum_{b \in B} \kappa(b) y(b).$$

In the case of SL_2 we had $C = \mathbb{Z}/2\mathbb{Z}$ so this formula became

$$\frac{1}{2}\left((O_{\delta_0}(f) + O_{\delta_1}(f)) + (O_{\delta_0}(f) - O_{\delta_1}(f)\right).$$

In the general case we have $A = \mathrm{im}(\alpha)$ and $B = \ker(H^1(\mathbb{A}, T_{\delta_0}) \to H^1(\mathbb{A}, G))$, and would like to find a suitable finite abelian group C with the following property:

Condition 5.2.2. *An element $a \in ker(H^1(\mathbb{A}, T_{\delta_0}) \to H^1(\mathbb{A}, G))$ maps to 0 in C if and only if it lifts to $ker(H^1(F, T_{\delta_0}) \to H^1(F, G))$.*

This is very easy to do when G satisfies the Hasse principle:

Definition 5.2.3. A connected reductive group H defined over F is said to satisfy the *Hasse principle* if $\ker^1(F, H) = \{0\}$.

It is known that the Hasse principle is satisfied by all simply connected groups (this is a Theorem of Kneser, Harder, Chernousov, cf. [PR94, Theorem 6.6]) and all adjoint groups ([PR94, Theorem 6.22]). Counterexamples to the Hasse principle are known both for tori and for semi-simple groups (cf. [PR94, p.324]).

Exercise 5.2.4. *Assume that G satisfies the Hasse principle. Then the following statements are equivalent.*

1. *The $G(\mathbb{A})$-conjugacy class of δ has an F-point.*

2. *The element $inv(\delta_0, \delta) \in ker(H^1(\mathbb{A}, T_{\delta_0}) \to H^1(\mathbb{A}, G))$ lies in the image of $ker(H^1(F, T_{\delta_0}) \to H^1(F, G))$.*

3. *The image of $inv(\delta_0, \delta)$ in $H^1(\mathbb{A}/F, T_{\delta_0})$ vanishes.*

Thus when G satisfies the Hasse principle we can take, just as in the case of $G = \mathrm{SL}_2$, $C = H^1(\mathbb{A}/F, T_{\delta_0})$. We are using here the finiteness of $H^1(\mathbb{A}/F, T_{\delta_0})$, which is a classical result in Galois cohomology, and follows for example from global Tate-Nakayama duality [PR94, Theorem 6.3]. According to the above exercise (5.1.2) becomes

$$\sum_{\delta_0 \in G(F)_{\text{sr.ell}}/\text{st.conj}} \tau(T_{\delta_0})|\ker(\alpha)| \cdot |H^1(\mathbb{A}/F, T_{\delta_0})|^{-1} \sum_{\kappa \in H^1(\mathbb{A}/F, T_{\delta_0})^*} O^{\kappa}_{\delta_0}(f), \quad (5.2.1)$$ {eq:regell3}

where

$$O^{\kappa}_{\delta_0}(f) = \sum_{\substack{\delta \in G(\mathbb{A})/\sim \\ \delta \text{ st. conj. } \delta_0}} \kappa(\text{inv}(\delta_0, \delta)) O_{\delta}(f).$$

This is the so-called *pre-stabilization* of the regular elliptic part, i.e. its expression as a sum of κ-parts, in which the $\kappa = 0$ part is a stable distribution, because it is a sum of stable adelic orbital integrals.

5.3 Pre-stabilization 2: without the Hasse principle

When G does not satisfy the Hasse principle the situation becomes a bit more complicated and we need to resort to the simply connected cover G_{sc} of the derived subgroup of G and rely on the fact that G_{sc} does satisfy the Hasse principle according the above mentioned theorem of Kneser, Harder, Chernousov.

{lem:scconj}

Lemma 5.3.1. *Two strongly regular semi-simple elements $\delta_0 \in G(F)$ and $\delta \in G(\mathbb{A})$ are conjugate under $G(\bar{\mathbb{A}})$ if and only if they are conjugate under $G_{sc}(\bar{\mathbb{A}})$.*

Proof. If $g \in G_{\text{sc}}(\bar{\mathbb{A}})$ conjugates δ_0 to δ, then its image in $G(\bar{\mathbb{A}})$ also does, hence the one implication. For the opposite implication, let $g \in G(\bar{\mathbb{A}})$ conjugate δ_0 to δ. Unfortunately, unlike the case of algebraically closed field, the natural map $G_{\text{sc}}(\bar{\mathbb{A}}) \to G(\bar{\mathbb{A}})$ need not be surjective, and so g need not lie in the image of $G_{\text{sc}}(\bar{\mathbb{A}})$. However, one can prove that $G_{\text{sc}}(\bar{\mathbb{A}}) \cdot T_{\delta_0}(\bar{\mathbb{A}}) = G(\bar{\mathbb{A}})$, so there is an element of $G_{\text{sc}}(\bar{\mathbb{A}})$ whose action by conjugation on δ_0 is the same as that of g. To prove that, it suffices to show that for a sufficiently large finite Galois extension E/F and almost all places w of E we have $G_{\text{sc}}(O_{E_w}) \cdot T_{\delta_0}(O_{E_w}) = G(O_{E_w})$. This can be proved using Bruhat-Tits theory, cf. [Kot84a, (3.3.4)]. $\square$

Let us abbreviate $T = T_{\delta_0}$ and write T_{sc} for the preimage of T in G_{sc}. If $g \in G_{\text{sc}}(\bar{\mathbb{A}})$ satisfies $g\delta_0 g^{-1} = \delta$ then the class of $\sigma \mapsto g^{-1}\sigma(g)$ gives an invariant

$$\text{inv}_{\text{sc}}(\delta_0, \delta) \in \ker(H^1(\mathbb{A}, T_{\text{sc}}) \to H^1(\mathbb{A}, G_{\text{sc}})),$$

which is again independent of the choice of g. It is a refinement of $\text{inv}(\delta_0, \delta)$ in the sense that the image of $\text{inv}_{\text{sc}}(\delta_0, \delta)$ under $H^1(\mathbb{A}, T_{\text{sc}}) \to H^1(\mathbb{A}, T)$ is equal to $\text{inv}(\delta_0, \delta)$.

Exercise 5.3.2. *The following statements are equivalent.*

1. *The $G_{sc}(\mathbb{A})$-conjugacy class of δ has an F-point.*
2. *The element $inv_{sc}(\delta_0, \delta) \in ker(H^1(\mathbb{A}, T_{sc}) \to H^1(\mathbb{A}, G_{sc}))$ lies in the image of $ker(H^1(F, T_{sc}) \to H^1(F, G_{sc}))$.*
3. *The image of $inv_{sc}(\delta_0, \delta)$ in $H^1(\mathbb{A}/F, T_{sc})$ vanishes.*

Define Θ to be the image under $H^1(\mathbb{A}, T_{\mathrm{sc}}) \to H^1(\mathbb{A}/F, T_{\mathrm{sc}})$ of the subgroup $\ker(H^1(\mathbb{A}, T_{\mathrm{sc}}) \to H^1(\mathbb{A}, T))$.

{pro:obs}

Proposition 5.3.3. *The following statements are equivalent.*

1. *The $G(\mathbb{A})$-conjugacy class of δ has an F-point.*
2. *The element $inv(\delta_0, \delta) \in ker(H^1(\mathbb{A}, T) \to H^1(\mathbb{A}, G))$ lies in the image of $ker(H^1(F, T) \to H^1(F, G))$.*
3. *The image of $inv_{sc}(\delta_0, \delta)$ in $H^1(\mathbb{A}/F, T_{sc})$ lies in Θ.*

Proof. The equivalence of the first two statements is immediate and left as an exercise. The equivalence between them and third statement on the other hand requires some thought. It is given in [Kot86, Theorem 6.6] in the more general situation of non-regular elliptic elements, but under the additional assumption that the derived subgroup of G is simply connected. We adapt the argument here to our case – we are assuming that δ is strongly regular, but are not assuming that the derived subgroup of G is simply connected.

Decompose the $G(\mathbb{A})$-conjugacy class of δ into $G_{\mathrm{sc}}(\mathbb{A})$-conjugacy classes. The first statement is equivalent to the statement that at least one of these $G_{\mathrm{sc}}(\mathbb{A})$-conjugacy classes has an F-point. By Lemma 5.3.1 the elements in the $G(\mathbb{A})$-conjugacy class of δ are $G_{\mathrm{sc}}(\bar{\mathbb{A}})$-conjugate, so the set of $G_{\mathrm{sc}}(\mathbb{A})$-conjugacy classes in the $G(\mathbb{A})$-conjugacy class of δ is in bijection with the set

$$S = \ker\left(H^1(\mathbb{A}, T_{\delta,\mathrm{sc}}) \to H^1(\mathbb{A}, G_{\mathrm{sc}})\right) \cap \ker\left(H^1(\mathbb{A}, T_{\delta,\mathrm{sc}}) \to H^1(\mathbb{A}, T_\delta)\right),$$

via the map $\delta' \mapsto \mathrm{inv}_{\mathrm{sc}}(\delta, \delta')$. By the previous exercise, the first statement is equivalent to the existence of δ' for which $\mathrm{inv}_{\mathrm{sc}}(\delta_0, \delta')$ maps to the trivial element of $H^1(\mathbb{A}/F, T_{\mathrm{sc}})$. The admissible isomorphism $\varphi_{\delta_0,\delta} : T \to T_\delta$ lifts to an isomorphism $\varphi_{\delta_0,\delta} : T_{\mathrm{sc}} \to T_{\delta,\mathrm{sc}}$ and we have the identity $\mathrm{inv}_{\mathrm{sc}}(\delta_0, \delta') = \mathrm{inv}_{\mathrm{sc}}(\delta_0, \delta) \cdot \varphi^{-1}_{\delta_0,\delta}(\mathrm{inv}_{\mathrm{sc}}(\delta, \delta'))$. The first statement is thus equivalent to the image of $\mathrm{inv}_{\mathrm{sc}}(\delta_0, \delta)^{-1}$ in $H^1(\mathbb{A}/F, T_{\mathrm{sc}})$ belonging to the image of $\varphi^{-1}_{\delta_0,\delta}(S)$ there. The set S is contained in the group

$$S^* = \ker\left(H^1(\mathbb{A}, T_{\delta,\mathrm{sc}}) \to H^1(\mathbb{A}, T_\delta)\right)$$

and Θ is the image in $H^1(\mathbb{A}/F, T)$ of $\varphi^{-1}_{\delta_0,\delta}(S^*)$.

The problem that we now have is that in general $S \subsetneq S^*$. This is caused by the infinite places and the fact that for a connected and simply connected semisimple group H defined over $\mathbb{R}$ the set $H^1(\mathbb{R}, H)$ may not vanish and may not have a natural group structure. In the special case when $H^1(\mathbb{R}, G_{\text{sc}})$ vanishes we have $S = S^*$ and the proof is complete.

To treat the general case, we need to show that the images in $H^1(\mathbb{A}/F, T_{\text{sc}})$ of $\varphi_{\delta_0,\delta}^{-1}(S^*)$ and $\varphi_{\delta_0,\delta}^{-1}(S)$ are equal. The group $\varphi_{\delta_0,\delta}^{-1}(S^*)$ and the set $\varphi_{\delta_0,\delta}^{-1}(S)$ lie in $H^1(\mathbb{A}, T_{\text{sc}})$ and have the same defining formulas as S^* and S, but with T_δ replaced by T. Therefore from now on we will work with T and suppress $\varphi_{\delta_0,\delta}$ from the notation.

Under the isomorphism $H^1(\mathbb{A}, T_{\text{sc}}) = \bigoplus_v H^1(F_v, T_{\text{sc}})$ the subgroup S^* decomposes as $\bigoplus S_v^*$ and the subset S decomposes as $\bigoplus_v S_v$, where

$$S_v^* = \ker\left(H^1(F_v, T_{\text{sc}}) \to H^1(F_v, T)\right)$$

and

$$S_v = \ker\left(H^1(F_v, T_{\text{sc}}) \to H^1(F_v, G_{\text{sc}})\right) \cap \ker\left(H^1(F_v, T_{\text{sc}}) \to H^1(F_v, T)\right).$$

When v is finite Kneser's theorem implies $H^1(F_v, G_{\text{sc}}) = 1$ and therefore $S_v = S_v^*$. When v is infinite we can have $S_v \subsetneq S_v^*$. Write $S_\infty^* = \bigoplus_{v|\infty} S_v^*$ and $S_\infty = \bigoplus_{v|\infty} S_v$ and set

$$S_\infty^* = \ker\left(H^1(F_\infty, T_{\text{sc}}) \to H^1(F_\infty, T)\right),$$

where $F_\infty = \prod_{v|\infty} F_v$. Then $S^* = S \cdot S_\infty^*$.

We now claim that the localization map $\ker\left(H^1(F, T_{\text{sc}}) \to H^1(F, T)\right) \to S_\infty^*$ is surjective. This claim would imply that the image of the total localization map $\ker\left(H^1(F, T_{\text{sc}}) \to H^1(F, T)\right) \to S^*$, when multiplied by S, equals all of S^* and the proof would be complete.

To prove the claim we consider the commutative diagram

$$\begin{array}{ccc} H^1(F, T_{\text{sc}} \to T) & \longrightarrow & \ker\left(H^1(F, T_{\text{sc}}) \to H^1(F, T)\right) \\ \downarrow & & \downarrow \\ H^1(F_\infty, T_{\text{sc}} \to T) & \longrightarrow & \ker\left(H^1(F_\infty, T_{\text{sc}}) \to H^1(F_\infty, T)\right) \end{array}$$

The horizontal maps are the connecting homomorphisms in the corresponding long exact sequences for the cohomology of complexes of tori. They are surjective. Applying restriction of scalars $F/\mathbb{Q}$ to T and T_{sc} we can reduce the base field to $\mathbb{Q}$ and then use [KS99, Lemma C.5.A] to see that the image of the left vertical map is dense. Since the target of the right vertical map is finite the proof is complete. □

The above Proposition tells us what the correct definition of the group C should be, namely the quotient $H^1(\mathbb{A}/F, T_{\text{sc}})/\Theta$. This group is traditionally denoted by $\mathfrak{K}(T/F)^*$. We write $\text{obs}(\delta) \in \mathfrak{K}(T/F)^*$ for the image of $\text{inv}_{\text{sc}}(\delta_0, \delta)$. In fact, we note that there is a well-defined map

$$\ker(H^1(\mathbb{A}, T) \to H^1(\mathbb{A}, G)) \to \mathfrak{K}(T/F)^*$$

and obs(δ) is the image of inv(δ_0, δ) under that map. Write $\mathfrak{K}(T/F)$ for the Pontryagin dual of $\mathfrak{K}(T/F)^*$. With this (5.1.2) becomes

$$\mathrm{TF}_{\mathrm{sr.ell}}(f) = \sum_{\delta_0 \in G(F)_{\mathrm{sr.ell}}/\mathrm{st.conj}} \tau(T_{\delta_0})|\ker(\alpha)| \cdot |\mathfrak{K}(T_{\delta_0}/F)^*|^{-1} \sum_{\kappa \in \mathfrak{K}(T_{\delta_0}/F)} O^{\kappa}_{\delta_0}(f), \qquad (5.3.1)$$ {eq:regell4}

where

$$O^{\kappa}_{\delta_0}(f) = \sum_{\substack{\delta \in G(\mathbb{A})/\sim \\ \delta \text{ st. conj. } \delta_0}} \kappa(\mathrm{obs}(\delta)) O_{\delta}(f).$$

This generalizes (5.2.1) to the case when G does not necessarily satisfy the Hasse principle.

5.4 Pre-stabilization 3: Stable classes in G_0

Our next step will be to show that the κ-parts for $\kappa \neq 1$ are transfers of stable trace formulas for endoscopic groups. Before we can do this, we must however switch from a sum of stable classes in $G(F)$ to a sum of stable classes in the quasi-split inner form $G_0(F)$. For this, we need to generalize yet again the obstruction we defined in the previous section. Its purpose was to detect, given $\delta_0 \in G(F)$ and $\delta \in G(\mathbb{A})$ stably conjugate to δ_0 whether the $G(\mathbb{A})$-class of δ has an F-point.

We fix an inner twist $\xi : G_0 \to G$ with G_0 quasi-split. We wish to generalize the obstruction to the case where $\delta_0 \in G_0(F)$, while still $\delta \in G(\mathbb{A})$. It will turn out that the group $\mathfrak{K}(T/F)^*$ from the previous section is still the right one to use, where now T is the centralizer of δ_0 in G_0. But the construction of the obstruction becomes a bit stranger, because inv$(\delta_0, \delta) \in H^1(\mathbb{A}, T)$ is not really defined.

Namely, Lemma 5.3.1 applies to this case as well and gives $g \in G_{0,\mathrm{sc}}(\bar{\mathbb{A}})$ with $\delta = \psi(g\delta_0 g^{-1})$. Let $\bar{z}_\sigma \in Z^1(F, G_{0,\mathrm{ad}})$ be the element $\psi^{-1}\sigma(\psi)$ and let $u_\sigma \in C^1(F, G_{0,\mathrm{sc}})$ be an arbitrary lift; here C^1 is the set of 1-cochains, and Z^1 the set of 1-cocycles. Then $g^{-1}u_\sigma\sigma(g) \in C^1(\mathbb{A}, T_{\mathrm{sc}})$. The image in $C^1(\mathbb{A}/F, T_{\mathrm{sc}})$ is in fact an element of $Z^1(\mathbb{A}/F, T_{\mathrm{sc}})$ and is independent of the choice of lift u_σ. Its class in $H^1(\mathbb{A}/F, T_{\mathrm{sc}})$ is also independent of the choice of g, and its image in $\mathfrak{K}(T/F)^*$ is obs(δ).

What makes this construction more difficult to work with is the use of 1-cochains that may not be 1-cocycles. This becomes necessary because in general $\bar{z}_\sigma$ may not lift as a 1-cocycle.

Proposition 5.4.1. *The element obs(δ) is trivial if and only if the $G(\mathbb{A})$-class of δ contains an F-point.*

Proof. The proof is a slight generalization of Proposition 5.3.3. It is given in [Kot86, Theorem 6.6] under the assumption that G_{der} is simply connected, and is contained in the arguments of [KS99, §6.4] without this assumption. The reader may also consult [KT18, Proposition 4.1.7]. □

We now see that (5.3.1) becomes

$$\sum_{\delta_0\in G_0(F)_{\text{sr.ell}}/\text{st.conj}} \tau(T_{\delta_0})|\ker(\alpha)|\cdot|\mathfrak{K}(T_{\delta_0}/F)^*|^{-1}\sum_{\kappa\in\mathfrak{K}(T_{\delta_0}/F)} O^{\kappa}_{\delta_0}(f),$$

where

$$O^{\kappa}_{\delta_0}(f)=\sum_{\substack{\delta\in G(\mathbb{A})/\sim\\ \delta \text{ st. conj. } \delta_0}}\kappa(\text{obs}(\delta))O_{\delta}(f).$$

This formula can be simplified further using a cohomological formula for the Tamagawa number of a connected reductive group G derived by Kottwitz. The Tamagawa number $\tau(G)$ is by definition [Wei82, Appendix 2] the volume of $G(\mathbb{A})^1/G(F)$ with respect to the Tamagawa measure. Weil conjectured that $\tau(G)=1$ when G is simply connected and this was proved by work of Weil, Tamagawa, Langlands, Lai, and Kottwitz, cf. [Kot88]. This makes unconditional the cohomological formula of Kottwitz [Kot84b, (5.1.1)], which builds on work of Ono and Sansuc and asserts that

$$\tau(G)=|\pi_0(Z(\widehat{G})^{\Gamma})|\cdot|\ker^1(\Gamma,Z(\widehat{G}))|. \tag{5.4.1}$$ {eq:kot_tamagawa}

From this formula one derives $|\ker(\alpha)|\cdot|\mathfrak{K}(T_{\delta_0}/F)^*|^{-1}=\tau(G)\tau(T_{\delta_0})^{-1}$, and plugging it into (5.3.1) we arrive at the following expression for the regular elliptic part of the trace formula:

$$\text{TF}_{\text{sr.ell}}(f)=\tau(G)\sum_{\delta_0\in G_0(F)_{\text{sr.ell}}/\text{st.conj}}\ \sum_{\kappa\in\mathfrak{K}(T_{\delta_0}/F)} O^{\kappa}_{\delta_0}(f). \tag{5.4.2}$$ {eq:regell5}

An alternative expression for the Tamagawa number, using abelianized cohomology, is given in [Lab99, Corollary 1.7.4].

5.5 Transfer 1: Construction of an endoscopic group

In order to rewrite the κ-part as a transfer from an endoscopic group we need to relate the obstruction obs(δ) to the local transfer factors discussed in the previous two Sections.

Fix an elliptic maximal torus $T\subset G_0$ defined over F and an element $\kappa\in\mathfrak{K}(T/F)$. The first step is to produce a global endoscopic triple (H,s,η). In fact, we will obtain that together with an embedding $T\to H$ up to stable conjugacy. This begins with a dual interpretation of $\mathfrak{K}(T/F)$. First, Tate-Nakayama duality gives an isomorphism

$$H^1(\mathbb{A}/F,T_{\text{sc}})\cong\pi_0([\widehat{T}/Z(\widehat{G})]^{\Gamma})^*.$$

Since $\mathfrak{K}(T/F)^*$ was defined as a quotient of $H^1(\mathbb{A}/F,T_{\text{sc}})$, its character group $\mathfrak{K}(T/F)$ is a subgroup of $\pi_0([\widehat{T}/Z(\widehat{G})]^{\Gamma})$. One can compute that it is given by the kernel of the composition of the connecting homomorphism $\pi_0([\widehat{T}/Z(\widehat{G})]^{\Gamma})\to H^1(\Gamma,Z(G))$ and total localization map:

Lemma 5.5.1.

$$\mathfrak{K}(T/F) = \mathit{ker}\left(\pi_0([\widehat{T}/Z(\widehat{G})]^\Gamma) \to \bigoplus_v H^1(\Gamma_v, Z(\widehat{G}))\right).$$

Proof. This is an exercise in Tate-Nakayama duality and we suggest the reader go through it on their own. The details are as follows. The group $\mathfrak{K}(T/F)$ is the annihilator of Θ and we recall that Θ is the image of $K = \ker(H^1(\mathbb{A}, T_{\mathrm{sc}}) \to H^1(\mathbb{A}, T))$ under the map $H^1(\mathbb{A}, T_{\mathrm{sc}}) \to H^1(\mathbb{A}/F, T_{\mathrm{sc}})$. The latter map dualizes to the diagonal map $\pi_0([\widehat{T}/Z(\widehat{G})]^\Gamma) \to \prod_v \pi_0([\widehat{T}/Z(\widehat{G})]^{\Gamma_v})$ and therefore $\mathfrak{K}(T/F)$ is the preimage of the annihilator of K. The annihilator of K is in turn the image of the map $\prod_v \pi_0(\widehat{T}^{\Gamma_v}) \to \prod_v \pi_0([\widehat{T}/Z(\widehat{G})]^{\Gamma_v})$. This map is the product over v of the individual maps $\pi_0(\widehat{T}^{\Gamma_v}) \to \pi_0([\widehat{T}/Z(\widehat{G})]^{\Gamma_v})$ and the image of each such map is the kernel of the connecting homomorphism $\pi_0([\widehat{T}/Z(\widehat{G})]^{\Gamma_v}) \to H^1(\Gamma_v, Z(\widehat{G}))$. Therefore $\mathfrak{K}(T/F)$ is the kernel of the composition of the diagonal map $\pi_0([\widehat{T}/Z(\widehat{G})]^\Gamma) \to \prod_v \pi_0([\widehat{T}/Z(\widehat{G})]^{\Gamma_v})$ and the product $\prod_v \pi_0([\widehat{T}/Z(\widehat{G})]^{\Gamma_v}) \to \prod_v H^1(\Gamma_v, Z(\widehat{G}))$ of the connecting homomorphisms. But that composition also equals the composition of the global connecting homomorphism $\pi_0([\widehat{T}/Z(\widehat{G})]^\Gamma) \to H^1(\Gamma, Z(\widehat{G}))$ and the total localization map. □

Therefore κ gives an element of $\pi_0([\widehat{T}/Z(\widehat{G})]^\Gamma)$. In fact, the ellipticity of T implies that $[\widehat{T}/Z(\widehat{G})]^\Gamma$ is already finite, hence equal to its component group.

Recall that there is a canonical $\widehat{G}$-conjugacy class of embeddings $\widehat{T} \to \widehat{G}$. Let $R^H \subset X^*(\widehat{T})$ be the root system of the centralizer of κ in $\widehat{G}$. Choose a set of simple roots $\Delta^H \subset R^H$ and for every $\sigma \in \Gamma$ let $w^H_\sigma \in \Omega(R^H)$ be the unique element such that $w^H_\sigma \sigma$ preserves Δ^H. There is a unique quasi-split connected reductive group H defined over F and equipped with a minimal Levi subgroup T^H_0 for which $X_*(T^H_0) := X^*(\widehat{T})$ and σ acts on $X_*(T^H_0)$ by $w^H_\sigma \sigma$. Then $w^H_\sigma \in Z^1(F, \Omega(T^H_0, H))$ and a theorem of Steinberg implies that this element lifts to an element of $\ker(H^1(F, N(T^H_0, H)) \to H^1(F, H))$, which gives the canonical stable class of embeddings $T \to H$.

Given $\delta_0 \in T(F)$ we therefore obtain $\gamma \in H(F)$ up to stable conjugacy. More precisely, we have [Kot86, Lemma 9.7]

{pro:tns}

Proposition 5.5.2. *Given a strongly regular semi-simple $\delta_0 \in T(F)$ and $\kappa \in \mathfrak{K}(T/F)$ there exists an endoscopic triple (H, s, η) and $\gamma \in H(F)$ related to δ_0 and unique up to stable conjugacy, such that under the embedding $Z(\widehat{H}) \to \widehat{T}$ the elements s and κ correspond. If (H', s', η', γ') is another such quadruple then there exists an isomorphism $(H, s, \eta) \to (H', s', \eta')$ mapping γ to a stable conjugate of γ'. This isomorphism is unique up to the action of $H_{ad}(F)$.*

5.6 Transfer 2: Completion under a coherence assumption

We now complete the stabilization of the elliptic part of the trace formula under the following assumption:

{asm:coh}

Assumption 5.6.1. For each elliptic extended endoscopic triple $(H, s, {}^L\eta)$ we have chosen, for each place v, a transfer factor Δ with the following property: For each pair of related strongly regular semi-simple elliptic elements $\gamma \in H(F)$ and $\delta \in G(F)$ the following equality holds:

$$\prod_v \Delta(\gamma_v, \delta_v) = \langle \text{obs}(\delta), s \rangle.$$

This assumption implies

$$O^\kappa_{\delta_0}(f) = \sum_{\substack{\delta \in G(\mathbb{A})/\sim \\ \delta \text{ st. conj. } \delta_0}} \left(\prod_v \Delta(\gamma, \delta_v) \right) O_\delta(f) = \prod_v \sum_{\substack{\delta_v \in G(F_v)/\sim \\ \delta_v \text{ st. conj. } \delta_0}} \Delta(\gamma_v, \delta_v) O_{\delta_v}(f_v),$$

where the elliptic endoscopic triple (H, s, η) and the element γ are provided by Proposition 5.5.2 for δ_0 and κ, and one has chosen an arbitrary extension ${}^L\eta$ of η.

Theorem 3.6.4 provides a Δ-matching function f^H and we can continue this equation chain by

$$\prod_v SO_{\gamma_v}(f^H_v) = SO_\gamma(f^H).$$

We can now state the final result of the stabilization. The notion of isomorphism of endoscopic triples gives a notion of an automorphism of a given such triple (H, s, η). Such an automorphism is called inner if it is given by conjugation by an element of $H_{\text{ad}}(F)$. Let $\text{Out}(H, s, \eta)$ be the quotient of the group of automorphisms of (H, s, η) by the subgroup of inner automorphisms. Collecting the terms of (5.4.2) associated to a single endoscopic triple (H, s, η) by Proposition 5.5.2 we obtain

$$\tau(G)\text{Out}(H, s, \eta)^{-1} \sum_\gamma SO_\gamma(f^H),$$

where the sum runs over the set of stable classes of strongly G-regular elliptic elements of $H(F)$. Define the G-regular elliptic part of the stable trace formula for the group H by

$$\text{STF}_{\text{G.sr.ell}}(f^H) = \tau(H) \sum_\gamma SO_\gamma(f^H).$$

Define further the constant

$$\iota(G, H) = \tau(G)\tau(H)^{-1}|\text{Out}(H, s, \eta)|^{-1}.$$

Then the final result of the stabilization of the elliptic regular term of the trace formula for G is

$$\text{TF}_{\text{sr.ell}}(f) = \sum_{(H,s,\eta)} \iota(G, H) \cdot \text{STF}_{\text{G.sr.ell}}(f^H),$$

where the sum is over isomorphism classes of elliptic endoscopic triples, and for each such triple one chooses an arbitrary extension ${}^L\eta$ of η.

Example 5.6.2. Consider the case of $G = \mathrm{SL}_2$ and (H, s, η) a non-trivial elliptic endoscopic datum. In particular H is a one-dimensional anisotropic torus. Then $\tau(G) = 1$ and $\tau(H) = 2$, while $|\mathrm{Out}(H, s, \eta)| = |\mathrm{Aut}(H, s, \eta)|$. Therefore $\iota(G, H) = \frac{1}{4}$, which is the constant that appeared in (2.3.1).

Remark 5.6.3. In principle we could end the discussion of the stabilization at this point, by simply remarking that a collection of local transfer factor satisfying Assumption 5.6.1 always exists. This would however not be satisfactory. The reason is that in order to make effective us of the stabilization identity one often needs a spectral interpretation of the summands $\mathrm{STF}_{\mathrm{G.sr.ell}}(f^H)$, or rather of certain closely related terms $\mathrm{STF}_{\mathrm{disc}}(f^H)$. Using an arbitrary collection of local transfer factors that satisfies Assumption 5.6.1 introduces arbitrariness in this spectral interpretation.

Instead, what we shall do over the ensuing subsections is show that the canonical local normalizations of the transfer factors of Definition 4.3.11 satisfy Assumption 5.6.1. This will bring definiteness to the spectral interpretation of the stable trace formula, via Conjecture 4.3.14, and would ultimately lead to the stable multiplicity formula, which we will discuss in the setting of SL_2 and its inner form, cf. (6.5.2).

5.7 Coherence 1: Quasi-split G

We begin with the following theorem ([LS87, Theorem 6.4.A]) that we will use for arbitrary G:

{thm:tfprod}

Theorem 5.7.1. *Let $\mathfrak{w}$ be a Whittaker datum for G_0 over the global field F, and let ${}^L\eta$ be an extension of η. Then*

$$\prod_v \Delta[\mathfrak{w}_v](\gamma, \delta_0) = 1.$$

Assume now that $G = G_0$ and $\xi = \mathrm{id}$. Let $\delta \in G(\mathbb{A})$ be stable conjugate to δ_0. Recall that by Proposition 3.5.9 and the discussion following it that

$$\Delta[\mathfrak{w}_v](\gamma_v, \delta_v) = \Delta[\mathfrak{w}](\gamma_v, \delta_{0,v}) \cdot \langle \mathrm{inv}_{\mathrm{sc}}(\delta_{0,v}, \delta_v), s_v\rangle.$$

Theorem 5.7.1 then implies

$$\prod_v \Delta[\mathfrak{w}_v](\gamma_v, \delta_v) = \prod_v \langle \mathrm{inv}_{\mathrm{sc}}(\delta_{0,v}, \delta_v), s_v\rangle = \langle \mathrm{inv}_{\mathrm{sc}}(\delta_0, \delta), s\rangle = \langle \mathrm{obs}(\delta), s\rangle.$$

In this equation chain we have used that under the isomorphism $H^1(\mathbb{A}, T_{\mathrm{sc}}) \cong \bigoplus_v H^1(F_v, T_{\mathrm{sc}})$ the adelic invariant $\mathrm{inv}_{\mathrm{sc}}(\delta_0, \delta) \in H^1(\mathbb{A}, T_{\mathrm{sc}})$ corresponds to the tuple $(\mathrm{inv}_{\mathrm{sc}}(\delta_{0,v}, \delta_v))_v$ of local invariants and that under Tate-Nakayama duality the map $\bigoplus_v H^1(F_v, T_{\mathrm{sc}}) \to H^1(\mathbb{A}, T_{\mathrm{sc}}) \to H^1(\mathbb{A}/F, T_{\mathrm{sc}})$ dualizes to the diagonal embedding $[\widehat{T}/Z(\widehat{G})]^\Gamma \to \prod_v [\widehat{T}/Z(\widehat{G})]^{\Gamma_v}$.

We have thus verified Assumption 5.6.1 in the case that G is quasi-split.

5.8 Coherence 2: Non-quasi-split G satisfying the Hasse principle

Let now $\xi : G_0 \to G$ be an arbitrary inner twist, but assume that G (equivalently G_0) satisfies the Hasse principle.

To prove Assumption 5.6.1 for the canonical local normalizations of the transfer factors one needs a global version of $H^1_{\text{bas}}(\mathcal{E}_F^{\text{rig}}, G)$. This has been done in [Kal18]. There is a pro-finite algebraic group P_F^{rig} equipped with a canonical class in $H^2(\Gamma, P_F^{\text{rig}})$ giving rise to a group extension

$$1 \to P_F^{\text{rig}} \to \mathcal{E}_F^{\text{rig}} \to \Gamma \to 1.$$

The group $H^1_{\text{bas}}(\mathcal{E}_F^{\text{rig}}, G)$ is defined by the same formalism as before. It comes equipped with localization maps

$$H^1_{\text{bas}}(\mathcal{E}_F^{\text{rig}}, G) \to H^1_{\text{bas}}(\mathcal{E}_{F_v}^{\text{rig}}, G)$$

for all places v. The natural map $H^1(\mathcal{E}_F^{\text{rig}}, G) \to H^1(F, G_{\text{ad}})$ is surjective. This allows us to fix $z \in Z^1_{\text{bas}}(\mathcal{E}_F^{\text{rig}}, G_0)$ lifting $\xi^{-1}\sigma(x)$.

Kottwitz's reinterpretation of the Hasse principle asserts that the total localization map $H^1(\Gamma, Z(\widehat{G})) \to \prod_v H^1(\Gamma_v, Z(\widehat{G}))$ is injective. Recall that the endoscopic element s lies in $[Z(\widehat{H})/Z(\widehat{G})]^\Gamma$ and lifts to $Z(\widehat{H})^{\Gamma_v}$ for each place v. The Hasse principle now implies that it lifts to an element of $Z(\widehat{H})^\Gamma$. Then it lifts further to an element of $Z(\widehat{\bar{H}})^+$. We fix such a lift $\dot{s}$. Then $\dot{\mathfrak{e}} = (H, \dot{s}, \eta)$ is a rigid refined global endoscopic datum. At each place v we have the normalized transfer factor $\Delta[\mathfrak{w}_v, \dot{\mathfrak{e}}_v]$ of Definition 4.3.11. These satisfy

{thm:tfrglob}

Theorem 5.8.1.

$$\prod_v \Delta[\mathfrak{w}_v, \dot{\mathfrak{e}}](\gamma_v, \delta_v) = \langle obs(\delta), s \rangle.$$

The proof of this theorem proceeds again via Theorem 5.7.1, which using Definition 4.3.11 reduces to showing

$$\prod_v \langle \text{inv}(\delta_{0,v}, \delta_v), \dot{s}_v \rangle = \langle \text{obs}(\delta), s \rangle.$$

This is not as straightforward as in the quasi-split case, essentially for the same reason that made it necessary to define obs(δ) in an ad-hoc way. We refer the reader to [KT18, §4].

5.9 Coherence 3: General G

We continue with a general inner twist $\xi : G_0 \to G$ and now drop the assumption that G satisfies the Hasse principle.

To overcome this we now make arrangements slightly differently. We fix $z \in Z^1_{\text{bas}}(\mathcal{E}_F^{\text{rig}}, G_{0,\text{sc}})$ lifting $\xi^{-1}\sigma(x)$ and a lift $s_{\text{sc}} \in Z(\widehat{H}_{G_{\text{sc}}})$ of s and use the transfer factor $\Delta[\mathfrak{w}_v]$ of Definition 4.4.1. Then we have

Theorem 5.9.1.
$$\prod_v \Delta[\mathfrak{w}_v](\gamma_v, \delta_v) = \langle obs(\delta), s\rangle.$$

This theorem is very closely related to Theorem 5.8.1.

6 Stabilization of the full trace formula for the group $\mathbf{SL}_2$

In this section we will review the stabilization of the full trace formula for the group SL_2, following the exposition in [LL79, §5]. We will not include all details of all computations. Rather, we will try to emphasize the structure of the argument and refer back to [LL79] for technical results.

Before we begin the stabilization we will review the (non-invariant) trace formula for SL_2. This review will be short and keep close to the exposition of [LL79]. For further background on the trace formula we refer to the lectures of Abhishek Parab, as well as [Gel96] and [Art05].

6.1 Basic notation

Let $G = \mathrm{SL}_2$ over a number field F. In fact, we will assume $F = \mathbb{Q}$ to simplify the discussion. We have the standard Borel pair (T_0, B_0) consisting of the diagonal maximal torus T_0 and the upper triangular Borel subgroup B_0. We have $B_0 = U_0 \rtimes T_0$, where U_0 is the subgroup of upper triangular matrices with diagonal entries equal to 1. We have the isomorphism

$$\alpha^\vee : \mathbb{G}_m \to T_0, \qquad x \mapsto \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}$$

and the positive root

$$\alpha : T_0 \to \mathbb{G}_m, \qquad \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \mapsto x^2.$$

Let $\mathfrak{a} = X_*(T_0) \otimes \mathbb{R}$. We have the isomorphisms $\alpha^\vee : \mathbb{R} \to \mathfrak{a}$ and $\alpha : \mathfrak{a} \to \mathbb{R}$ whose composition is multiplication by 2. For any place v of F we have the Iwasawa decomposition

$$G(F_v) = U_0(F_v) \cdot T_0(F_v) \cdot K_v,$$

where $K_v = G(O_{F_v})$ when v is finite, $K_v = SO(2)$ when $F_v = \mathbb{R}$. This decomposition leads to the function

$$H : G(F_v) \to \mathfrak{a}, \qquad \langle \alpha, H(utk)\rangle = \log|\alpha(t)|_v.$$

There is also the adelic version

$$G(\mathbb{A}) = U_0(\mathbb{A}) \cdot T_0(\mathbb{A}) \cdot K,$$

where $K = \prod_v K_v$, and hence the adelic map

$$H : G(\mathbb{A}) \to \mathfrak{a}, \qquad \langle \alpha, H(utk) \rangle = \log |\alpha(t)|_{\mathbb{A}}.$$

Evidently we have $H(g) = \sum_v H(g_v)$ for $g = (g_v)_v \in G(\mathbb{A})$.

Finally choose a non-trivial character $\psi : \mathbb{A}/F \to \mathbb{C}^\times$.

6.2 The trace formula for a torus

{sub:tftorus}

In the course of stabilizing the trace formula for SL_2 will will apply the trace formula for all of its non-split maximal tori. We will now review this formula, which turns out to be an immediate consequence of the Poisson summation formula.

The Poisson summation formula applies to the general setting of an exact sequence

$$1 \to A \to B \to C \to 1$$

of locally compact abelian groups and a nice function $f : B \to \mathbb{C}^\times$. In that setting we have the Pontryagin dual $\check{B} = \mathrm{Hom}_{\mathrm{cts}}(B, \mathbb{S}^1)$ and the Fourier transform $\check{f} : \check{B} \to \mathbb{C}^\times$ defined by

$$\check{f}(\check{b}) = \int_B f(b) \langle \check{b}, b \rangle db,$$

depending on the choice of Haar measure db, where $\langle -, - \rangle : \check{B} \times B \to \mathbb{S}^1$ is the canonical pairing. Once db is chosen it determines a unique measure $d\check{b}$ on $\check{B}$ such that the Fourier inversion formula holds

$$\check{\check{f}}(b) = f(b^{-1}).$$

Given the exact sequence above fix arbitrarily Haar measures da and db on A and B and let dc be the quotient measure, i.e. the one which makes true the integration by fibers formula

$$\int_C \int_A f(ac) da dc = \int_B f(b) db.$$

Equip $\check{A}$, $\check{B}$, and $\check{C}$ with the dual measures and note that $d\check{a}$ equals the quotient measure $d\check{b}/d\check{c}$. Then the Poisson summation formula says

{thm:poisson}

Theorem 6.2.1.

$$\int_A f(a) da = \int_{\check{C}} \check{f}(\check{c}) d\check{c}.$$

Now consider an algebraic torus T defined over F. We will apply the Poisson summation formula to the exact sequence of locally compact groups

$$1 \to T(F) \to T(\mathbb{A})^1 \to T(\mathbb{A})^1 / T(F) \to 1.$$

Here $T(\mathbb{A})^1$ is the intersection of the kernels of $t \mapsto |\chi(t)|_{\mathbb{A}}$, where χ runs over $X^*(T)^\Gamma$. The group $T(\mathbb{A})^1/T(F)$ is compact. The Poisson summation formula then immediately gives

{cor:tortf}

Corollary 6.2.2.

$$\sum_{t \in T(F)} f(t) = \sum_{\xi : T(\mathbb{A})^1 / T(F) \to \mathbb{S}^1} \check{f}(\xi),$$

which is the trace formula for the torus T, the left hand side being the geometric side, i.e. the sum of the orbital integrals of f, which are now simply evaluations, and the right hand side is the spectral side, i.e. the sum of the contributions of the automorphic representations, which are now simply generalizations of Hecke characters. Note that when the torus T is anisotropic we have $X^*(T)^\Gamma = \{0\}$ and hence $T(\mathbb{A})^1 = T(\mathbb{A})$.

6.3 The non-invariant trace formula for SL_2

The non-invariant trace formula is the identity of two distributions

$$J_{\text{geom}}(f) = J_{\text{spec}}(f) \tag{6.3.1}$$ {eq:tf1}

one of geometric nature and one of spectral nature, for a test function $f \in \mathcal{C}_c^\infty(G(\mathbb{A}))$. These distributions are obtained by introducing a truncation parameter $T \in \mathfrak{a}$ and modifying the usual kernel of the trace formula associated to a test function $f \in \mathcal{C}_c^\infty(G(\mathbb{A}))$

$$k_f(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y), \qquad x, y \in G(\mathbb{A}),$$

which is not integrable along the diagonal in $G(F) \setminus G(\mathbb{A}) \times G(F) \setminus G(\mathbb{A})$, to obtain a truncated kernel $k_f^T(x,y)$ which is integrable along the diagonal. Integrating the truncated kernel along the diagonal and deriving a geometric and spectral expression for the result gives a formula

$$\sum_o J_o^T(f) = \sum_\chi J_\chi^T(f), \tag{6.3.2}$$ {eq:tf2}

where the sum on the left is over geometric quantities, and the sum on the right over spectral quantities. Each summand turns out to depend polynomially (in fact, a polynomial of degree 1 in the case of SL_2) on the parameter T.

For each value of T one thus obtains an equation of a geometric term and a spectral term. There is a particular value of T for which these terms are better behaved. While for general groups this value need not be 0 (see [Art05, Equation (9.4)]), for $G = \mathrm{SL}_2$ this value is 0. Setting $T = 0$ we obtain the identity (6.3.1). In the following we shall describe the individual summands in (6.3.2) explicitly for the case $T = 0$, before turning to their stabilization.

6.3.1 The geometric side

{sub:jgeom}

In general the geometric side is a sum over equivalence classes o of elements of $G(F)$, where two elements are called equivalent if their semi-simple parts are conjugate in $G(F)$. In our case $G = \mathrm{SL}_2(F)$. There are four kinds of equivalence classes

1. Each conjugacy class of elliptic regular semi-simple elements is an equivalence class. If $\gamma \in G(F)$ is an elliptic regular element, then its contribution to the geometric side is the usual orbital integral

$$\mathrm{vol}(S(F) \setminus S(\mathbb{A})) \int_{S(\mathbb{A})\setminus G(\mathbb{A})} f(g^{-1}\gamma g)dg,$$

   where S is the centralizer of γ.

   The eigenvalues of γ do not lie in F and generate a quadratic extension E/F. We have $S(F) = E^1$ and $S(\mathbb{A}) = \mathbb{A}_E^1$, where as before the superscript 1 indicates that we are taking the kernel of the norm map for the extension E/F. On G and S we take the Tamagawa measures. The volume factor is then the Tamagawa number of S, which equals 2, cf. (5.4.1). Note that the Tamagawa measures are straightforward in this situation – for both G and S they are given by choosing an F-rational top form and the ψ-self dual measure on $\mathbb{A}$.

2. Each conjugacy class of split (aka hyperbolic) regular semi-simple elements is an equivalence class. If $\gamma \in G(F)$ is such an element, its contribution to the geometric side is the weighted orbital integral

$$\mathrm{vol}(S(F) \setminus S(\mathbb{A})^1) \int_{S(\mathbb{A})\setminus G(\mathbb{A})} f(g^{-1}\gamma g)v(g)dg,$$

   where again S is the centralizer of γ. We may of course assume that S is the diagonal torus. We have $S(\mathbb{A}) = \mathbb{A}^1$ the group of ideles of norm 1, where now norm refers to the idele norm $\mathbb{A}^\times \to \mathbb{R}_{>0}$ that is the product of all the local absolute values $|-|_v : F_v^\times \to \mathbb{R}_{>0}$. The measures are again the Tamagawa measures and the volume term now equals 1.

   The weight factor $v(g)$ is described as follows. Let $T \in \mathfrak{a}$ be a non-negative truncation parameter. Define $v_T(g)$ to be the volume of the convex hull of the set of points

$$\{w^{-1}T - w^{-1}H(wg) | w \in W\},$$

   where W is the Weyl group of S. In our special case this set contains two points and equals

$$\{T - H(g), -T + H(wg)\}.$$

   We use the isomorphism $\alpha : \mathfrak{a} \to \mathbb{R}$ and compute the length of the resulting line segment as

$$v_T(g) = \langle \alpha, 2T - (H(g) + H(wg))\rangle.$$

   After setting $T = 0$ we obtain

$$v(g) = -\langle \alpha, H(g) + H(wg)\rangle.$$

We can further express the weighted adelic orbital integral in terms of local weighted orbital integrals. The relation $H(g) = \sum_v H(g_v)$ noted earlier leads immediately to the expression $v(g) = \sum_v v(g_v)$ of the adelic weight factor as a sum of local weight factors. Since the adelic integral itself factorizes as a product of local integrals we obtain

$$\mathrm{vol}(F^\times \setminus \mathbb{A}_F^1) \sum_v \int_{S(F_v)\setminus G(F_v)} f_v(g_v^{-1}\gamma_v g_v) v(g_v) dg_v \prod_{w\neq v} \int_{S(F_w)\setminus G(F_w)} f_w(g_w^{-1}\gamma_w g_w) dg_w.$$

Since the local weight factor is zero on the maximal compact subgroup K_v the sum is actually finite – indeed, any test function f is a finite sum of factorizable test functions $f = \otimes f_v$, and for those the sum goes only over those places where f_v is not the unit in the unramified Hecke algebra.

Let us define the normalized local orbital integral (we apologize for the double use of F, but prefer to keep with the terminology used in [LL79])

$$F(\gamma_v, f_v) = |\alpha_v - \alpha_v^{-1}|_v \int_{S(F_v)\setminus G(F_v)} f_v(g_v^{-1}\gamma_v g_v) dg_v,$$

where α_v, α_v^{-1} are the eigenvalues of γ_v, and we take on $S(F_v) = F_v^\times$ the measure given by a chosen differential form on S over F. Let us further define the normalized local weighted orbital integral

$$A_1(\gamma_v, f_v) = |\alpha_v - \alpha_v^{-1}|_v \int_{S(F_v)\setminus G(F_v)} f_v(g_v^{-1}\gamma_v g_v)\lambda(g_v) dg_v,$$

where $\lambda : G(F_v) \to \mathbb{R}$ is defined as $\lambda(g_v) = \langle \alpha, H(g_v) - H(wg_v)\rangle$. With this notation the adelic weighted orbital integral becomes

$$-\mathrm{vol}(F^\times \setminus \mathbb{A}_F^1) \sum_v \frac{A_1(\gamma_v, f_v)}{L(1, F_v)} \prod_{w\neq v} \frac{F(\gamma_w, f_w)}{L(1, F_w)}.$$

Here the normalizing factors $|\alpha_v - \alpha_v^{-1}|_v$ disappear due to the product formula, while the factors $L(1, F_v)$ translate between the local and global measures on S. More precisely, if we fix the standard top form on $S = \mathbb{G}_m$, the volume of $O_{F_v}^\times = \mathbb{Z}_p^\times$ with respect to the resulting measure equals $1 - 1/p$. The product of these terms over all primes p is equal to zero, so the top form does not produce a good measure on $S(\mathbb{A}) = \mathbb{A}^\times$. Therefore, one multiplies this measure by $\zeta_p(1) = \frac{1}{1-1/p}$ to achieve that the product over all p produces a measure on $\mathbb{A}^{\mathrm{fin}}$ that assigns to $\widehat{\mathbb{Z}}$ volume 1, rather than 0. This is the Tamagawa measure on $S(\mathbb{A})^\times = \mathbb{A}^\times$, with the factor $\mathbb{R}^\times$ equipped with the usual Lebesgue measure.

3. The set of unipotent elements constitutes a single equivalence class. Its contribution breaks up into the sum of contributions of each unipotent

conjugacy class. The trivial element contributes its "orbital integral", which is simply

$$f(1).$$

This would be multiplied by the volume of $G(F) \setminus G(\mathbb{A})$, but in our case this volume is 1. Consider now a regular unipotent element, which up to conjugation is of the form

$$\gamma = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

with $x \neq 0$. Like in the hyperbolic case, the truncation of the kernel has the effect that this conjugacy class does not contribute its usual orbital integral (which is divergent), but rather a regularized version of it. The regularization takes the following form. Introduce a complex parameter s into the orbital integral by considering

$$\int_{Z(\mathbb{A})U_0(\mathbb{A})\setminus G(\mathbb{A})} f(g^{-1}\gamma g)\beta(g)^{-s} dg,$$

where $\beta(g) = e^{\langle \alpha, H(g) \rangle}$. The value at $s = 0$, if it were to exist, would be the non-regularized orbital integral of γ. This value does exist locally, i.e. if we replace $\mathbb{A}$ by F_v, but it does not exist globally. Instead, the contribution of γ is the "finite part" of this expression at $s = 0$, i.e. the constant term of the Laurent expansion at $s = 0$.

Let us look a bit more closely. Consider a finite place $v = p$ and take as a test function f_v the characteristic function of K_v, and assume that $x \in O_{F_v} = \mathbb{Z}_p$. Then a simple computation shows that the usual orbital integral equals $L(1, F_v) = \zeta_p(1) = (1 - 1/p)^{-1}$. This not only explains why the global unipotent orbital integral doesn't converge, but also gives a way to re-express its regularization. Namely, define

$$\theta[x, f](s) = L(1 + s, F)^{-1} \int_{Z(\mathbb{A})U_0(\mathbb{A})\setminus G(\mathbb{A})} f(g^{-1}\gamma g)\beta(g)^{-s} dg.$$

The local computation we just alluded to shows that this function is analytic at $s = 0$. Therefore the constant term we are looking for equals

$$\lambda_{-1}\theta[x, f]'(0) + \lambda_0\theta[x, f](0),$$

where

$$\lambda_{-1}(s-1)^{-1} + \lambda_0 + \dots$$

is the Laurent expansion at $s = 1$ of $L(s, F)$. In our $F = \mathbb{Q}$ case we have $\lambda_{-1} = 1$ and λ_0 is the Euler-Mascheroni constant, usually denoted by γ.

4. The product of the set of unipotent elements with the non-trivial central element constitutes a single equivalence class. The contributions are the same as for the previous point, since none of the arguments are affected

by a central translation of the element γ. Thus, for $a \in \{\pm 1\}$ and $x \in F^\times$ the contribution of the conjugacy class of

$$\gamma = a \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

is given by

$$\lambda_{-1}\theta[a,x,f]'(0) + \lambda_0 \theta[a,x,f](0),$$

where

$$\theta[a,x,f](s) = L(1+s,F)^{-1} \int_{Z(\mathbb{A})U_0(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\beta(g)^{-s} dg.$$

Collecting all terms we obtain the following expression for $J_{\text{geom}}(f)$:

$$\begin{array}{rcl} (cnt) & & f(1) + f(-1) \\ (ell) & + & \displaystyle\sum_{\gamma} \text{vol}(E^1 \setminus \mathbb{A}_E^1) \int_{S(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g) dg \\ (hyp) & - & \displaystyle\frac{1}{2}\text{vol}(F^\times \setminus \mathbb{A}_F^1) \sum_{\gamma \in F^\times \smallsetminus \{\pm 1\}} \sum_v \frac{A_1(\gamma_v, f_v)}{L(1,F_v)} \prod_{w \neq v} \frac{F(\gamma_w, f_w)}{L(1,F_w)} \\ (uni) & + & \displaystyle\sum_{a \in \{\pm 1\}} \sum_{x \in F^{\times,2}\backslash F^\times} \lambda_{-1}\theta[a,x,f]'(0) + \lambda_0\theta[a,x,f](0) \end{array}$$

In (ell) the sum runs over the set of elliptic conjugacy classes in $\mathrm{SL}_2(F)$. In (hyp) the element $\gamma \in F^\times$ is identified with the diagonal matrix γ with diagonal entries γ and γ^{-1}. The factor $1/2$ comes from the elements γ and γ^{-1} leading to the same conjugacy class. In (uni) we have collected the contributions of the unipotent classes and their central translates. Every unipotent class has a representative of the form

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

and two such matrices for $x, x' \in F^\times$ represent the same class if and only if $x^{-1}x' \in F^{\times,2}$.

The reader will recognize these four terms as the terms (5.1)-(5.4) of [LL79]. The reader will also recognize the terms in (ell) and (hyp) as the formula of [Art05, Theorem 11.2]. On the other hand, the part for $a = 1$ of (uni) matches the formula of [Art05, Theorem 19.1]. The latter formula in the special case of $G = \mathrm{SL}_2$ is the sum of three terms

$$\frac{1}{2}\text{vol}(F^\times/\mathbb{A}^1)J_T(1,f) + \text{vol}(G(F)/G(\mathbb{A}))f(1) + \sum_u a^G(S,u)J_G(u,f),$$

where a finite set of places S has been fixed and the sum over u runs over the $G(\mathbb{A}_S)$-conjugacy classes of unipotent elements of $G(F)$. Reconciling the two formulas is less straightforward. We give a brief indication. Let $Z(s) = Z[x,f](s) = \theta[1,x,f](s)L(s,F)$, so that the contribution of the unipotent matrix γ with upper right corner x is given by the finite part of $Z[x,f]$ at $s = 0$. This finite part can be expressed as the derivative at $s = 0$ of $sZ(s)$. We have the product decomposition $Z(s) = Z_S(s) \cdot Z^S(s)$, where the first factor contains all places in S, and the second factor all places away from S. We are assuming that away from S the test function f is the unit in the unramified Hecke algebra. Then the contribution becomes $Z'_S(0) \cdot [sZ^S(s)]|_{s=0} + Z_S(0) \cdot [sZ^S(s)]'|_{s=0}$. Now $Z_S(0)$ is the usual orbital integral $J_G(u,f)$ over the conjugacy class in $G(\mathbb{A}_S)$ of the unipotent element γ. This integral converges because S is finite. The term $[sZ^S(s)]'|_{s=0}$ is the factor $a^G(S,u)$. It does not depend on f. The term $[sZ^S(s)]|_{s=0}$ can be recognized as the volume factor vol$(F^\times \setminus \mathbb{A}^1)$. Finally, the term $Z'_S(0)$ is the distribution $J_T(1,f)$. For more details on these calculations we refer the reader to the work of Chaudouard [Cha17] (esp. Theorem 8.5.1 and §12.7), [Cha18] (esp. Theorems 5.1.1 and 6.2.1).

6.3.2 The spectral side

In the spectral side of the non-invariant trace formula the sum runs over the set χ of *cuspidal automorphic data*. These are pairs (M,σ) consisting of a Levi subgroup M and a cuspidal automorphic representation σ of $M(\mathbb{A})^1$. Two pairs are equivalent if they are conjugate by $G(F)$. For $G = \mathrm{SL}_2$ there are two kinds of cuspidal automorphic data –

1. pairs (G,σ), where σ is a cuspidal automorphic representation of $G(\mathbb{A})$
2. pairs (T,ξ), where T is the standard diagonal maximal torus and $\eta : \mathbb{A}^1/F^\times \to \mathbb{C}^\times$ is a continuous character. The pair (T,η^{-1}) is equivalent.

Most of the cuspidal data of the second kind are what Arthur calls "unramified" (this may be confusing given the concept of an unramified representation; the term "regular" might be a good alternative), which in our setting simply means $\eta^2 \neq 1$.

We can arrange the contributions of the cuspidal data to the spectral side of the trace formula into four spectral terms. The first term, which we call $r(f)$, is the trace of f on the discrete spectrum of G. It contains the contributions of all cuspidal data of the form (G,σ) as well as the contribution of the trivial representation.

The next term is also of discrete nature, but it contains contributions from the continuous spectrum. It is

$$(cts.0) \qquad \frac{1}{4} \sum_{\eta: F^\times \setminus \mathbb{A}^\times \to \{\pm 1\}} \mathrm{tr}(M(\eta)\rho(f,\eta)).$$

Here the sum runs over all quadratic characters η of $F^\times \setminus \mathbb{A}^\times$, including the trivial character, and $\rho(\eta)$ is the (normalized) parabolic induction of η, and $M(\eta)$

is the un-normalized intertwining operator $\rho(\eta) \to \rho(\eta^{-1}) = \rho(\eta)$ defined by

$$M(\eta)\varphi(g) = \int_{U_0(\mathbb{A})} \varphi(wng)dn,$$

where $w \in G(F)$ is any lift of the Weyl element and φ is a generic element of the induced representation. The measure dn on $U_0(\mathbb{A}) = \mathbb{A}$ gives $\mathbb{A}/F$ volume 1.

The next two terms are also contributions from the continuous spectrum, and the terms themselves are continuous. Let D^0 denote the set of unitary characters η of $F^\times \setminus \mathbb{A}^\times$. It is a disjoint union of connected components, each component being a torsor for $i\mathbb{R}$ defined by

$$\eta_s(a) = \eta(a)|a|^{2s}, \qquad \eta \in D^0, s \in i\mathbb{R}.$$

This torsor structure transfers the Lebesgue measure on $\mathbb{R}$ to each component. The two continuous terms are

$$(cts.1) \qquad -\frac{1}{4\pi}\int_{D^0} \frac{L(1,\eta^{-1})}{L(1,\eta)} \mathrm{tr}(\rho(f,\eta))d\eta$$

and

$$(cts.2) \qquad -\sum_v \frac{1}{4\pi}\int_{D^0} \mathrm{tr} R^{-1}(\eta_v)R'(\eta_v)\rho(f_v,\eta_v)\prod_{w\neq v}\mathrm{tr}(\rho(f_w,\eta_w))d\eta.$$

Here we are using the normalized intertwining operator defined on page 745 of [LL79]

$$R(\eta)\varphi = \epsilon(0,\eta)\frac{L(1,\eta)}{L(0,\eta)}M(\eta)\varphi$$

and its derivative. This derivative is taken in the variable s that parameterizes η.

The reader will again recognize the terms $(cts.0)$, $(cts.1)$, and $(cts.2)$ as the terms of [LL79, §5] denoted by (5.5), (5.6), and the term that appears at the top of page 755 of [LL79] without a number. We have however switched their signs, because in our discussion they will appear on the same side of the equality as the term $r(f)$, while in the discussion of [LL79] they appear on the opposite side of the equality, together with the geometric terms, so as to give an expression for $r(f)$.

The terms $(cts.1)$ and $(cts.2)$ match the contribution of the the unramified (i.e. regular) data as given in [Art05, Theorem 15.4], the translation being explained in [Gel96, §V,Proposition 1.3]. The term $(cts.0)$ comes in addition when considering the non-unramified data, i.e. the quadratic characters.

6.4 Stabilization for $\mathbf{SL}_2$

6.4.1 The elliptic regular term

Our discussion of the elliptic regular term from §2 resulted in the following equation (2.3.1) {eq:jgeom}

$$\mathrm{TF}^G_{\text{reg.ell}}(f) = \mathrm{STF}^G_{\text{reg.ell}}(f) + \frac{1}{4}\sum_T \mathrm{STF}^T_{\text{G-reg}}(f^T),$$

where the second sum runs over the set of elliptic maximal tori of G up to stable conjugacy, i.e. the set of quadratic extensions E/F.

The term $\mathrm{STF}^G_{\text{reg.ell}}$ is stable by design. The term $\mathrm{STF}^T_{\text{G-reg}}(f^T)$ is not a stable distribution of f. Furthermore, in its current form it is not of much use. In order for it to be useful, it must be completed by adding the missing contributions of the elements $+1, -1 \in T(F)$.

Fix T. The contribution we are missing is

$$\mathrm{vol}(T(F) \setminus T(\mathbb{A}))(f^T(1) + f^T(-1)).$$

Let us recall from (2.5.1) that for G-regular $\gamma_v \in T(F_v)$ we have defined

$$f^T_v(\gamma_v) = \lambda(E_v/F_v, \psi_v)\kappa_v\left(\frac{\gamma_v - \bar{\gamma}_v}{\eta_v}\right)|\gamma_v - \bar{\gamma}_v|_v \cdot f^T_{v,\text{naive}}(\gamma_v),$$

and

$$f^T_{v,\text{naive}}(\gamma_v) = O_{\gamma_v}(f_v) - O_{\bar{\gamma}_v}(f_v),$$

where $\gamma_v \in E^1_v = T(F_v)$ is embedded into G as (2.2.1). Recall also that f^T_v is smooth at $\gamma_v = +1, -1$.

To understand what is going on we need to look more carefully at the normalized orbital integral

$$|\gamma_v - \bar{\gamma}_v|_v O_{\gamma_v}(f_v)$$

and its behavior around the singular points $\gamma_v = 1$ and $\gamma_v = -1$. We shall do this by example in the case $v = \infty$ and $\gamma_v = 1$. A direct computation then shows that

$$\lim_{\theta\downarrow 0} |e^{i\theta} - e^{-i\theta}| O_{e^{i\theta}}(f) = \int_0^\infty \int_K f\left(k\begin{bmatrix}1 & u\\ 0 & 1\end{bmatrix}k^{-1}\right) dk du$$

up to a universal constant depending on measures. A simple substitution shows

$$\lim_{\theta\uparrow 0} |e^{i\theta} - e^{-i\theta}| O_{e^{i\theta}}(f) = \int_{-\infty}^0 \int_K f\left(k\begin{bmatrix}1 & u\\ 0 & 1\end{bmatrix}k^{-1}\right) dk du.$$

Putting these together we see

$$\begin{aligned}\lim_{\theta\downarrow 0} |e^{i\theta} - e^{-i\theta}| \cdot f^T_{\infty,\text{naive}}(e^{i\theta}) &= \int_{-\infty}^{+\infty} \mathrm{sgn}(u) \int_K f\left(k\begin{bmatrix}1 & u\\ 0 & 1\end{bmatrix}k^{-1}\right) dk du \\ &= \int\limits_{U_0(F_v)\backslash \mathrm{PGL}_2(F_v)} \mathrm{sgn}(\det(g)) f\left(g^{-1}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}g\right) dg.\end{aligned}$$

Note that $\kappa_\infty = \mathrm{sgn}$, so what we have on the right is a *unipotent κ-orbital integral*. This works for any local field; in the p-adic case one has the Shalika germ expansion [Kot05, Theorem 6.1]. We record more precisely

$$\lim_{e_v \to a} f_v^T(e_v) = \lambda(E_v/F_v, \psi_v) L(1, \kappa_v)^{-1} \int\limits_{U_0(F_v)\backslash \mathrm{PGL}_2(F_v)} \kappa_v(\det(g)) f\left(g^{-1} n_a g\right) dg,$$

where $L(1, \kappa_v)$ comes from measure considerations and for $a = +1, -1$ we have

$$n_a = a \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The upshot: *The regular elliptic κ-orbital integral degenerates towards the singular elements $a = +1, -1$ to a unipotent κ-orbital integral.*

This suggests that the terms we are missing from STF^T are hiding in the unipotent part of the geometric side of TF^G. In order to see this one first performs a sequence of manipulations which allow us to replace the group $G(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A})$ by the group

$$G' = \{g \in \mathrm{GL}_2(\mathbb{A}) | \det(g) \in \mathbb{A}^{\times,2} F^\times\}.$$

and thus write the unipotent contribution to the trace formula for G (i.e the term (uni) on page 59) as the constant term of the Laurent expansion at $s = 0$ of

$$\int_{Z' U_0(\mathbb{A})\backslash G'} f(g^{-1} a n_a g) \beta(g)^{-s} dg.$$

We will not repeat these computations, as they are fairly straightforward yet not particularly enlightening. They are contained in pages 758-759 of [LL79]. Note that the summation over $F^\times/F^{\times,2}$ has disappeared, because the unipotent elements in $\mathrm{SL}_2(F)$ form a single conjugacy class under G'.

The integrand as a function of g is well-defined for any $g \in \mathrm{GL}_2(\mathbb{A})$. The quotient $\mathrm{GL}_2(\mathbb{A})/G'$ is isomorphic via the determinant to the compact abelian group $\mathbb{A}^\times/F^\times \mathbb{A}^{\times,2}$ and performing Fourier inversion on that group we can write the above integral as

$$\sum_\kappa \int_{U_0(\mathbb{A})\backslash \mathrm{PGL}_2(\mathbb{A})} \kappa(\det(g)) f(g^{-1} n_a g) \beta(g)^{-s} dg. \qquad (6.4.1)$$ {eq:unikappa}

Consider one $\kappa : \mathbb{A}^\times/F^\times \mathbb{A}^{\times,2} \to \mathbb{C}^\times$. It corresponds via global class field theory to a quadratic extension E/F. At a place v where E/F is unramified and f_v is the unit in the unramified Hecke algebra one can compute explicitly

$$\int_{U_0(F_v)\backslash \mathrm{PGL}_2(F_v)} \kappa_v(\det(g_v)) f_v(g_v^{-1} n_a g_v) \beta(g_v)^{-s} dg = f_v(a) L(s+1, \kappa_v)$$

up to a volume factor, as we have already remarked. But the essential observation now is that the global function $L(s, \kappa)$ is analytic at $s = 1$ when $\kappa \neq 1$.

Therefore the constant term at $s = 0$ of the above global expression equals

$$L(1,\kappa)\prod_v L(1,\kappa_v)^{-1}\int_{U_0(F_v)\backslash \mathrm{PGL}_2(F_v)} \kappa_v(\det(g_v))f_v(g_v^{-1}n_a g_v)dg$$

which equals by the previous computation

$$L(1,\kappa)f^T(a).$$

Note finally $L(1,\kappa) = \mathrm{vol}(T(F)\setminus T(\mathbb{A}))$ for T given by the quadratic extension E/F.

We have thus seen that the summand for $\kappa \neq 1$ in (6.4.1) matches precisely the contribution of the singular element a in $\mathrm{STF}^T(f^T)$ that is missing in $\mathrm{STF}^T_{G\text{-reg}}(f^T)$. This leaves for now as unaccounted for the summand for $\kappa = 1$, which we will denote by $(uni)_{\kappa=1}$.

6.4.2 The discrete contribution of the continuous spectrum

{subsub:disccont}

We have now arrived at the equality of the geometric distribution

$$(cnt) + \mathrm{STF}^G_{\text{reg.ell}}(f) + \frac{1}{4}\sum_T \mathrm{STF}^T(f^T) + (hyp) + (uni)_{\kappa=1}$$

and the spectral distribution that has not yet changed

$$r(f) + (cts.0) + (cts.1) + (cts.2).$$

As we have already remarked the distribution $\mathrm{STF}^T(f^T)$ is not stable. Being the full trace formula for the group T, it has both a geometric and a spectral interpretation. Namely, as discussed in §6.2, the geometric side is

$$\sum_{\gamma\in T(F)} \mathrm{vol}(T(F)\setminus T(\mathbb{A}))f^T(\gamma)$$

while the spectral side is

$$\sum_{\eta:T(F)\backslash T(\mathbb{A})\to\mathbb{C}^\times} \eta(f^T).$$

We have already used the geometric side in order to extract $\mathrm{STF}^T(f^T)$ from the geometric side of the trace formula for G. We now use the spectral side of $\mathrm{STF}^T(f^T)$ and relate it to the spectral side of the trace formula for G.

For this we recall the term $(cts.0)$ from page 60, which is the sum over all quadratic characters $\eta : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ of

$$\frac{1}{4}\mathrm{tr}(M(\eta)\rho(f,\eta)).$$

It is not hard to see that the distribution tr$\rho(f, \eta)$ is stable for any character η, quadratic or not. Indeed, $\rho(\eta)$ is the restriction to $\mathrm{SL}_2(\mathbb{A})$ of a representation of $\mathrm{GL}_2(\mathbb{A})$. However, the occurrence of the operator $M(\eta)$ might break this stability, because this operator might act differently on different constituents of this restriction.

In order to study this we recall the normalized intertwining operator

$$R(\eta) = \epsilon(0, \eta) \frac{L(1, \eta)}{L(0, \eta)} M(\eta).$$

Using the functional equation $L(V, s) = \epsilon(V, s) L(V^*, 1 - s)$ of the Artin L-function we can rewrite this as

$$R(\eta) = \frac{L(1, \eta)}{L(1, \eta^{-1})} M(\eta)$$

so that the term we are studying is

$$\frac{1}{4} \frac{L(1, \eta^{-1})}{L(1, \eta)} \mathrm{tr}(R(\eta) \rho(\eta, f)).$$

The constant $L(1, \eta^{-1})/L(1, \eta)$ is evidently equal to 1 when $\eta \neq 1$, because we are considering the case $\eta = \eta^{-1}$. When $\eta = 1$ neither the numerator nor the denominator is defined, because then the L-function has a pole at 1. The quotient is defined as the limit of $L(1, \eta_s)/L(1, \eta_s^{-1})$ as $s \to 0$, where η_s is the unramified character $| - |^s$ for a complex variable s. It is an amusing exercise to compute that this limit is -1.

More interesting however is to compute the action of the normalized intertwining operator $R(\eta)$ on the induced representation $\rho(\eta)$. This is a local computation because both $R(\eta) = \otimes_v R(\eta_v)$ and $\rho(\eta) = \otimes_v \rho(\eta_v)$ factor as tensor products of local terms. Here $R(\eta_v)$ is the operator $\rho(\eta_v) \to \rho(\eta_v)$ defined by

$$\epsilon(0, \eta_v, \psi_v) \frac{L(1, \eta_v)}{L(0, \eta_v)} M(\eta_v), \tag{6.4.2}$$ {eq:locnormintop}

where $M(\eta_v)$ is the un-normalized local intertwining operator. One has to be a bit careful here. Both the local operator $M(\eta_v)$ and the global operator $M(\eta)$ are defined by integrals that generally diverge. They converge absolutely only when η is replaced by a twist $\eta \cdot | - |^s$ for a sufficiently large positive real number s. The resulting (operator valued) function in s has analytic continuation to $s = 0$ in the global case and its value at $s = 0$ is the definition of $M(\eta)$. In the local case the resulting function in s may have a pole at $s = 0$; therefore $M(\eta_v)$ is not defined. But this pole is canceled by the corresponding zero of the normalizing factor, so that the function in s defining $R(\eta_v)$ has analytic continuation to $s = 0$, and its value at $s = 0$ is the definition of $R(\eta_v)$. Thus (6.4.2) is only a valid equation for η_v replaced by $\eta_v \cdot | - |^s$ with $s >> 0$, and in general is to be taken symbolically.

Besides being actually defined, the operator $R(\eta_v)$ has another virtue – it satisfies $R(\eta_v)^2 = 1$. Since for $\eta_v \neq 1$ the representation $\rho(\eta_v)$ decomposes as a direct sum of two inequivalent irreducible representations, $R(\eta_v)$ must act on each of them by multiplication by a scalar that is either 1 or -1. In order to distinguish between the two possibilities we note that $R(\eta_v)$ depends on the additive character $\psi_v : F_v \to \mathbb{C}^\times$ that is the local component of ψ at the place v, because the local ϵ-factor does. At the same time, ψ_v also distinguishes one of the two constituents of $\rho(\eta_v)$, because exactly one of them is generic (i.e. has a Whittaker model) for the Whittaker datum determined by the standard pinning of SL_2 and the additive character ψ_v.

Lemma 6.4.1. *1. In the case $\eta_v = 1$ the operator $R(\eta_v)$ acts as the identity.* {lem:iop}

2. In the case $\eta_v \neq 1$ the operator $R(\eta_v)$ acts by the scalar $+1$ on the unique constituent of $\rho(\eta_v)$ that is generic for ψ_v, and by the scalar -1 on the other constituent.

This is proved in [LL79, Lemmas 3.5,3.6]. We will not reproduce the proof. In the notation of loc. cit., the constituent of $\rho(\eta_v)$ that is generic for ψ_v is denoted by π^+ and the other constituent by π^-.

This lemma allows us to connect the term $(cts.0)$ on the spectral side of the trace formula of SL_2 with the term $\frac{1}{4}\sum_T \mathrm{STF}^T(f^T)$ on its partially stabilized geometric side. The summand of $(cts.0)$ corresponding to the trivial character $\eta = 1$ is the stable distribution

$$-\frac{1}{4}\mathrm{tr}(\rho(f, \eta)),$$

which we shall denote by $(cts.0)_{\eta=1}$. On the other hand, consider a summand corresponding to a non-trivial character $1 \neq \eta : \mathbb{A}^\times/F^\times \to \{\pm 1\}$. This character corresponds to a quadratic extension E/F by global class field theory and hence to an endoscopic torus T for G. Let $\theta : T(\mathbb{A})/T(F) \to \mathbb{C}^\times$ be the trivial character. It corresponds to the trivial Langlands parameter $W_F \to {}^LT$, which, via the canonical embedding ${}^LT \to {}^LG$, gives a (non-trivial) Langlands parameter $W_F \to {}^LG$, which in turn corresponds to an L-packet for G. Despite coming from an elliptic endoscopic group, this packet is not discrete. Its local components corresponds to the irreducible constituents of the reducible principal series representation $\rho(\eta_v)$. The above lemma shows that the summand of $(cts.0)$ corresponding to η has v-component equal to

$$\pi_v^+(f_v) - \pi_v^-(f_v).$$

The endoscopic character identity (2.5.2) shows that this equals

$$\theta_v(f_v^T).$$

As is emphasized in [LL79], having the correct normalization of $R(\eta_v)$ is crucial here. This is an example of what Arthur calls a "local intertwining relation". Globally we obtain

$$\mathrm{tr}(R(\eta)\rho(\eta, f)) = \theta(f^T).$$

We thus see that the contribution to $(cts.0)$ of a non-trivial quadratic character $\eta : \mathbb{A}^\times/F^\times \to \{\pm 1\}$ matches precisely the contribution to the spectral side of the trace formula for the endoscopic group T determined by η of the trivial character θ of $T(\mathbb{A})$.

6.4.3 Final stabilization

We now have the following geometric side

$$(cnt) + \mathrm{STF}^G_{\text{reg.ell}}(f) + (hyp) + (uni)_{\kappa=1}$$

equal the following spectral side

$$r(f) + (cts.0)_{\eta=1} - \frac{1}{4}\sum_T \mathrm{STF}^T(f^T)_{\theta\neq 1} + (cts.1) + (cts.2).$$

The terms (cnt) and $\mathrm{STF}^G_{\text{reg.ell}}(f)$ on the geometric side are stable. The terms $(cts.0)_{\eta=1}$ and $(cts.1)$ on the spectral side are stable. The final step is to show that

$$(hyp) + (uni)_{\kappa=1} - (cts.2)$$

is also stable. Assuming this, the "stable trace formula" for $G = \mathrm{SL}_2$ has the following meaning:

1. The claim that $r(f) - \frac{1}{4}\sum_T \mathrm{STF}^T(f^T)_{\theta\neq 1}$ is a stable distribution.
2. The equality of that stable distribution with

   $$(cnt) + \mathrm{STF}^G_{\text{reg.ell}}(f) + \{(hyp) + (uni)_{\kappa=1} - (cts.2)\} - (cts.1) - (cts.0)_{\eta=1},$$

   where each summand is stable.

For the remainder of this subsection we will discuss the computation showing that $(hyp) + (uni)_{\kappa=1} - (cts.2)$ is a stable distribution. In the following subsection we will make some comments on the result.

Given $g \in \mathrm{GL}_2(\mathbb{A})$ we want to show that

$$(hyp)^g - (hyp) + (uni)^g_{\kappa=1} - (uni)_{\kappa=1} - (cts.2)^g + (cts.2) = 0.$$

The three individual differences are computed locally. We will not reproduce the local computations, we will just quote them.

We begin with the term $(cts.2)$ and recall that it is

$$\sum_v \frac{1}{4\pi}\int_{D^0} \mathrm{tr}R^{-1}(\eta_v)R'(\eta_v)\rho(f_v,\eta_v)\prod_{w\neq v}\mathrm{tr}\rho(f_w,\eta_w)d\eta.$$

We want to compute the difference of this term evaluated once at f^g and once at f. The product over $w \neq v$ is a stable distribution, so that difference is

$$\sum_v \frac{1}{4\pi}\int_{D^0}[\mathrm{tr}R^{-1}(\eta_v)R'(\eta_v)\rho(f^g_v,\eta_v) - \mathrm{tr}R^{-1}(\eta_v)R'(\eta_v)\rho(f_v,\eta_v)]\prod_{w\neq v}\mathrm{tr}\rho(f_w,\eta_w)d\eta.$$

The difference in the square brackets is computed locally [LL79, Lemma 3.4] and this term becomes

$$\sum_v \frac{1}{4\pi}\int_{D^0} [\mathrm{tr}\rho(f_v,\eta_v)N(g_v) + \mathrm{tr}\rho(f_v,\eta_v^{-1})N(g_v)] \prod_{w\neq v} \mathrm{tr}\rho(f_w,\eta_w)d\eta.$$

Here $N(g_v)$ is the operator on the space $\rho(\eta_v)$ defined as follows. For any $k_v \in K_v$ decompose $k_v g_v = n_v a_v l_v$ according to the Iwasawa decomposition and consider $\beta(a_v) = |a_v|_v^2$, where we identify the diagonal torus with $F_v^\times$. Then $N(g_v)$ is multiplication by $\ln \beta(a_v)$.

Since the integral will meet both η_v and η_v^{-1} we can split the two summands, make a substitution, and arrive at

$$\sum_v \frac{1}{2\pi}\int_{D^0} \mathrm{tr}\rho(f_v,\eta_v)N(g_v) \prod_{w\neq v} \mathrm{tr}\rho(f_w,\eta_w)d\eta.$$

Following Labesse-Langlands we introduce the notation

$$H_v(\eta_v) = \mathrm{tr}\rho(f_v,\eta_v)N(g_v) \qquad \text{and} \qquad I_w(\eta_w) = \mathrm{tr}\rho(f_w,\eta_w).$$

Thus we arrive at

$$\boxed{(cts.2)^g - (cts.2) = \sum_v \frac{1}{2\pi}\int_{D^0} H_v(\eta_v) \prod_{w\neq v} I_w(\eta_w)d\eta.}$$

The next term we turn to is (hyp), given by

$$-\frac{\lambda}{2}\sum_v \sum_{\gamma\in F^\times\setminus\{\pm 1\}} \frac{A_1(\gamma,f_v)}{L(1,F_v)} \prod_{w\neq v} \frac{F(\gamma,f_w)}{L(1,F_w)}.$$

The unweighted hyperbolic orbital integral $F(\gamma,f_w)$ is stable, but its weighted brother $A_1(\gamma,f_v)$ is unstable. We get

$$(hyp)^g - (hyp) = -\frac{\lambda}{2}\sum_v \sum_{\gamma\in F^\times\setminus\{\pm 1\}} \frac{A_1(\gamma,f_v^g) - A_1(\gamma,f_v)}{L(1,F_v)} \prod_{w\neq v} \frac{F(\gamma,f_w)}{L(1,F_w)}.$$

Again there is a local computation for the instability of the local weighted hyperbolic orbital integral [LL79, Lemma 3.2] which leads to

$$(hyp)^g - (hyp) = -\frac{\lambda}{2}\sum_v \sum_\gamma{}' \frac{\check{H}(\gamma_v) + \check{H}(w\gamma_v)}{L(1,F_v)} \prod_{w\neq v} \frac{F(\gamma,f_w)}{L(1,F_w)}.$$

The summation index γ runs over $F^\times$ and the prime on the sum indicates that $\{\pm 1\}$ are omitted form the sum. Again since both γ and $w\gamma$ show up in the summation index we can combine the two terms and obtain

$$(hyp)^g - (hyp) = -\lambda\sum_v \sum_\gamma{}' \frac{\check{H}(\gamma_v)}{L(1,F_v)} \prod_{w\neq v} \frac{F(\gamma,f_w)}{L(1,F_w)}.$$

Since the Fourier transform of $I_w(\eta_w)$ is $F(\gamma_w, f_w)$, all in all we obtain

$$\boxed{(hyp)^g - (hyp) = -\lambda \sum_v \sum_\gamma{}' \frac{\check{H}(\gamma_v)}{L(1, F_v)} \prod_{w \neq v} \frac{\check{I}(\gamma_w)}{L(1, F_w)}.}$$

Finally we come to $(uni)_{\kappa=1}$, whose a-part is the constant term of the Laurent expansion at $s = 0$ of

$$\int_{I_F U_0(\mathbb{A}) \backslash \mathrm{GL}_2(\mathbb{A})} f\left(g^{-1} a \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} g\right) \beta(g)^{-s} dg.$$

Define the global function

$$\theta(a, s, f) = \frac{1}{L(1+s, F)} \int_{I_F U_0(\mathbb{A}) \backslash \mathrm{GL}_2(\mathbb{A})} f\left(g^{-1} a \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} g\right) \beta(g)^{-s} dg.$$

It is analytic, and the constant term of the Laurent expansion at $s = 0$ we are computing equals $\lambda_{-1}\theta'(a, 0, f) + \lambda_0 \theta(a, 0, f)$.

But we are not really computing $(uni)_{\kappa=1}$. Rather, we are computing $(uni)^g_{\kappa=1} - (uni)_{\kappa=1}$. Its a-summand ends up being

$$\lambda_{-1}(\theta'(a, 0, f^g) - \theta'(a, 0, f)) + \lambda_0(\theta(a, 0, f^g) - \theta(a, 0, f)).$$

It is evident that $\theta(a, s, f)$ is stable, so the second difference vanishes. On the other hand, since the global function $\theta(a, s, f)$ is the product $\prod_v \theta(a_v, s, f_v)$ of local functions, its derivative becomes a sum, so we end up with

$$(uni)^g_{\kappa=1} - (uni)_{\kappa=1} = \lambda_{-1} \sum_v (\theta'(a_v, 0, f_v^g) - \theta'(a_v, 0, f_v)) \prod_{w \neq v} \theta(a_w, 0, f_w),$$

which again uses the stability of $\theta(a_w, 0, f_w)$. A local computation [LL79, Lemma 3.3] expresses the difference of derivatives of the local θ map and we get

$$(uni)^g_{\kappa=1} - (uni)_{\kappa=1} = -\lambda_{-1} \sum_a \sum_v \frac{\check{H}(a_v)}{L(1, 1_v)} \prod_{w \neq v} \theta(a_w, 0, f_w).$$

At the same time by definition $\theta(a_w, 0, f_v) = \frac{F(a, f_w)}{L(1, 1_w)}$. We conclude

$$\boxed{(uni)^g_{\kappa=1} - (uni)_{\kappa=1} = \lambda_{-1} \sum_a \sum_v \frac{\check{H}(a_v)}{L(1, 1_v)} \prod_{w \neq v} \frac{\check{I}(a_w)}{L(1, 1_w)}.}$$

We are now ready to show that the three boxed differences cancel out. Adding the boxed differences for (hyp) and (uni) we arrive at

$$\lambda_{-1} \sum_\gamma \sum_v \frac{\check{H}(\gamma_v)}{L(1, 1_v)} \prod_{w \neq v} \frac{\check{I}(\gamma_w)}{L(1, 1_w)}.$$

where γ now runs over all elements of the diagonal torus – the regular ones coming from (hyp) and the singular central ones from (uni).

We now apply the Poisson summation formula to see this matches the boxed formula for the difference for $(cts.2)$. More precisely, we consider the exact sequence

$$1 \to F^\times \to \mathbb{A}^\times \to \mathbb{A}^\times / F^\times \to 1$$

of locally compact abelian groups. The function

$$\gamma \mapsto \check{H}(\gamma_v) \prod_{w \neq v} \check{I}(\gamma_w)$$

is a function on all of $\mathbb{A}^\times$ and its Fourier transform is the function

$$\eta \mapsto H(\eta_v) \prod_{w \neq v} I(\eta_w).$$

In our case we are taking the counting measure on the discrete group $F^\times$. The proof of the stability of $(hyp) + (uni)_{\kappa=1} - (cts.2)$ is thus complete.

6.5 The stabilized trace formula for SL_2 and its spectral interpretation

We have shown that the distribution

$$r(f) - \frac{1}{4} \sum_T \mathrm{STF}^T(f^T)_{\theta \neq 1}$$

is stable. This is a distribution purely of spectral nature. We have furthermore obtained an alternative expression for this stable distribution, i.e. "a formula". That expression however is not of purely geometric nature. Rather, it is a mix of geometric and spectral terms. This is an example of a general phenomenon that occurs not just during the stabilization of the trace formula, but already in the process of making it invariant (this is a process that we skipped, as we passed directly from the non-invariant to the stable trace formula). The reason is that not only are individual terms in the geometric (resp. spectral) side of the trace formula non-invariant, or non-stable, but rather the entire geometric (resp. spectral) side is non-invariant and non-stable in general. This makes it necessary to combine geometric and spectral terms in order to cancel out the failure of invariance and stability. The goal is to do this in a way that is as minimal as possible, so as to retain as much as possible of the structure of each side.

We can express the stabilization of the trace formula slightly differently. Consider the distribution

$$I^G_{\mathrm{disc}}(f) := r(f) + (cts.0).$$

This is an invariant distribution on $G(\mathbb{A})$ of a spectral nature. It is discrete, not in the sense that it only receives contributions from the traces of discrete automorphic representations, but in the sense that it is a discrete sum of contributions of individual representations. This is true for both summands $r(f)$ and $(cts.0)$. In contrast, the terms $(cts.1)$ and $(cts.2)$ are given by integrals and are of continuous nature. Thus I^G_{disc} is the discrete part of the spectral side of the trace formula. Consider further the stable distribution

$$\mathrm{STF}^G_{\text{disc}}(f) := r(f) - \frac{1}{4}\sum_T \mathrm{STF}^T(f^T)_{\theta\neq 1} + (cts.0)_{\eta=1},$$

which by the same token can be seen as the discrete part of the stabilized trace formula for G. By reversing the computation we did in §6.4.2 we can also write it as

$$\mathrm{STF}^G_{\text{disc}}(f) = r(f) - \frac{1}{4}\sum_T \mathrm{STF}^T(f^T) + (cts.0).$$

The stabilization of the trace formula can be written in this notation as

$$\mathrm{STF}^G_{\text{disc}}(f) = I^G_{\text{disc}}(f) - \frac{1}{4}\sum_T \mathrm{STF}^T(f^T) = I^G_{\text{disc}}(f) - \sum_{H\neq G}\iota(G,H)\,\mathrm{STF}^H(f^H),$$

and conversely

$$I^G_{\text{disc}}(f) = \mathrm{STF}^G_{\text{disc}}(f) + \sum_{H\neq G}\iota(G,H)\,\mathrm{STF}^H(f^H) = \sum_H \iota(G,H)\,\mathrm{STF}^H_{\text{disc}}(f^H). \tag{6.5.1}$$

{eq:stabid}

This is the stabilization identity for the discrete part of the trace formula for $G = \mathrm{SL}_2$, i.e. [Art05, Corollary 29.10]. Note that $\mathrm{STF}^T = \mathrm{STF}^T_{\text{disc}}$ because the spectral side of the trace formula for an anisotropic torus is a discrete sum.

The stabilization process derived an expression for $\mathrm{STF}^G_{\text{disc}}(f)$ in terms of a mixture of geometric and spectral terms. A further interpretation of $\mathrm{STF}^G_{\text{disc}}(f)$ is given by Arthur's stable multiplicity formula [Art13, §4.1] which we shall now review. The tempered discrete automorphic representations of $G(\mathbb{A})$ are expected to be partitioned into packets Π_φ indexed by irreducible L-homomorphisms $L_F \to \mathrm{PGL}_2(\mathbb{C})$, where L_F is the hypothetical Langlands group of F, an extension of the Weil group W_F by a complex connected pro-algebraic group. The group L_F should come equipped with morphisms $L_{F_v} \to L_F$, well-defined up to conjugation, and the packet Π_φ is defined as the restricted tensor product $\otimes_v \Pi_{\varphi_v}$ of the local L-packets corresponding to the restrictions φ_v of φ to L_{F_v}. An element of this restricted tensor product is an adelic representation $\otimes_v \pi_v$ with $\pi_v \in \Pi_{\varphi_v}$ and π_v unramified for almost all v. To a global parameter φ we associate the stable distribution $S\Theta_\varphi(f)$ which on factorizable test-functions $f = \prod_v f_v$ is defined by

$$S\Theta_\varphi(f) = \prod_v S\Theta_{\varphi_v}(f_v),$$

where $S\Theta_{\varphi_v}$ is the stable character of Definition 3.6.6. Since f_v is the unit in the unramified Hecke-algebra for almost all v and an unramified local L-packet has a unique member that is spherical for a fixed hyperspecial maximal compact subgroup, the term on the right is a finite sum of characters of irreducible representations once the test function f is fixed. The stable multiplicity formula states that the contribution of φ to the distribution $\mathrm{STF}^G_{\mathrm{disc}}(f)$ is

$$|S_\varphi|^{-1} S\Theta_\varphi(f), \tag{6.5.2}$$ {eq:stabmult}

where S_φ is the centralizer in $\widehat{G} = \mathrm{PGL}_2(\mathbb{C})$ of the image of φ.

There is a related conjectural formula describing the distribution I^G_{disc}. Granting the existence and properties of L_F consider a representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ occurring in the discrete spectrum that is not the trivial representation. The Ramanujan conjecture implies that all local components π_v are tempered. For any L-homomorphism $\varphi : L_F \to \mathrm{PGL}_2(\mathbb{C})$ we have by definition $\pi \in \Pi_\varphi$ if and only if $\pi_v \in \Pi_{\varphi_v}$ for all v. Given such φ we obtain embeddings $S_\varphi \to S_{\varphi_v}$ for all v. The representation π_v corresponds to $\rho_v \in \mathrm{Irr}(S_{\varphi_v})$ via Conjecture 3.6.7, which is known in the case of SL_2. For almost all v, the representation π_v is generic (this follows from the Casselman-Shalika formula, since it is unramified) and hence ρ_v is trivial. Therefore $\rho_{\varphi,\pi} := \otimes_v(\rho_v|_{S_\varphi})$ is a finite-dimensional representation of S_φ. Let $m(\varphi,\pi)$ be the multiplicity of the trivial representation in $\rho_{\varphi,\pi}$. Then it is conjectured [Kot84b, (12.3)] that the multiplicity $m(\pi)$ of π in the discrete automorphic spectrum of G is equal to

$$\sum_\varphi m(\varphi,\pi), \tag{6.5.3}$$ {eq:mult}

where φ runs over all discrete global parameters for which $\pi_v \in \Pi_{\varphi_v}$. Hence

$$r(f) = \mu(f) + \sum_\varphi \sum_{\pi\in\Pi_\varphi} m(\varphi,\pi)\pi(f),$$

where $\mu(f)$ is the contribution of the trivial representation, i.e. the integral of f over $G(\mathbb{A})$. This latter conjecture can be reformulated in terms that do not involve the hypothetical group L_F and then proved, as follows. Consider a given φ. The image of the connected pro-algebraic group contained in L_F is either trivial or all of $\mathrm{PGL}_2(\mathbb{C})$. Therefore if it is non-trivial then $S_\varphi = \{1\}$ and we expect $m(\varphi,\pi) = 1$. If it is trivial then φ is an L-homomorphism $W_F \to \mathrm{PGL}_2(\mathbb{C})$ and its image is a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$. All possibilities are known, as well as their centralizers. The only case with non-trivial centralizer is when φ factors through an endoscopic group H, which is an anisotropic torus. With this information, the conjectural multiplicity formula for $G = \mathrm{SL}_2$ says the following: Let π and π' be two representations of $G(\mathbb{A})$ such that for all v the local components π_v and π'_v belong to the same local L-packet Π_{φ_v} (which is defined non-conjecturally). Then $m(\pi) = m(\pi')$ unless there exists $\varphi : W_F \to {}^LH \to {}^LG$ whose localization at each v is the local parameter φ_v in whose packet π_v and π'_v lie. In that latter case the statement of the conjectural multiplicity formula (6.5.3) is unconditional and the claim is that it is true.

In [LL79, §6] both of these statements are proved using the stabilized trace formula. Instead of reproducing the argument we will outline how the stabilization identity (6.5.1), the stable multiplicity formula (6.5.2), and the multiplicity formula (6.5.3), all fit together. For this we observe that we can decompose each of the three terms of (6.5.1) as a sum over the parameters φ. The contribution of φ to the left-hand side of (6.5.1) is, according to (6.5.2), given by

$$|S_\varphi|^{-1} S\Theta_\varphi(f),$$

where $S\Theta_\varphi$ is the global stable distribution given as the product of the local stable distributions $S\Theta_{\varphi_v}$ of Definition 3.6.6. That is, for a factorizable test-function $f = \prod_v f_v$ we have

$$S\Theta_\varphi(f) = \prod_v S\Theta_{\varphi_v}(f_v) = \prod_v \sum_{\pi_v \in \Pi_{\varphi_v}} \pi_v(f_v) = \sum_{\pi \in \Pi_\varphi} \pi(f).$$

Even though the set Π_φ is infinite, for a fixed f the component f_v is the unit in the spherical Hecke algebra for almost all v, which means that $\pi(f) \neq 0$ implies that for those v the local component π_v is the unique unramified member of Π_{φ_v}; therefore the sum is finite for a fixed f.

The contribution of φ to $r(f)$ is given, according to (6.5.3), by

$$|S_\varphi|^{-1} \sum_{\pi \in \Pi_\varphi} \sum_{s \in S_\varphi} \rho_{\varphi,\pi}(s)\pi(f).$$

Again the sum over π is finite, so we can switch the order of the two sums. The contribution of $1 \in S_\varphi$ then matches the contribution of φ to $\mathrm{STF}^G_{\mathrm{disc}}(f)$. Therefore (6.5.1) reduces to

$$|S_\varphi|^{-1} \sum_{1 \neq s \in S_\varphi} \sum_{\pi \in \Pi_\varphi} \rho_{\varphi,\pi}(s)\pi(f) = \frac{1}{4} \sum_T \mathrm{STF}^T(f^T)_\varphi. \qquad (6.5.4)$$ {eq:stabid2}

The subscript φ on the right signifies that we have taken only the contributions of those characters of $T(\mathbb{A})/T(F)$ whose parameter, when composed with the embedding ${}^LT \to {}^LG$, are equivalent to φ.

When φ does not factor through any endoscopic T then S_φ is trivial so the sum on the left is empty, while the sum on the right is empty by assumption. Now assume that φ factors through some T. Then $S_\varphi \neq \{1\}$ and there are two possibilities: $S_\varphi \cong \mathbb{Z}/2\mathbb{Z}$ and $S_\varphi \cong (\mathbb{Z}/2\mathbb{Z})^2$, referred to as "type (a) and type (b)" in [LL79, §6].

Consider the case $S_\varphi \cong \mathbb{Z}/2\mathbb{Z}$. Then the endoscopic T through which φ factors is unique. The left side of (6.5.4) is

$$\frac{1}{2}(\pi^+(f) - \pi^-(f)),$$

while the right side is

$$\frac{1}{4}[\theta(f^T) + \theta^{-1}(f^T)],$$

because the only characters whose parameter composes to a parameter for G equivalent to φ is a pair of mutually inverse, but not equal to each other, characters. We have $\theta(f^T) = \pi^+(f) - \pi^-(f) = \theta^{-1}(f^T)$ by the global version of (2.5.2), which is an immediate consequence of the local version.

Consider now the case $S_\varphi = (\mathbb{Z}/2\mathbb{Z})^2$. Let $s_1, s_2, s_3 \in S_\varphi$ be the non-trivial elements. Each gives an endoscopic torus T_i, so there are exactly three endoscopic tori T_1,T_2,T_3 through which the parameter φ factors, giving rise to three characters θ_1, θ_2, θ_3, one on each T_i. These characters each have order 2. So the right side of (6.5.4) equals

$$\frac{1}{4}\sum_{i=1}^{3}\theta_i(f^{T_i}).$$

Applying the character identity of Conjecture 3.6.7, i.e. the generalization of (2.5.2) to the case where the local L-packets can have size 4, we see that the i-th summand equals $\sum_{\pi\in\Pi_\varphi}\rho(s_i)\pi(f)$, so the entire expression becomes

$$\frac{1}{4}\sum_{i=1}^{3}\sum_{\pi\in\Pi_\varphi}\rho_{\varphi,\pi}(s_i)\pi(f),$$

i.e. the left-hand side of (6.5.4).

For the analog of this computation in the setting of general reductive groups we refer the reader to [Kot84b, §12].

7 Stabilization of the full trace formula for inner forms of SL_2

7.1 Basic notation

As in the previous section F denotes a number field, which we take to be $\mathbb{Q}$ for simplicity. Let D be a quaternion algebra over F, i.e. a division algebra of degree 2. For each place v of F the algebra $D_v = D \otimes_F F_v$ is central simple of degree 2, hence either a quaternion algebra (we call v non-split) or the split algebra $M_2(F_v)$ (we call v split). All but finitely many places are split, and the number of non-split places is even.

We write D^1 for the subgroup of $D^\times$ consisting of those non-zero elements whose reduced norm is 1. Analogously we write D^1_v. In the split case we have $D^\times_v = \mathrm{GL}_2(F_v)$ and $D^1_v = \mathrm{SL}_2(F_v)$.

There is an inner form G of $G_0 = \mathrm{SL}_2$ with $G(F) = D^1$ and every inner form arises this way. The group G is anisotropic – it has no proper parabolic subgroups. Therefore every element of $G(F)$ is semi-simple and elliptic. The same is true for the group $G(F_v)$ at the non-split places v.

Again we choose a non-trivial character $\psi : \mathbb{A}/F \to \mathbb{C}^\times$.

7.2 The trace formula

The fact that G is anisotropic makes the trace formula very simple. The geometric side has two parts:

1. The central contribution, consisting of $f(1) + f(-1)$
2. The regular elliptic contribution, consisting of

$$\sum_\gamma \mathrm{vol}(E^\times \setminus \mathbb{A}_E^\times) O_\gamma(f),$$

   where the sum runs over the $G(F)$-conjugacy classes of regular elliptic elements, E/F is the quadratic extension generated by the eigenvalues of γ, and O_γ is the adelic orbital integral.

Note that not all quadratic extensions E/F will contribute elements γ. A necessary and sufficient condition is that E embeds into D, equivalently E/F is non-split at every non-split place v.

The spectral side has the single summand $r(f)$, i.e. the trace of f on the representation $L^2(G(F) \setminus G(\mathbb{A}))$. This representation decomposes as a discrete sum of irreducible representations with finite multiplicity.

7.3 Stabilization of the trace formula

We begin with the geometric side. The central contribution is already stable. The regular elliptic contribution was stabilized for general groups in §5. Specializing to the case at hand we obtain

$$\mathrm{TF}^G_{\text{reg.ell}}(f) = \mathrm{STF}^G_{\text{reg.ell}}(f) + \frac{1}{4}\sum_T \mathrm{STF}^T_{\text{G-reg}}(f^T). \qquad (7.3.1) \quad \{eq:tfq1\}$$

This looks exactly like what we encountered in the case of SL_2, i.e. (2.3.1). Indeed, the groups G and G_0 share the same endoscopic data, so the sum over T has the same indexing set. But note that f is now a function of $G(\mathbb{A})$, and not of $\mathrm{SL}_2(\mathbb{A})$, and f^T is not defined in the same way as it was for SL_2, i.e. not via the product over v of (2.5.1).

Instead, we follow the discussion of §4. Let $\xi : G_0 \to G$ be an inner twist and choose an element $z \in Z^1_{\text{bas}}(\mathcal{E}^{\text{rig}}_F, G_0)$ such that its image $\bar z \in Z^1(\Gamma, G_0/Z(G_0))$ satisfies $\xi^{-1}\sigma(\xi) = \mathrm{Ad}(\bar z_\sigma)$. Under the localization map we obtain at each place v an element $z_v \in Z^1(\mathcal{E}^{\text{rig}}_{F_v}, G_0)$ well-defined up to $B^1(\Gamma_v, Z(G_0))$.

Recall from Example 4.3.17 the description of $H^1_{\text{bas}}(\mathcal{E}^{\text{rig}}_{F_v}, G_0)$. When v is non-archimedean we have $H^1_{\text{bas}}(\mathcal{E}^{\text{rig}}_{F_v}, G_0) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore the class of z_v is trivial when v is split and non-trivial otherwise. When v is real then $H^1_{\text{bas}}(\mathcal{E}^{\text{rig}}_{F_v}, G_0)$ consists of three points, the trivial point and two non-trivial points that are an orbit for the action of $H^1(\Gamma_v, Z(G_0))$ by multiplication. For such v the class of z_v is the trivial element when v is split and one of the two non-trivial elements when v is non-split. Finally, because G_0 satisfies the Hasse principle, the class of z is uniquely determined by the collection of classes of z_v for all v.

Each z_v determines a normalized transfer factor Δ_v by Definition 4.3.11. We are using here the localization at v of the global Whittaker datum determined by ψ and the standard pinning of G_0. We let f_v^T be the function matching f with respect to Δ_v, and $f^T = \prod_v f_v^T$. The local components f_v^T depend on the choice of z, but their product f^T does not, due to Theorem 5.8.1.

We now continue with the discussion of (7.3.1). In the case of SL_2 this was just the beginning of a lengthy discussion involving the unipotent and hyperbolic contributions, as well as the auxiliary spectral contributions. In the case at hand the discussion is over as soon as it has begun, because none of these terms appear. Indeed, the only term by which the full geometric side differs from the left-hand side of (7.3.1) is the central term $f(1) + f(-1)$. We add this term fo $\mathrm{STF}^G_{\text{reg.ell}}(f)$ and call the result $\mathrm{STF}^G(f)$. It is still a stable distribution. At the same time, for each T, the distribution STF^T differs from $\mathrm{STF}^T_{G\text{-reg}}$ by the two evaluation distributions at the elements ± 1 of $T(F)$. But the function f^T has the special property that its values at these elements are zero. This comes again from the degeneration formulas for orbital integrals, which played a role already for SL_2, together with the observation that at every non-split place v the local function f_v is supported only on elliptic elements so the degeneration formula implies that $f_v^T(1) = f_v^T(-1) = 0$. Therefore (7.3.1) immediately becomes

$$\mathrm{TF}^G(f) = \mathrm{STF}^G(f) + \frac{1}{4}\sum_T \mathrm{STF}^T(f^T). \qquad (7.3.2)$$ {eq:tfq2}

This is the stabilization identity for the full trace formula for G.

7.4 The spectral side

There is again a spectral interpretation of STF^G, namely the "stable multiplicity formula", as we already discussed in the case of SL_2. In fact, the formula is exactly the same as (6.5.2). The indexing set of the sum is unchanged (G_0 and G have the same dual group), but of course $S\Theta_\varphi$ is now different – it is the stable character associated to the global L-packet $\Pi_\varphi(G)$ on the group G, rather than the group G_0. Again the definition of this global distribution is simply as the product $S\Theta_\varphi(f) = \prod_v S\Theta_{\varphi_v}(f_v)$ of the local stable characters, but these are now defined by Definition 4.3.13 with respect to the group G_v, i.e. the base change of G to F_v. Note that now at each place the sign $e(G_v)$ appears. This sign is $+1$ when v is split and -1 when v is non-split. Since the non-split places are a finite even number, the product of these signs is $+1$.

It is also not hard to check, using the geometric sides, that if one takes f_0 to be a function of $G_0(\mathbb{A})$ whose stable orbital integrals match those of f, then

$$\mathrm{STF}^G(f) = \mathrm{STF}^{G_0}(f_0).$$

Therefore the validity of the stable multiplicity formula for G follows from the validity for G_0 and the local character identities of Conjecture 4.3.14 applied to the special case $\dot{s} = 1$, in which case $H = G_0$. These identities can be proved by hand in our special case.

Accepting this, the same analysis as in the case of SL_2 will yield the following multiplicity formula: Let $\pi = \otimes_v \pi_v$ be an irreducible admissible representation of $G(\mathbb{A})$ occurring in $L^2(G(F) \setminus G(\mathbb{A}))$. Let φ be a global parameter such that $\pi_v \in \Pi_{\varphi_v}(G)$. Conjecture 4.3.14 produces from π_v and z_v a representation ρ_v of $\pi_0(S^+_{\varphi_v})$. Here in our special case $S^+_{\varphi_v}$ is the preimage of $S_{\varphi_v} \subset \mathrm{PGL}_2(\mathbb{C})$ in $\mathrm{SL}_2(\mathbb{C})$. Let $\rho_{\varphi,\pi} = \otimes_v(\rho_v|_{S^+_\varphi})$, where S^+_φ is again the preimage in $\mathrm{SL}_2(\mathbb{C})$ of S_φ. It turns out that $\rho_{\varphi,\pi}$ restricts trivially to the center of $\mathrm{SL}_2(\mathbb{C})$ and therefore factors through a representation of S_φ. Again let $m(\varphi,\pi)$ be the multiplicity of the trivial representation in $\rho_{\varphi,\pi}$. Then the multiplicity $m(\pi)$ of π in the discrete spectrum of G should be equal to

$$\sum_\varphi m(\varphi,\pi).$$

The same caveats apply to this formula as in the case of SL_2 – because of the appearance of the parameters φ even its statement is conjectural. However, for parameters factoring through W_F all objects are well-defined, so the conjecture can be stated precisely. Furthermore, the analysis we made in the case of SL_2 holds here as well and shows its compatibility with the stable multiplicity formula. In this way one can obtain the validity of this formula.

Let us now observe one interesting implication. For the group $\mathrm{SL}_2(\mathbb{A})$ it is known that every discrete automorphic representation occurs with multiplicity 1; that is $m(\pi) \in \{0,1\}$ for an adelic representation $\pi = \otimes_v \pi_v$. This is not so any more for the group G. Indeed in Example 4.3.17 we saw that at a finite place v there exists a supercuspidal parameter φ_v such that $S^+_{\varphi_v}$ is the quaternion group. This group has a 2-dimensional irreducible representation, and this corresponds to the unique member of the L-packet for φ_v for the non-split inner form of SL_2 over F_v. For any finite set S of finite places one can find a discrete automorphic representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ that has this particular supercuspidal representation as a local component at every place of the given finite set, see e.g. [Shi12]. We can also arrange that, at another finite place, the local component is the trivial representation, whose parameter $W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ is trivial on W_F and the natural map on the second component. This forces the local centralizer at that place to be trivial, hence the global centralizer to also be trivial. It follows that the contribution to $\rho_{\varphi,\pi}$ of these places is a representation of dimension $2^{|S|}$, forcing the multiplicity $m(\pi)$ to be at least $2^{|S|}$.

Appendix

Remark .0.1. Transfer factors were first defined in [LS87] in the case of ordinary endoscopy, then in [KS99] in the more general case of twisted endoscopy. The latter reference contains some mistakes that were corrected in [KS]. As explained in [KS, §5] in the case of ordinary endoscopy there are four possible transfer factors, denoted there by Δ, Δ', Δ_D, and Δ'_D. The two factors Δ and Δ' are adapted to the "usual" normalization of local class field theory, which maps the arithmetic Frobenius elements of W_F (i.e. those inducing $x \mapsto x^q$ on the residue field) to uniformizing elements (i.e. elements of minimal positive valuation), while the two factors Δ_D and Δ'_D are adapted to the "Deligne" normalization, which maps geometric Frobenius elements ($x \mapsto x^{-q}$) to uniformizing elements. The factor Δ is the one defined in [LS87] as the product $\Delta_I \cdot \Delta_{II} \cdot \Delta_{III_1} \cdot \Delta_{III_2} \cdot \Delta_{IV}$. It does not involve the ϵ-term, which was only introduced in [KS99]. It does not depend on any choices and is defined without the assumption of G being quasi-split, but it is only well-defined up to a non-zero scalar multiple. When G is quasi-split a choice of pinning for G makes it well-defined. This version is denoted by Δ_0 in [LS87] and [KS]. The product of Δ_0 and the ϵ-term does not depend on the choice of pinning any more, but it does depend on the choice of Whittaker datum. This product is called Δ_λ in [KS].

To obtain Δ' from Δ, one inverts the endoscopic element s, which has the same effect as inverting Δ_I and Δ_{III_1}. The same remark obtains Δ'_D from Δ_D. To obtain Δ_D from Δ one inverts Δ_{III_2} and Δ_{II}. Thus, all in all, the four variations are given by

$$\begin{aligned}
\Delta &= \Delta_I \cdot \Delta_{II} \cdot \Delta_{III_1} \cdot \Delta_{III_2} \cdot \Delta_{IV}, \\
\Delta' &= \Delta_I^{-1} \cdot \Delta_{II} \cdot \Delta_{III_1}^{-1} \cdot \Delta_{III_2} \cdot \Delta_{IV}, \\
\Delta_D &= \Delta_I \cdot \Delta_{II}^{-1} \cdot \Delta_{III_1} \cdot \Delta_{III_2}^{-1} \cdot \Delta_{IV}, \\
\Delta'_D &= \Delta_I^{-1} \cdot \Delta_{II}^{-1} \cdot \Delta_{III_1}^{-1} \cdot \Delta_{III_2}^{-1} \cdot \Delta_{IV}.
\end{aligned}$$

The five pieces in these definitions are all defined in [LS87]. All of the four versions can be taken either as well-defined up to scalar and available for all connected reductive groups, or normalized by a pinning (so decorated with a subscript 0), or normalized by a Whittaker datum, so decorated by a subscript λ.

In the case of twisted endoscopy there are only two versions. They are denoted by Δ' and Δ_D, because they specialize to those two versions in the case of ordinary endoscopy. The reason that Δ and Δ'_D do not have analogues in the twisted case is that the operation $s \mapsto s^{-1}$ is not valid. Indeed, it is the product $s\widehat{\theta}$ of s with the dual twisting automorphism $\widehat{\theta}$ that is to be considered as the endoscopic element in the twisted case, and not simply s. Another expression of the same phenomenon is that in the twisted case the two terms Δ_{III_1} and Δ_{III_2} are linked in a way that does not allow one of them to be inverted and the other not. These two linked terms make up a single term called Δ_{III}. It is important to note that $\Delta_{III} = \Delta_{III_1} \cdot \Delta_{III_2}^{-1}$.

Therefore it is clear that, even in the case of ordinary endoscopy, the factors Δ' and Δ_D are more natural than the factors Δ and Δ'_D. In these notes we are using the "usual" normalization of local class field theory, and hence we are using the factor Δ'. However, we are dropping the prime notation and are simply calling it Δ. We are furthermore using the Whittaker normalization, but we are not using the subscript λ either. This is meant to lighten the notation as much as possible and to not distract readers new to the theory with these technicalities.

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