Stability and Long Time Evolution of the Periodic Solutions to the Two Coupled Nonlinear Schrödinger Equations

Benkui Tan

Department of Geophysics, Peking University, Beijing 100871, China

John P. Boyd

Department of Atmospheric, Oceanic and Space Science, University of Michigan, Ann Arbor, MI48109, USA

Submitted: October 30, 1999

Revised: January 20, 2000

Corresponding author address: Prof. John P. Boyd, Dept. of atmospheric, Oceanic and Space Science, University of Michigan, 2455 Hayward Aveneue, Ann Arbor MI 48109, USA. Email: jpboyd@engin.umich.edu. Fax: 734-764-5137. Tel: 734-764-3338.

Abstract

The system of two coupled nonlinear Schrödinger equations has wide applications in physics. In the past, the main attention has been their solitary waves. Here we turn our attention to their periodic wave solutions. In this paper, the stability of the periodic solutions is studied analytically and the criteria for the stability are obtained.

The long time evolution of the solutions to the coupled system is studied numerically for the unstable case emphasizing wave-wave interactions in nonlinear optics. Different kinds of evolution are observed depending on the coefficients of the system and the parameters of the unperturbed waves and perturbation. For certain ranges of parameters, the evolution appears to be periodic, while for some other ranges of parameters, solitary wave or solitary wave pairs can be excited among the irregular background although often the evolution is completely chaotic.

1. **Introduction**

The two coupled nonlinear Schrödinger equations (CNLS, hereafter) are

$$\begin{aligned} i\frac{\partial A_1}{\partial t}+\alpha_1\frac{\partial^2 A_1}{\partial x^2}+\left(\sigma_1|A_1|^2+\nu_{12}|A_2|^2\right)A_1=0,\\ i\frac{\partial A_2}{\partial t}+\alpha_2\frac{\partial^2 A_2}{\partial x^2}+\left(\sigma_2|A_2|^2+\nu_{21}|A_1|^2\right)A_2=0. \end{aligned} \tag{1}$$

These have many applications including, for example, nonlinear optics (see, e.g., [1-4]) and geophysical fluid dynamics [5,6]. In Eqs. (1), i is the imaginary number, t and x are time and space variables, respectively. The parameters α_j are the dispersion coefficients, σ_j the Landau constants which describe the self-modulation of the wave packets, and ν_{12} and ν_{21} the wave-wave interaction coefficients which describe the cross-modulations of the wave packets. They are all real parameters and their values vary for different polarizations in nonlinear optics or for different kinds of basic flows in geophysical fluid dynamics.

The unknowns $A_1(x,t)$, $A_2(x,t)$ are the envelopes of wave packets in two different degrees of freedom of the underlying physical systems which we shall call “modes”. The CNLS system is derived as an approximation to a more complex set of equations by singular perturbation theory.

Manakov [2] solved the system (1) by the inverse scattering method for the case where $\alpha_1=\alpha_2=1,\ \ \sigma_1=\sigma_2=\nu_{12}=\nu_{21}=\kappa$. Later Zakharov and Schuman [7] studied the integrability of the system (1) thoroughly and showed that only when

$$\alpha_1=\alpha_2,\sigma_1=\sigma_2=\nu_{12}=\nu_{21} \tag{2}$$

or

$$\alpha_1=-\alpha_2,\sigma_1=\sigma_2=-\nu_{12}=-\nu_{21} \tag{3}$$

the system (1) is integrable, i.e., they can be solved by the inverse scattering method. Otherwise, they are nonintegrable.

There has been a lot of previous work on the solitary wave solutions to the system (1) under the infinite boundary condition $A_j\to 0,\ \ at\ |x|\to\infty$ (see, [5,6,8-12] among others.). Little attention has been give to the solutions under periodic boundary conditions. In this paper we shall pay our attention to the periodic solutions. In the next section the linear stability of the periodic solutions is studied

analytically and the criteria for stability are obtained. In section 3, the long time evolution of the perturbed periodic solutions for the unstable case is studied and different kinds of evolution are observed. Finally a brief discussion and summary are given in section 4.

2. Stability analysis

It is well-known that the system (1) has the exact periodic solutions

$$\begin{aligned} A_1 &= a_1 e^{i(k_1 x - \varpi_1 t)}, \\ A_2 &= a_2 e^{i(k_2 x - \varpi_2 t)}, \end{aligned} \tag{4}$$

with the dispersion relations

$$\begin{aligned} \varpi_1 &= \alpha_1 k_1^2 - (\sigma_1 a_1^2 + \nu_{12} a_2^2), \\ \varpi_2 &= \alpha_2 k_2^2 - (\sigma_2 a_2^2 + \nu_{21} a_1^2). \end{aligned} \tag{5}$$

Here the amplitudes a_1 and a_2 have been assumed to be real without loss of generality. Eqs. (5) show that the dispersion relation for each wave contains not only its own amplitude, which is characteristic of nonlinear waves in general, but also the amplitude of another wave, which is a reflection of the wave-wave interactions.

Now we analyze the stability of the finite amplitude sine waves (4) to perturbations that are infinitesimal sine waves of different wave numbers. It is convenient, following Newton and Keller [13], to write the perturbed solution as

$$\begin{aligned} A_1 &= e^{i(k_1 x - \varpi_1 t)}(a_1 + \varepsilon\phi_1(x,t)), \\ A_2 &= e^{i(k_2 x - \varpi_2 t)}(a_2 + \varepsilon\phi_2(x,t)), \end{aligned} \tag{6}$$

where $\varepsilon \prec\prec 1$. Substituting the perturbed solution (6) into (1) and neglecting the nonlinear terms gives linear equations for the perturbations ϕ_1 and ϕ_2

$$\begin{aligned} i\frac{\partial\phi_1}{\partial t} + \alpha_1\left(2k_1 i\frac{\partial\phi_1}{\partial x} + \frac{\partial^2\phi_1}{\partial x^2}\right) + \sigma_1 a_1^2\left(\phi_1 + \phi_1^*\right) + \nu_{12} a_1 a_2\left(\phi_1 + \phi_2^*\right) = 0, \\ i\frac{\partial\phi_2}{\partial t} + \alpha_2\left(2k_2 i\frac{\partial\phi_2}{\partial x^2} + \frac{\partial^2\phi_2}{\partial x^2}\right) + \nu_{21} a_1 a_2\left(\phi_1 + \phi_2^*\right) + \sigma_2 a_2^2\left(\phi_2 + \phi_2^*\right) = 0, \end{aligned} \tag{7}$$

where '*' denotes the complex conjugate. The perturbations ϕ_1 and ϕ_2 are now assumed to be sinusoidal

$$\begin{aligned}\phi_1 &= B_1 e^{i(lx-\mu t)} + C_1 e^{-i(lx-\mu t)},\\ \phi_2 &= B_2 e^{i(lx-\mu t)} + C_2 e^{-i(lx-\mu t)}.\end{aligned} \tag{8}$$

Substitution of Eqs. (8) into Eqs. (7) gives

$$\begin{aligned}&\left[\left(\sigma_1 a_1^2 - \alpha_1 l^2\right) + \left(\mu - 2\alpha_1 k_1 l\right)\right]B_1 + \nu_{12} a_1 a_2 B_2 + \sigma_1 a_1^2 C_1^{\ *} + \nu_{12} a_1 a_2 C_2^{\ *} = 0,\\ &\nu_{21} a_1 a_2 B_1 + \left[\left(\sigma_2 a_2^2 - \alpha_2 l^2\right) + \left(\mu - 2\alpha_2 k_2 l\right)\right]B_2 + \nu_{21} a_1 a_2 C_1^{\ *} + \sigma_2 a_2^2 C_2^{\ *} = 0,\\ &\sigma_1 a_1^2 B_1^{\ *} + \nu_{12} a_1 a_2 B_2^{\ *} + \left[\left(\sigma_1 a_1^2 - \alpha_1 l^2\right) - \left(\mu - 2\alpha_1 k_1 l\right)\right]C_1 + \nu_{12} a_1 a_2 C_2 = 0,\\ &\nu_{21} a_1 a_2 B_1^{\ *} + \sigma_2 a_2^2 B_2^{\ *} + \nu_{21} a_1 a_2 C_1 + \left[\left(\sigma_2 a_2^2 - \alpha_2 l^2\right) - \left(\mu - 2\alpha_2 k_2 l\right)\right]C_2 = 0.\end{aligned} \tag{9}$$

The first two equations and the complex conjugates of the last two equations of Eqs. (9) form the closed equations for B_1, B_2, $C_1^{\ *}$ and $C_2^{\ *}$. They are homogeneous algebraic equations. Non-zero solutions are impossible unless the determinant of the matrix in (9) is zero, i.e.,

$$\begin{aligned}&\left[\left(\alpha_1^2 l^2 - 2\alpha_1 \sigma_1 a_1^2\right) - \left(c - 2\alpha_1 k_1\right)^2\right]\times\\ &\qquad\left[\left(\alpha_2^2 l^2 - 2\alpha_2 \sigma_2 a_2^2\right) - \left(c - 2\alpha_2 k_2\right)^2\right] = 4\alpha_1 \alpha_2 \nu_{12} \nu_{21} a_1^2 a_2^2,\end{aligned} \tag{10}$$

where

$$c \equiv \frac{\mu}{l} \tag{11}$$

is the phase velocity of the perturbation. Equation (10) is the dispersion relation for the perturbation, a special case of which with $k_1 = k_2 = 0$ was obtained by Roskes [14]. It is a fourth-order algebraic equation for the phase velocity c. The sinusoidal solution (4) is unstable provided at least one pair of the roots for c are complex.

Set

$$\bar{k} = \alpha_1 k_1 + \alpha_2 k_2,\ \ \hat{k} = \alpha_1 k_1 - \alpha_2 k_2,\ \ \bar{c} = c - \bar{k}, \tag{12}$$

then, the dispersion relation (10) reduces to a quartic algebraic equation for $\bar{c}$ without cubic term

$$\bar{c}^4 + P_0 \bar{c}^2 + P_1 \bar{c} + P_2 = 0, \tag{13}$$

where

$$\begin{aligned}P_0 &= -(\alpha_1^2 l^2 - 2\alpha_1 \sigma_1 a_1^2) - (\alpha_2^2 l^2 - 2\alpha_2 \sigma_2 a_2^2) - 2\hat{k}^2,\\ P_1 &= -2\left[(\alpha_1^2 l^2 - 2\alpha_1 \sigma_1 a_1^2) - (\alpha_2^2 l^2 - 2\alpha_2 \sigma_2 a_2^2)\right]\hat{k},\\ P_2 &= \hat{k}^4 + \left[(\alpha_1^2 l^2 - 2\alpha_1 \sigma_1 a_1^2) + (\alpha_2^2 l^2 - 2\alpha_2 \sigma_2 a_2^2)\right]\hat{k}^2 +\\ &\qquad + \left(\alpha_1^2 l^2 - 2\alpha_1 \sigma_1 a_1^2\right)\left(\alpha_2^2 l^2 - 2\alpha_2 \sigma_2 a_2^2\right) - 4\alpha_1 \alpha_2 \nu_{12} \nu_{21} a_1^2 a_2^2.\end{aligned} \tag{14}$$

The major difficulty with the dispersion relation as it stands is that it is a function of eleven parameters:

$\{\alpha_1, \alpha_2, \sigma_1, \sigma_2, \nu_{12}, \nu_{21}, a_1, a_2, k_1, k_2, l\}$. Ferrari derived an analytical solution to the quartic equation more than four centuries ago, but it is very complicated with cube roots nested inside square roots. Fortunately, there are some interesting special cases.

(a) Special case one: Uncoupled NLS equations.

If any of the four quantities $(a_1, a_2, \nu_{12}, \nu_{21})$ are zero, then the dispersion relation collapses into two quadratic factors. Each of these factors is the dispersion relation for a nonlinear Shrödinger equation which is uncoupled from the other NLS equation. The roots of the dispersion relation are then

$$c = \alpha_1 \left\{ 2k_1 \pm \sqrt{l^2 - \frac{2\sigma_1}{\alpha_1} a_1^2} \right\}, \quad \alpha_2 \left\{ 2k_2 \pm \sqrt{l^2 - \frac{2\sigma_2}{\alpha_2} a_2^2} \right\}, \tag{15}$$

where the second pair of roots in identical to the first except for the replacement of all the subscript ones by twos. Equation (15) shows that the sine wave is unstable if and only if $\alpha_1 \sigma_1 \succ 0$ and simultaneously $l^2 \prec \frac{2\sigma_1}{\alpha_1} a_1^2$, and stable otherwise, and similarly for the other mode. This instability is usually called "modulational instability" or the "Benjamin-Feir" instability [15].

(b) Special two: Infinitesimal unperturbed waves or very short wavelength perturbation.

In the limit that both $a_1, a_2 \to 0,$ (as opposed to a single unperturbed amplitude being very small or zero, which gives the previous special case), or if $l \to \infty$, then these two physically distinct limits both give the same dispersion relation in the limit with the roots

$$c = \alpha_1 \{2k_1 \pm l\}, \quad \alpha_2 \{2k_2 \pm l\}. \tag{16}$$

The important point is that these limits are always stable. Thus, the unperturbed solution must be of finite amplitude in at least one mode and the perturbing wave must have a finite wavenumber if modulational instability is to develop, even when the two NLS equations are coupled.

(c) Special case three: Vanishing group velocity difference between the unperturbed modes.

The condition for this limit is

$$\alpha_1 k_1 = \alpha_2 k_2, \tag{17}$$

which includes $k_1 = k_2 = 0.$ To understand this limit, we need to recall the underlying physical

interpretation of the CNLS system.

The unknowns $A_1(x,t)$ and $A_2(x,t)$ are actually the envelopes of wave packets. The complete solution is the product of the envelope with functions in the coordinates perpendicular to x and a sine wave $\exp(ik_{central}x - \varpi(k_{central})t)$ where $k_{central}$ is the wavenumber of the sinusoidal factor and $\varpi(k_{central})$ is the corresponding frequency as given by infinitesimal amplitude wave theory. The sine wave factor propagates at the phase velocity; the envelope $A_1(x,t)$ travels at an amplitude-dependent group velocity. The coordinate x in the CNLS system is actually a coordinate in a frame of reference that travels at the group velocity of infinitesimal amplitude waves of wavenumber $k_{central}$. The spatial dependence of the envelope implies that the Fourier transform of the complete solution is not a delta-function at a single wavenumber, but rather is a smooth function of finite width. However, the envelope is assumed to vary slowly with space (in the singular perturbation theory used to derive the CNLS system from a more complicated set of equations), and therefore the Fourier transform is narrowly peaked about the wavenumber $k = k_{central}$. For this reason, the wavenumber of the sinusoidal factor, present in the complete wave solution but not explicitly in the CNLS system, is called the "central wavenumber of the packet", $k_{central}$.

When the envelope itself is assumed to be a sine wave and only a single wave packet is present, the wavenumber of the envelope can be absorbed onto the central packet wavenumber. The alteration in group velocity produced by the NLS dispersive term $\alpha_1 A_{1,xx}$ is absorbed into the speed at which the moving frame of reference is travelling. Even for the coupled NLS system, it is always legitimate to set either $k_1 = 0$ or $k_2 = 0$.

Unfortunately, it is not possible to set both wavenumbers equal to zero. Physically, $k_1 \neq k_2$ implies that the wavepackets in the two coupled modes are trvelling at slightly different group velocities. This obviously has profound implications for the coupled dynamics because crest in the envelopes of the two modes will gradually move into or out of spatial alignment as each travels at its own group speed.

Although we cannot *a priori* dismiss the possibility that this relative movement of envelope may amplify the modulational instability in at least some parameter ranges, it seems likely that group velocity differences usually reduces the instability.

The group velocity of our unperturbed sinusoidal envelopes is, for infinitesimal envelopes A_1 and A_2, $c_{g1} = 2\alpha_1 k_1$ and $c_{g2} = 2\alpha_2 k_2$. Thus, the condition for this special case is that the group velocities of the two packets match.

When the group velocity difference is zero, the quartic simplifies to a biquadratic equation which can be solved explicitly. Define

$$\Delta_1 \equiv \alpha_1^2 l^2 - 2\alpha_1 \sigma_1 a_1^2, \qquad \Delta_2 \equiv \alpha_2^2 l^2 - 2\alpha_2 \sigma_2 a_2^2,$$

$$S \equiv 4\alpha_1 \alpha_2 \nu_{12} \nu_{21} a_1^2 a_2^2. \tag{18}$$

Note that when the NLS equations are uncoupled, the condition for instability is that either Δ_1 or Δ_2 is negative; the parameter S measures the strength of the coupling between modes and is zero when the two NLS equations are uncoupled. Then the roots are

$$c = 2\alpha_1 k_1 \pm \sqrt{\frac{1}{2}(\Delta_1 + \Delta_2) \pm \sqrt{(\Delta_1 - \Delta_2)^2 + 4S}}\ . \tag{19}$$

For the cases where either $\Delta_1 + \Delta_1 \prec 0$ or $4S + (\Delta_1 - \Delta_2)^2 \prec 0$, the sine wave solution (4) is unstable. If both $\Delta_1 + \Delta_1$ and $4S + (\Delta_1 - \Delta_2)^2$ are larger than zero, the sine wave solution (4) is stable if $\Delta_1 \Delta_2 \succ S$; otherwise it is unstable.

(d) Special case four: $\Delta_1 = \Delta_2$.

This is a special case where the two unperturbed sine waves have the same stability if they were uncoupled. For this case, the quartic also simplifies to a biquadratic equation which can be solved explicitly. The roots are

$$c = \bar{k} \pm \sqrt{\Delta_1 + \hat{k}^2 \pm \sqrt{S}}. \tag{20}$$

For the cases where either $S \prec 0$ or $\Delta_1 + \hat{k}^2 \prec 0$, the sine wave solution (4) is unstable. If both S and $\Delta_1 + \hat{k}^2$ are larger than zero, the sine wave solution is unstable if $S \succ (\Delta_1 + \hat{k}^2)^2$; otherwise it is stable. Obviously, for the case where $S \succ 0$, the role of the wavenumber difference of the unperturbed sine waves is to increase the stability.

In the numerical case studies presented below, we shall consider only the last two special cases. The general case is interesting, but because of the large number of parameters, it is not possible to make a frontal attack even numerically.

The best strategy is to apply perturbation theory in context of particular physical application so that some parameters are known, or at least known to within certain intervals. For example, when the product of $\nu_{12}\nu_{21}$ is small, it is straightforward to derive a "week coupling expansion" using a perturbation parameter proportional to the product of the coupling coefficients. Similarly, when the parameter $\hat{k}$ is small but nonzero, one can easily derive a series in power of $\hat{k}$. However, the details lie beyond the scope of this work.

3. Behavior of the unstable periodic solutions

In this section we concern the long time behavior of the periodic solutions when they are destabilized by perturbations. This is an interesting problem. For a single nonlinear Schrödinger equation, Yuen and Ferguson [16] demonstrated that the Fermi-Pasta-Ulam recurrence occurs if the periodic solution of a nonlinear Schrödinger equation is destabilized by perturbations. What happens for the two coupled nonlinear Schrödinger equations? Does the FPU recurrence occur? If it does, what are the most significant characteristics of the solutions? Are there any other types of evolution to them? The problems will be answered in the following.

For convenience in our numerical study, we use the CNLS system in its parameter-reduced form [12]

$$\begin{aligned} i\frac{\partial A}{\partial t}+\frac{\partial^2 A}{\partial X^2}+|A|^2 A+\tilde{\nu}_{12}|B|^2 A=0,\\ i\frac{\partial B}{\partial t}+\alpha\frac{\partial^2 B}{\partial X^2}+|B|^2 B+\tilde{\nu}_{21}|A|^2 B=0, \end{aligned} \tag{21}$$

where the new unknowns and coefficients are

$$A\equiv\sqrt{\sigma_1}A_1,\qquad B\equiv\sqrt{\sigma_2}A_2,\qquad X\equiv\sqrt{1/\alpha_1}x,$$

$$\tilde{\nu}_{12}=\frac{\nu_{12}}{\sigma_2},\quad \tilde{\nu}_{21}=\frac{\nu_{21}}{\sigma_1},\quad \alpha=\frac{\alpha_2}{\alpha_1}. \tag{22}$$

We solve the system (17) numerically with the initial conditions

$$\begin{aligned} A=a_0(1-\varepsilon\cos lX),\\ B=b_0(1-\varepsilon\cos lX), \end{aligned} \tag{23}$$

where a_0 and b_0 are the initial amplitudes of the two perturbed periodic waves, respectively. The

wave numbers k_1 and k_2 of the perturbed periodic waves have been taken to be zero only for the purpose of showing the evolution of the solution clearly. $\varepsilon \prec\prec 1$ is a small parameter and it represents the strength of the perturbation and l is the wave number of the perturbation.

Generally the coefficients in CNLS system are variable parameters depending on the basic flow for the geophysical fluid dynamics case. For nonlinear optics, in contrast, the coefficients are some fixed constants. In the numerical experiments reported below, the coefficients are mainly taken from optics.

The numerical solutions were computed by expanding each unknown $u(x,t)$ and $v(x,t)$ as a Fourier series in space with time-dependent coefficients. Spatial derivatives are computed by exact differentiation of the truncated Fourier series. Collocation on an evenly spaced spatial grid gives a system of ordinary differential equations in time , which is then solved by the usual fourth-order Runge-Kutta initial-value method. (By ‘‘collocation” is meant the imposition of the constraint that the truncated series yield a residual of zero at each point on the grid.) If the exact solution is periodic and holomorphic for real x, it can be proved that the spatial discretization error decreases exponentially fast with N, the number of terms in the Fourier series. The pseudospectral algorithm is well-described in books such as [17] and [18] and articles such as [19], so we shall omit a complete description.

Calculations were repeated with different N and different timesteps until we were satisfied that increasing resolution further would produce no visible changes in any of our plots

In our first experiment the coefficients $\alpha = 1, \tilde{\nu}_{12} = \tilde{\nu}_{21} = 1$, which corresponds to $\alpha_1 = \alpha_2 = \sigma_1 = \sigma_2 = \nu_{12} = \nu_{21} = 1$. In nonlinear optics, it is a special case of elliptical polarization. This is an integrable case that Manakov [2] studied. According to the stability analysis, the solution (4) for this case is conditionally stable. It is stable if the perturbation wave number is above the critical value $l_c = \sqrt{2\left(a_0^{\,2} + b_0^{\,2}\right)}$; otherwise it is unstable. Typical evolution of the destabilized solution (4) for this case is a regular oscillation between the uniform state and hump state. Figs. 1-3 show the numerical tests with $\varepsilon = 0.1$, $a_0 = 0.5$ and $b_0 = 0.3$. The critical perturbation wave number $l_c = 0.8246$.

The perturbation to the uniform solution grows initially at an exponential rate as predicted by the stability theory, but its growth will be restricted after some period of time due to the nonlinearity and

eventually the solutions demodulate and return to the near-uniform state. Then another cycle begins. The evolution depends on the wave number of the perturbation. In Fig. 1, the wave number of the perturbation $l = 0.5$. For this case all its harmonics are stable. As can be seen, the solutions evolve in a simple way, changing alternately between the near-uniform state and the one-hump state.

Figures 2 and 3 show more complicated cases where not only the basic perturbation but also some of its harmonics generated by the nonlinearity are unstable. For these cases, the basic perturbation and all its unstable harmonics play very active roles in the evolution, each taking its turn dominating the solution profile. In Fig. 2, the wave number of the perturbation $l = 0.3$. For this case, the perturbation and one of its harmonics with a wave number of 0.6 are unstable, but all of its higher harmonics are stable. So, as anticipated, the solutions evolve alternately between the near-uniform state, the one-hump state and the two-hump state. In Fig. 3 the wave number of the perturbation $l = 0.2$. For this case, the perturbation and two of its harmonics are unstable, as anticipated; there will be three modes which are very active in the evolution of the solutions: the solutions evolve alternately between the near-uniform state, the one-hump state, the two-hump state and the three-hump state.

Not only the wave number of the perturbation affects the long-time evolution. Other factors such as the amplitudes of the initial uniform waves and the strength of the perturbation also have significant effects on the evolution. To demonstrate this clearly, we introduce the amplitude norm of the solutions defined by Yuen and Ferguson [16] as

$$E(t) = \frac{1}{N}\sum_{j=1}^{N} |A(t, j) - A(0, j)|, \tag{24}$$

where N is the number of the grid points used in the calculation and $A(t, j)$ is the amplitude of the solutions for wave A at the jth point at time t. The amplitude norm can reflect the main characteristic of the recurrence more clearly.

The top plot in Fig. 4 shows the effects of initial amplitude of the waves on the periods of the recurrence. In this figure, the perturbation parameters are: $l = 0.5$ and $\varepsilon = 0.1$. The solid line represents the evolution of the amplitude norm with initial amplitudes $a_0 = 0.5,\ b_0 = 0.3$, while the doted solid line indicates the evolution of the amplitude norm with initial amplitudes $a_0 = 0.6,$ $b_0 = 0.4$. The figure shows that the period of the recurrence decreases when the initial amplitudes increase.

The bottom plot in Fig. 4 shows how the strength of the perturbation influences the period of the recurrence. In this figure, the wave number of the perturbation $l = 0.5$ and the initial amplitudes of the waves $a_0 = 0.5$, $b_0 = 0.3$. The solid line shows the evolution of the amplitude norm when the strength of the perturbation $\varepsilon = 0.1$, while the doted solid line represents the evolution of the amplitude norm when the strength of the perturbation $\varepsilon = 0.2$. It is clear that the bigger the strength of the perturbation, the shorter the period of the recurrence.

In the second experiment, the coefficients $\alpha = 1$, $\tilde{\nu}_{12} = \tilde{\nu}_{21} = 2/3$ which corresponds to $\alpha_1 = \alpha_2 = 1$, $\sigma_1 = \sigma_2 = 1$ and $\nu_{12} = \nu_{21} = 2/3$. In nonlinear optics, this is the linearly polarized mode case and the CNSL system becomes non-integrable. For this set of coefficients, the solution (4) is also conditionally stable. The critical perturbation wave number $l_c = \left[a_0^{\,2} + b_0^{\,2} + \sqrt{\left(a_0^{\,2} + b_0^{\,2}\right)^2 - \frac{20}{9} a_0^{\,2} b_0^{\,2}} \right]^{\frac{1}{2}}$, above which the solution (4) is stable; otherwise it is unstable.

Numerical results show that two types of evolution are frequently observed: recurrence and generation of solitary wave-like pulse. If the initial amplitudes a_0 and b_0 are equal, recurrence is expected (Fig. 5a). Generally, generation of solitary wave-like pulse or pulses among the irregular background are observed (Fig. 5b). As can be seen from Fig. 5b, the evolution of wave B is rather irregular while in the evolution process of wave A, there are two oscillating pulses excited from a single pulse of large peak. Initially, one pulse propagates toward left, another propagates toward right. After a period of time, they remerge. Then separate and merge again for a couple of times. Finally, they move away in different directions.

In the third experiment, the coefficients $\alpha = 1$, $\tilde{\nu}_{12} = \tilde{\nu}_{21} = 2$ which corresponds to $\alpha_1 = \alpha_2 = 1$, $\sigma_1 = \sigma_2 = 1$ and $\nu_{12} = \nu_{21} = 2$. This is the circular polarized mode case in nonlinear optics. For this set of coefficients the CNLS system is also non-integrable and the interaction between the perturbed periodic waves is very strong for the wave-wave interaction coefficients are two times of the dispersion coefficients or the Landau constants. The solution (4) is again conditionally stable: it is stable if the perturbation wave number is above the critical value

$l_c = \left[a_0^{\,2} + b_0^{\,2} + \sqrt{\left(a_0^{\,2} + b_0^{\,2}\right)^2 + 12 a_0^{\,2} b_0^{\,2}} \right]^{\frac{1}{2}}$; otherwise it is unstable. As in experiment two, two types of evolution are also frequently observed: regular evolution and generation of solitary wave-like pulses. If the initial amplitudes a_0 and b_0 are equal, regular evolution can be expected (Fig. 6a). Generally, the typical evolution for this case is that among the irregular background, a pair of pulses belonging to two different wave modes are excited among the irregular background (see Fig.6b). The two pulses are coupled together and form a new bound state, remaining stationary and oscillating in their strength. In Yang and Benney [10] and Tan and Boyd [12], the analytical expression of the coupled–mode solitary wave pair is given; here their generation from a general initial condition is observed.

In the fourth experiment, the coefficients are arbitrarily chosen to be $\alpha = 1$, $\tilde{\nu}_{12} = -\tilde{\nu}_{21} = -1$ which corresponds to $\alpha_1 = \alpha_2 = 1$, $\sigma_1 = \sigma_2 = 1$ and $\nu_{12} = -\nu_{21} = -1$. This is a non-integrable case with the wave-wave interaction coefficients of opposite signs. This case occurs in geophysical fluid dynamics [9]. For this set of coefficients, the solution (4) is may be absolutely unstable or conditionally stable depending on the initial amplitudes. It is absolutely unstable i.e., it is always unstable to perturbations of any wave numbers, if the initial amplitudes a_0 and b_0 satisfy $a_0^{\,2} + b_0^{\,2} \prec 2\sqrt{2} a_0 b_0$. If $a_0^{\,2} + b_0^{\,2} \succ 2\sqrt{2} a_0 b_0$, the solution (4) is conditionally stable, i.e., it is stable when the perturbation wave number is above the critical value $l_c = a_0^{\,2} + b_0^{\,2} + \sqrt{a_0^{\,4} + b_0^{\,4} - 6 a_0^{\,2} b_0^{\,2}}$, and it is unstable if the perturbation wave number is below the critical perturbation wave number.

Tan and Boyd [12] shows numerically that the solitary wave pair solution to the CNLS system is unstable if the two wave-wave interaction coefficients have opposite signs. So solitary wave pairs cannot be expected to be excited as in Fig. 6 for this case. Indeed, the evolution of the solution is quite different for the absolutely unstable case and the conditionally unstable case. For the conditionally unstable case, the oscillation of the solution between the uniform state and the humps state can still be observed for some parameter ranges though the oscillation is not as perfect as in Figs. 1-3, 5a and 6a. (see Fig.7a). For the absolutely unstable case, the typical evolution for both wave A and wave B is chaotic (see Fig.7b).

4. Discussion and conclusions

In the recurrence-type figures (Figs.1-3, Fig. 5a, Fig. 6a), the evolution patterns for waves A and B are alike. This suggests that the CNLS system has a kind of solutions that waves A and B are simply in proportion. Indeed here we show that this is true when α in Eq. (21) is equal to unity and $\tilde{\nu}_{12}$ and $\tilde{\nu}_{21}$ meet certain conditions.

Set

$$A = rB, \tag{25}$$

where r is a proportionality parameter to be determined. Substitution of Eq. (25) into Eq. (21) shows that the two nonlinear Schrödinger equations in the CNLS system are no longer independent. They reduce to the same nonlinear Schrödinger equation provided that

$$(1-\tilde{\nu}_{21})|r|^2 = 1-\tilde{\nu}_{12}. \tag{26}$$

This means that the CNLS system, with any $\tilde{\nu}_{12}$ and $\tilde{\nu}_{21}$ such that $|r|$ in Eq. (26) has positive real roots, has solutions in which waves A and B are in proportion. Since a single nonlinear Schrödinger equation has, as Yuen and Ferguson [16] demonstrated, recurrence type solutions, the recurrence-type solutions are also expected for the CNLS system for this case. In fact, in experiment one, $\tilde{\nu}_{12} = \tilde{\nu}_{21} = 1$, Eq. (26) is unconditionally satisfied no mater what value r takes. In experiments two and three, $\tilde{\nu}_{12} = \tilde{\nu}_{21}$. For these two cases, $|r|$ has a positive real root 1. Figs. 5a and 6a are just the case where $r = 1$. So for these cases, the recurrence-type solutions are not unexpected. We have proved numerically that this is true even with some other $\tilde{\nu}_{12}$ and $\tilde{\nu}_{21}$ that enable $|r|$ has positive real roots.

The recurrence-type behavior is, however, not restricted only to the cases where waves A and B are in proportion. Fig. 8 shows such an example. In this figure, the two wavenumbers of the unperturbed waves are not equal. In Fig. 8a, the wavenumbers $k_1 = 0$ and $k_2 = 0.3$ while in Fig. 8b, $k_1 = 0.1$ and $k_2 = 0.2$. As can be seen, the two waves are not in proportion though, they do evolve periodically.

Our numerical tests (Fig. 7b and other numerical experiments not given here) also reveal another

significant characteristic of the CNLS system that the solutions are inevitably chaotic if the perturbed sine waves (4) are unconditionally unstable. This suggests that the chaos is connected to a infinite set of unstable spectrums.

In conclusion, the periodic solutions of the two coupled nonlinear Schrödinger equations may be stable or unstable depending on the different range of the parameters. In the long time evolution of the periodic solutions destabilized by the perturbation, very different kinds of evolution are observed. For certain parameter ranges, the Fermi-Pasta-Ulam phenomenon can be observed, as for a single uncoupled nonlinear Schrödinger equation, in the two coupled nonlinear Schrödinger equations. While for other ranges of parameters, solitary wave-like pulses or a pair of solitary wave-like pulses are excited, or the evolution is completely chaotic.

Acknowledgements. This work is supported by the Chinese NSF grant 4987014 and National Key Basic Research grant. Part of this work was done while one of the authors (B. Tan) visited the Department of Atmospheric, Oceanic and Space Science of the University of Michigan with partial support from the International Programs Office of the U. S. NSF through grant OCE9521133. He would like to thank the hospitality from AOSS department.

References

[1] Berkhoer, A. L. and Zakharov, V. E., 1970, Self excitation of waves with different polarizations in nonlinear media. ***Sov. Phys. JETP*** (Engl. Transl.) **31**, 486-490.

[2] Manakov, S. V., 1974, On the theory of two-dimensional stationary self-focusing of electromagnetic waves. ***Sov. Phys. JEPT***(Engl. Transl.), **38**, 248-253.

[3] Trillo, S. and Wabnitz, S., 1989, Ultrashort pulse train generation through induced modulational polarization instability in a birefringent Kerr-like medium. ***J. Opt. Soc. Am.*** **B6**, 238-249.

[4] Chen, C. J., Wai, P. A. and Menyuk, C. R., 1990, Soliton switch using birefringent optical fibers. Opt. Lett. **15**, 477-479.

[5] Tan, B. and Wu, R., 1993, Nonlinear Rossby waves and their interactions. ***Science in China***, **B**, **36**, 1367-1380.

[6] Tan, B. and Liu, S., 1995, Collision interactions of solitons in a baroclinic atmosphere. ***J. Atmos. Sci***. **52**, 1501-1512.

[7] Zakharov, V. E. and Schulman, E. I. , 1982, To the integrability of the system of two coupled nonlinear Schrödinger equations. ***Physica D***, **4**, 270-274.

[8] Malomed, B. and Wabnitz, S., 1991, Soliton annihilation and fusion from resonant inelastic collisions in birefringent optical fibers. ***Optics Letters***, **16**, 1388-1390.

[9] Tan, B., 1996, Collision interactions of envelope Rossby solitons in a barotropic atmosphere. ***J. Atoms. Sci***. 53, 1604-1616.

[10] Yang, J. and Benney, D. J., 1996, Some properties of nonlinear wave system. ***Stud***. ***Appl. Math***. **96**, 111-139.

[11] Yang, J. 1997, Classification of the solitary waves in coupled nonlinear Schrödinger equations. ***Physica D*** **108,** 92-112.

[12] Tan, B. and Boyd, J. P., 1999, Coulped-mode solitary wave in a pair of cubic Schrödinger Equations with cross modulations: Analytical solution and collisions. ***Chaos, Solitons and Fractals***. In press.

[13] Newton, P. K. and Keller J. B., 1988, Stability of plane wave solutions of nonlinear systems. ***Wave Motion*** **10**, 183-191.

[14] Roskies, G. J., 1976, Some nonlinear multiphase interactions. ***Stud. Appl. Math***. **55**, 231-238.

[15] Benjamin, T. B. and Feir J. E., 1967, The disintegration of wave trains on deep water. Part 1. Theory. J. Fluid Mech. 27, 417-430.

[16] Yuen, H. C. and Ferguson, W. E., 1978, Relationship between Benjamin-Feir instability and recurrence in the nonlinear Schrödinger equation. ***Phys. Fluids***, **21**, 1275-1278.

[17] Boyd, J. P., 2000, *Chebyshev and Fourier Spectral Methods, 2d ed.* Dover, New York, 662 pp.

[18] Fornberg, B., 1996, *A Practical Guide to Pseudospectral Methods,* Cambridge University Press, New York.

[19] Fornberg, B., and Whiham, G. B., 1978, A numerical and theoretical study of certain nonlinear wave phenomena. *Phil. Trans. R. Soc. Lond.*, **289**, 373-404.

Figure Captions

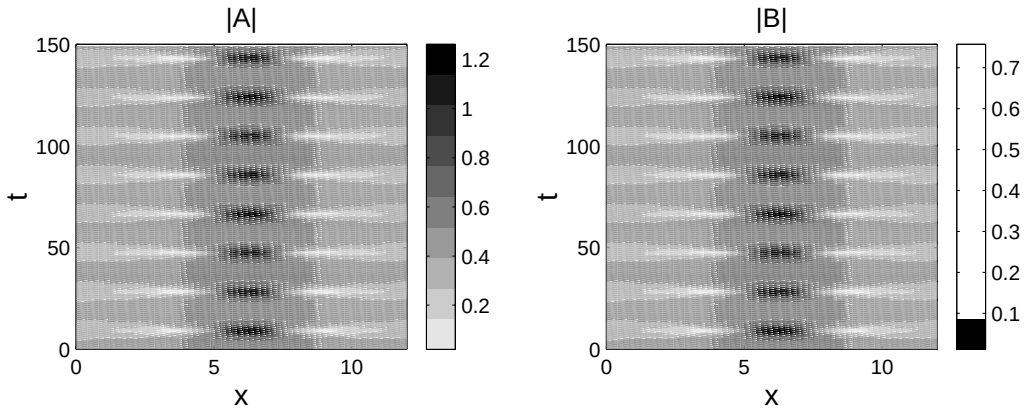

Fig. 1 Long time evolution of the destabilized sine wave solution to the CNLS system with $\alpha_1 = \alpha_2 = 1.0$, $\sigma_1 = \sigma_2 = 1.0,$ and $\nu_{12} = \nu_{21} = 1.0$. Other parameters are: $a_0 = 0.5,$ $b_0 = 0.3$, $l = 0.5$, $\varepsilon = 0.1$. Left and right plots show the evolution of waves A and B, respectively.

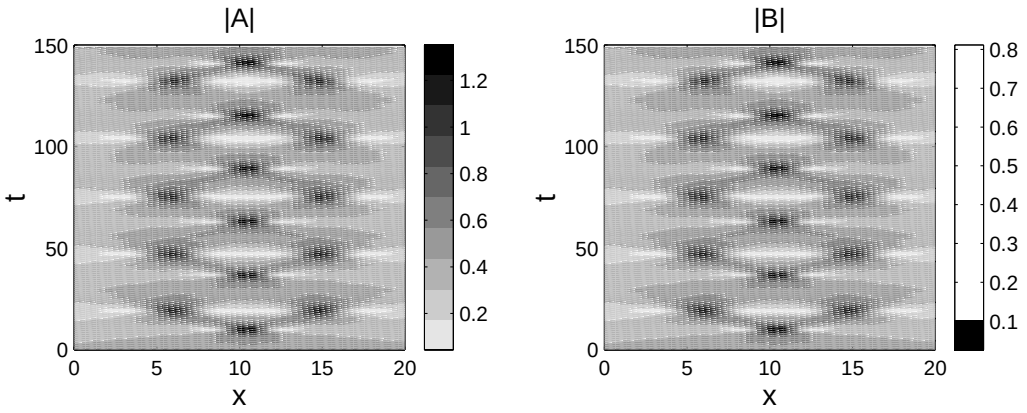

Fig. 2 As in Fig. 1 except $l = 0.3$.

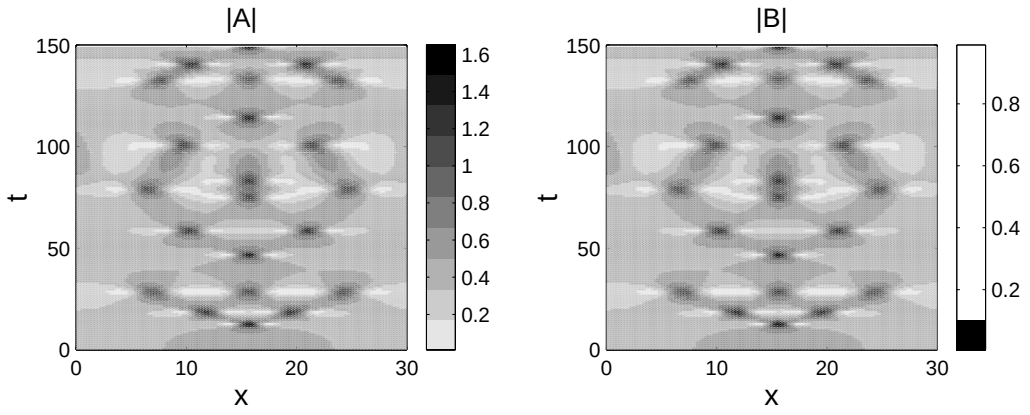

Fig. 3 As in Fig. 1 except $l = 0.2$.

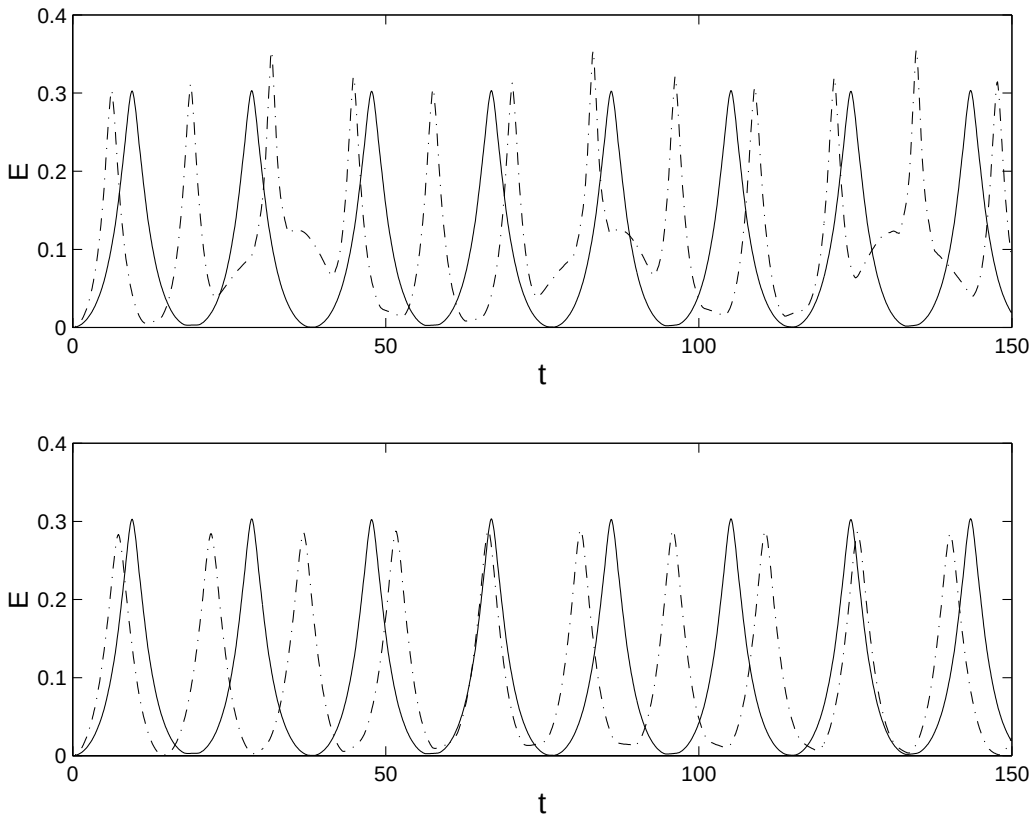

Fig. 4 Evolution of the amplitude norms of the destabilized sine wave solution to the CNLS system with the same set of coefficients as in Fig. 1. In top plot, $l = 0.5$, $\varepsilon = 0.1$. The solid line represents the norm when $a_0 = 0.5$ and $b_0 = 0.3$; the doted solid line presents the norm when $a_0 = 0.6$ and $b_0 = 0.4$. In bottom plot, $a_0 = 0.5$, $b_0 = 0.3$ and $l = 0.5$. The solid line represents the norm of when $\varepsilon = 0.1$; the solid line represents the norm when $\varepsilon = 0.2$.

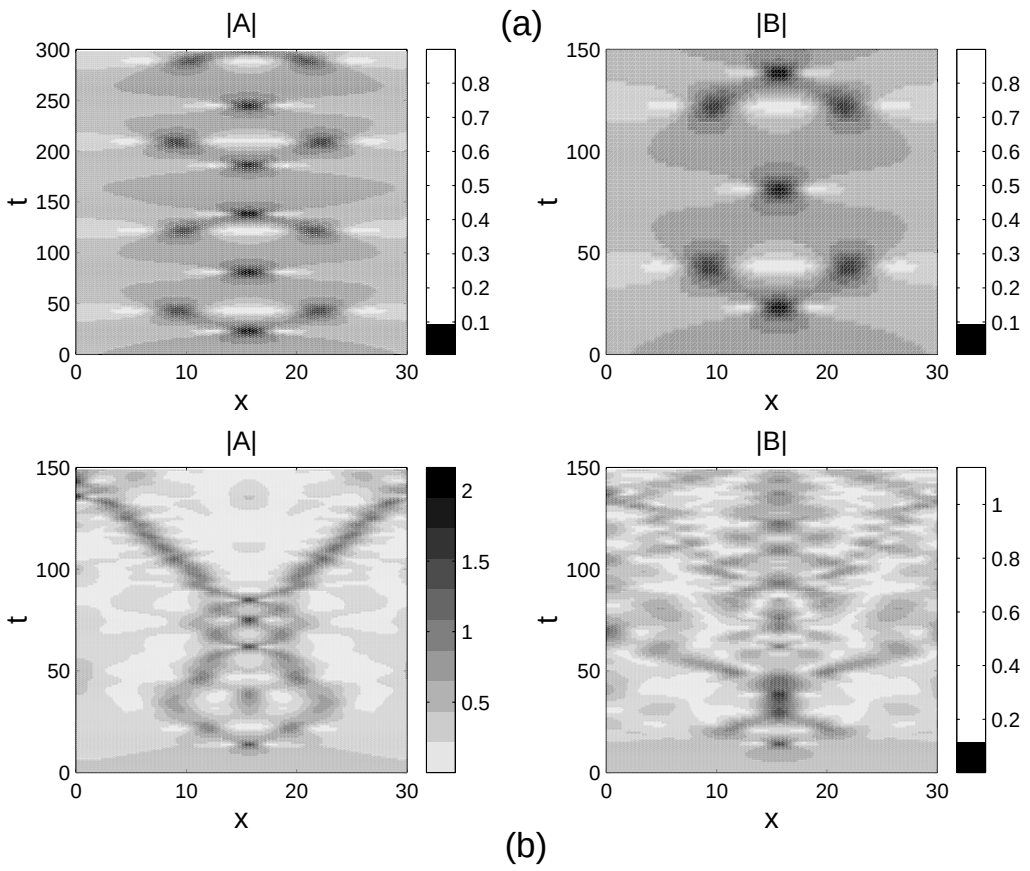

Fig. 5 Long time evolution of the destabilized sine wave solution to the CNLS system with coefficient $\alpha_1 = \alpha_2 = 1.0$, $\sigma_1 = \sigma_2 = 1.0,$ and $\nu_{12} = \nu_{21} = 2/3$. (a) $\varepsilon = 0.1$, $l = 0.2$, $a_0 = 0.3$ and $b_0 = 0.3$. (b) $\varepsilon = 0.1$, $l = 0.2$, $a_0 = 0.5$ and $b_0 = 0.3$.

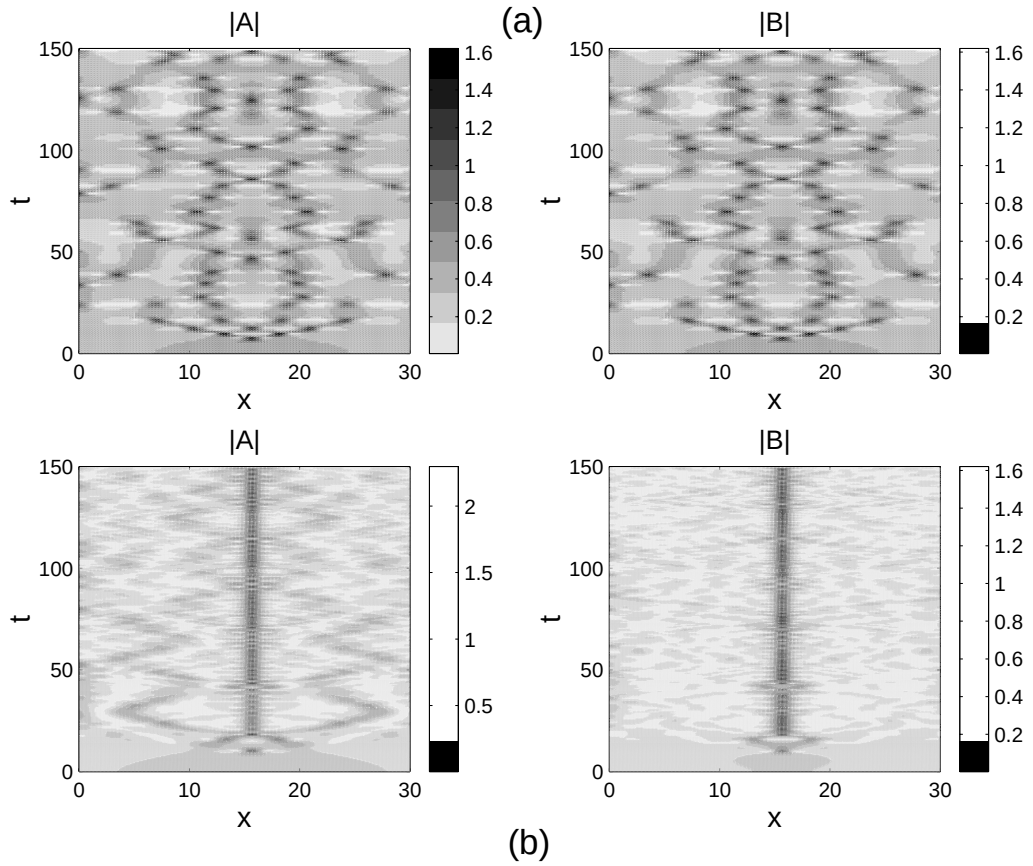

Fig. 6 Long time evolution of the destabilized sine wave solution to the CNLS system with $\alpha_1 = \alpha_2 = 1.0$, $\sigma_1 = \sigma_2 = 1.0$, and $\nu_{12} = \nu_{21} = 2$. (a) $\varepsilon = 0.1$, $l = 0.2$, $a_0 = 0.5$ and $b_0 = 0.5$. (b) $\varepsilon = 0.1$, $l = 0.2$, $a_0 = 0.5$ and $b_0 = 0.3$.

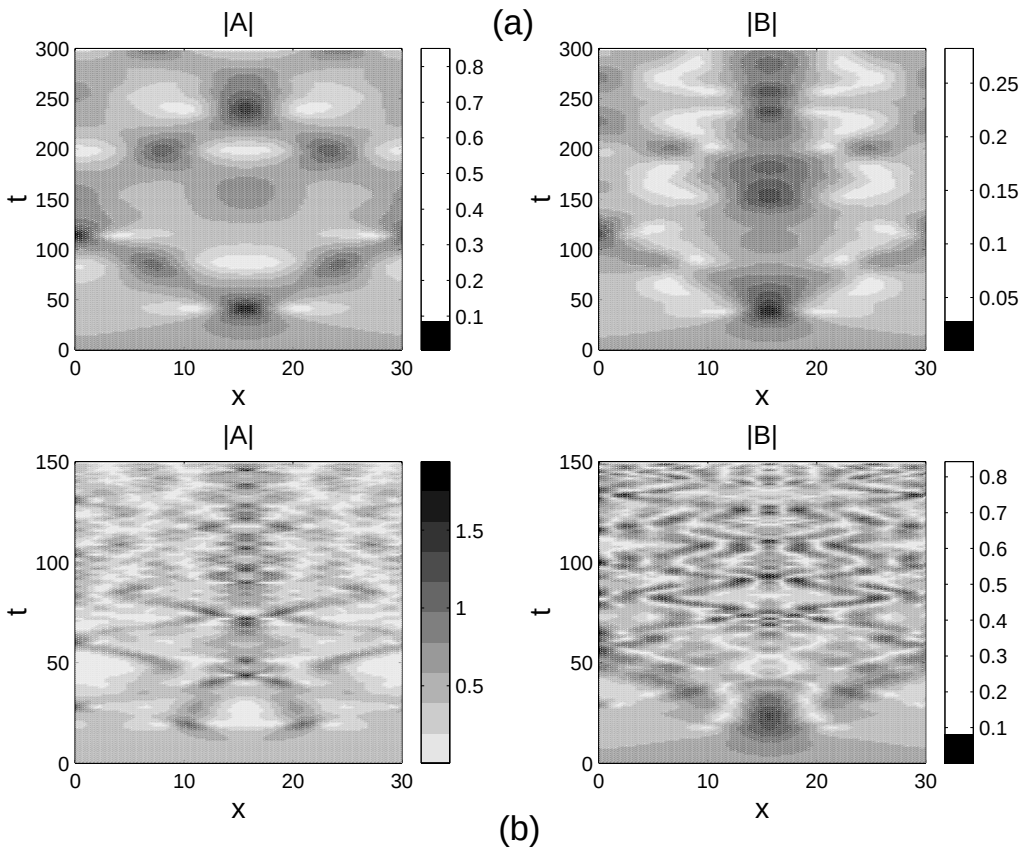

Fig. 7 Long time evolution of the destabilized sine wave solution to the CNLS system with $\alpha_1 = \alpha_2 = 1.0$, $\sigma_1 = \sigma_2 = 1.0$, and $\nu_{12} = -\nu_{21} = -1$. (a) $\varepsilon = 0.1$, $l = 0.2$, $a_0 = 0.3$ and $b_0 = 0.1$. (b) $\varepsilon = 0.1$, $l = 0.2$, $a_0 = 0.5$ and $b_0 = 0.3$.

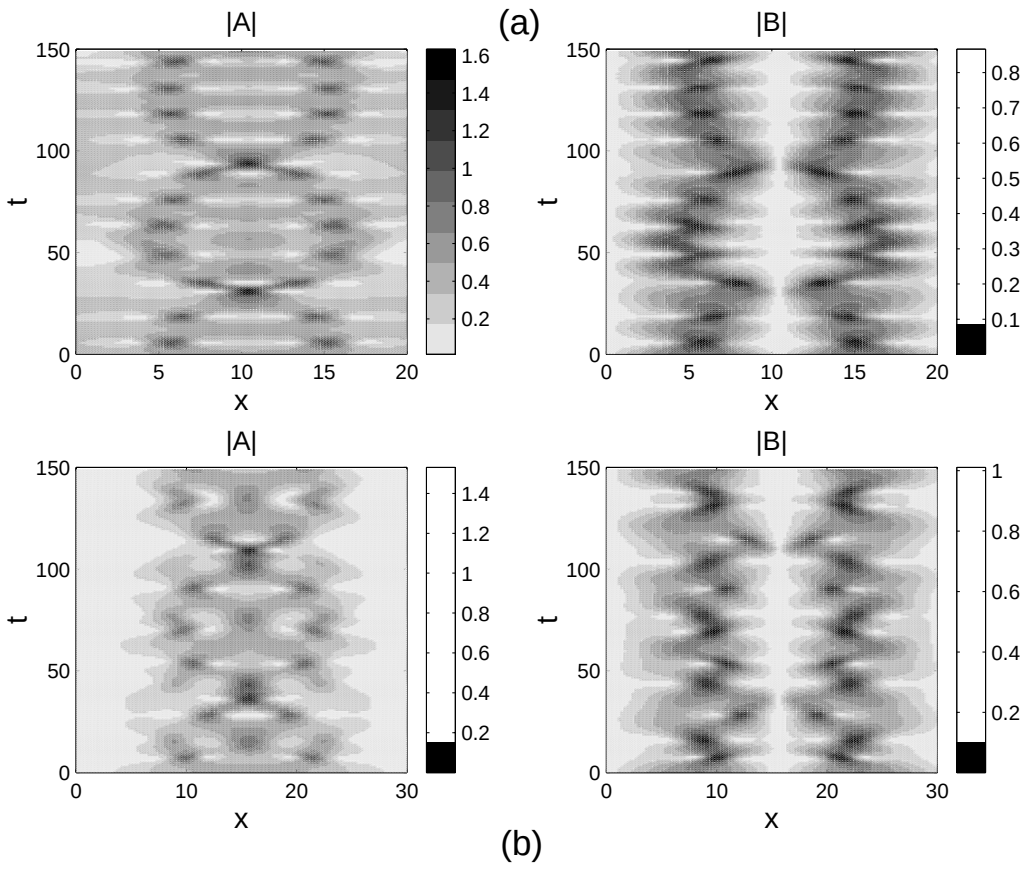

Fig. 8 Long time evolution of the destabilized sine wave solution to the CNLS system with $\alpha_1 = \alpha_2 = 1.0$, $\sigma_1 = \sigma_2 = 1.0$, and $\nu_{12} = \nu_{21} = 1.0$. (a) $A(0, X) = 0.5 \times (1 - 0.1\cos(0.3X))$ and $B(0, X) = 0.5\sin(0.3X)(1 - 0.1\cos(0.3X))$; (b) $A(0, X) = 0.5\sin(0.1X)(1 - 0.1\cos(0.2X))$ and $B(0, X) = 0.5\sin(0.2X)(1 - \cos(0.2X))$.