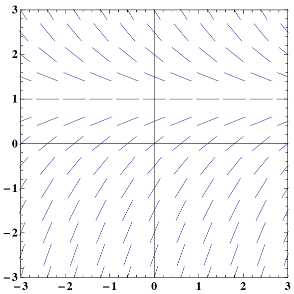
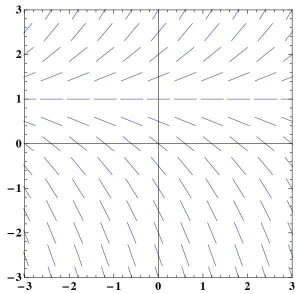
9. [12 points] Match each of the following differential equations with its possible slope field. Circle your answers. No justification is required.

(a) $y' = ay$ with $a > 0$. I II III IV

(b) $y' = k(y - a)$ with $a > 0$ and $k < 0$. I II III IV



I



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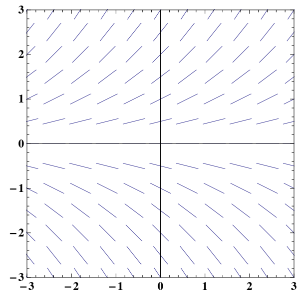

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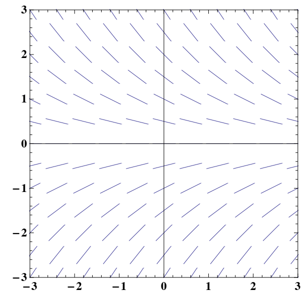
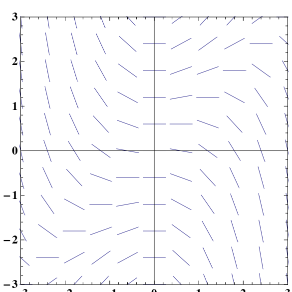


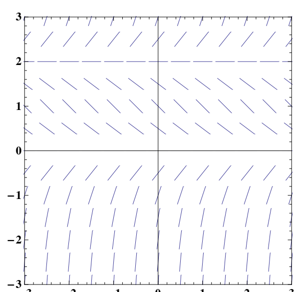
IV

(a) $y' = x(y - ax)$ with $a > 0$. V VI VII VIII

(b) $y' = ay^2 - y$ with $a > 0$. V VI VII VIII



V



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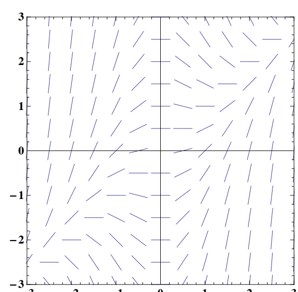
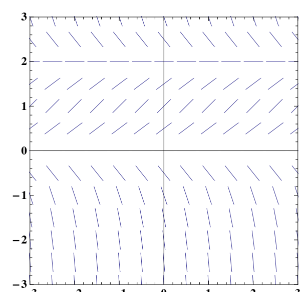


VII



VIII

4. [12 points] Another farmer notices the plague of grasshoppers has spread to his crop. He also visits the pest control company and requests a cheaper pesticide. This new pesticide is capable of eliminating the grasshoppers at a rate that decreases with time. Specifically, the rate at which grasshoppers are killed is given by the function $f(t) = \frac{3}{10}(4 - t)$ in thousands of grasshoppers per week at t weeks after the pesticide application. There is no pesticide remaining after 4 weeks. Suppose there are 3000 grasshoppers at the time the pesticide is applied.

Let $Q(t)$ the population of grasshoppers (in thousands) t weeks after this cheaper pesticide is applied to the crop. Then for $0 \leq t \leq 4$, $Q(t)$ satisfies

$$\frac{dQ}{dt} = \frac{Q}{5} - f(t).$$

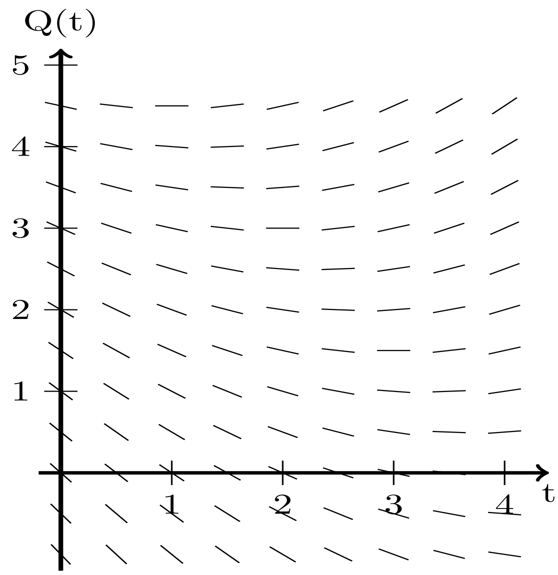
a. [1 point] Is this differential equation separable?

b. [7 points] Using Euler's method, fill the table with the amount of grasshoppers (in thousands) in the crop during the first week. Show all your computations.

t	0	$\frac{1}{2}$	1
$Q(t)$			

(problem 4 continued)

Use the slope field of the differential equation satisfied by $Q(t)$ to answer the following questions.



c. [2 points] Does this equation have any equilibrium solutions in the region shown? List each equilibrium solution and determine whether it is stable or unstable. **Justify your answer**.

d. [2 points] If the farmer's goal is to kill all the grasshoppers in his crop, will the pesticide be effective in this case? Draw the solution $Q(t)$ on the slope field.

3. [14 points] A farmer notices that a population of grasshoppers is growing at undesirable levels in his crop. He decides to hire the services of a pest control company. They offer the farmer a pesticide capable of eliminating the grasshoppers at a rate of 1 thousand grasshoppers per week. In the absence of pesticides, it is estimated that the grasshopper population grows at a rate of 20 percent every week. Let $P(t)$ be the number of grasshoppers (in thousands) t weeks after the pesticide is applied to the crop. Then $P(t)$ satisfies

$$\frac{dP}{dt} = \frac{P}{5} - 1.$$

Suppose there are P_0 thousand grasshoppers in the crop at the time the pesticide is applied in the crop.

a. [8 points] Find a formula for $P(t)$ in terms of t and P_0.

b. [3 points] Does the differential equation have any equilibrium solutions? List each equilibrium solution and determine whether it is stable or unstable. **Justify your answer**.

c. [3 points] Does the effectiveness of the pesticide depend on P_0? That is, is the pesticide guaranteed to eliminate the grasshopper population regardless of the value of P_0, or are there some values of P_0 for which the grasshoppers will survive? If so, determine these values of P_0.

4. [13 points]

a. [6 points] A cylindrical tank with height 8 m and radius of 8 m is standing on one of its circular ends. The tank is initially empty. Water is added at a rate of 2 m^3 /min. A valve at the bottom of the tank releases water at a rate proportional to the water's depth (proportionality constant $= k$). Let $V(t)$ be the volume of the water in the tank at time t, and $h(t)$ be the depth of the water at time t.

i. Find a formula for $V(t)$ in terms of $h(t)$. $V(t) =$ ____________________________

ii. Find the differential equation satisfied by $V(t)$. Include the appropriate initial conditions.

Differential equation: ____________________________ Initial condition: __________

b. [7 points] Let $M(t)$ be the balance in dollars in a bank account t years after the initial deposit. The function $M(t)$ satisfies the differential equation

$$\frac{dM}{dt} = \frac{1}{100}M - a.$$

where a is a positive constant. Find a formula for $M(t)$ if the initial deposit is $1,000$ dollars. Your answer may depend on a.

5. [14 points] A particle moves on the unit circle according to the parametric equations

$$x(t) = -\sin(bt^2) \quad , \quad y(t) = \cos(bt^2) \qquad \text{and } b > 0.$$

for $0 \le t \le \pi$. Make sure to show all your work.

a. [1 point] What is the starting point of the particle?

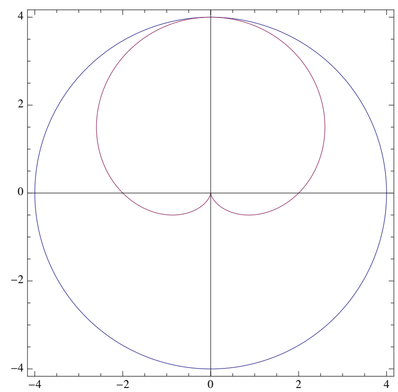
b. [2 points] In which direction (counterclockwise/clockwise) is the particle moving along the circle? Justify.

c. [5 points] Find an expression for the speed of the particle. Simplify it as much as possible.

d. [2 points] At what value of t in $[0, \pi]$ is the speed of the particle the largest?

e. [4 points] Find the equation of the tangent line to the parametric equation at $t = \sqrt{\frac{\pi}{3b}}$.

2. [14 points] The graph of the circle $r = 4$ and and the cardioid $r = 2\sin\theta - 2$ are shown below.

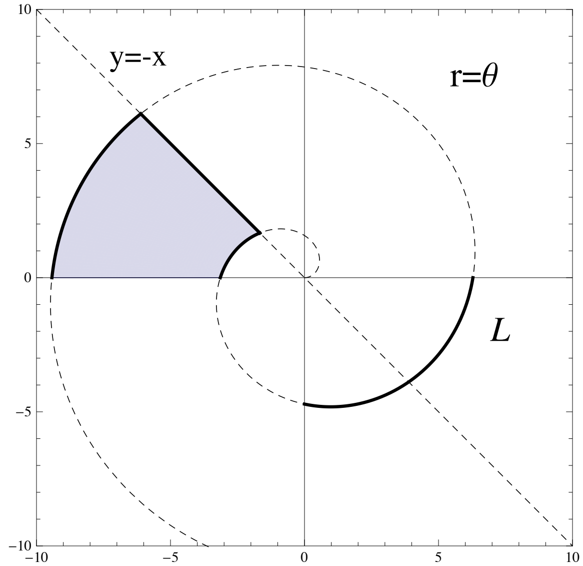


a. [3 points] Write a formula for the area inside the circle and outside the cardioid in the first quadrant.

b. [7 points] At what angles $0 \le \theta < 2\pi$ is the minimum value of the y coordinate on the cardioid attained? No credit will be given for answers without proper mathematical justification.

c. [4 points] Write an integral that computes the value of the length of the piece of the cardioid lying below the x-axis.

2. [11 points] Consider the graph of the spiral $r = \theta$ for $\theta \geq 0$.



In the following questions, write an expression (you do not need to evaluate any integrals) involving definite integrals that computes the values of the following quantities :

a. [4 points] The length of the arc L.

b. [7 points] The area of the shaded region.

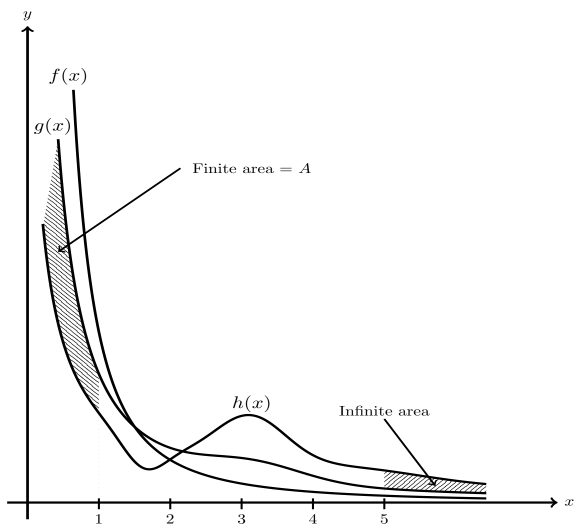
5. [11 points] Determine if the following integrals converge or diverge. If an integral diverges, explain why. If it converges, find the value to which it converges. Mathematical precision is important.

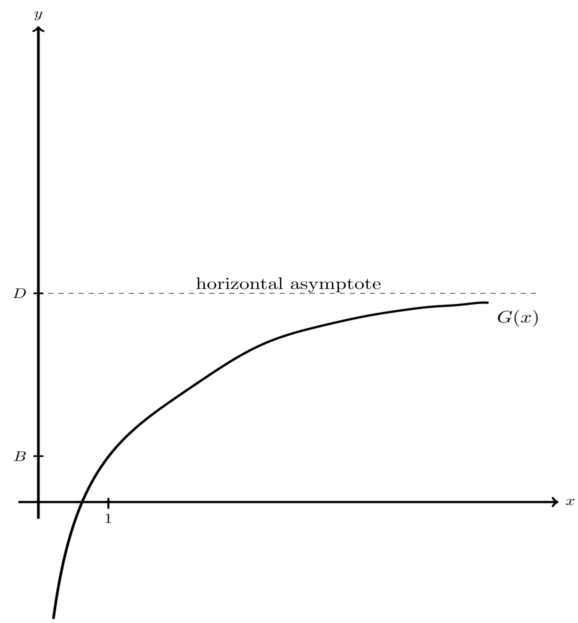
a. [4 points] $\displaystyle\int_{-1}^{2} \frac{1}{\sqrt{2-x}}\,dx$

b. [4 points] $\displaystyle\int_{10}^{\infty} \frac{5+2\sin(4\theta)}{\theta}\,d\theta$

c. [3 points] $\displaystyle\int_{1}^{\infty} \frac{x}{1+x}\,dx$

8. [15 points] Graphs of f, g and h are below. Each function is positive, is continuous on $(0, \infty)$, has a horizontal asymptote at $y = 0$ and has a vertical asymptote at $x = 0$. The area between $g(x)$ and $h(x)$ on the interval $(0, 1]$ is a finite number A, and the area between $g(x)$ and $h(x)$ on the interval $[5, \infty)$ is infinite. On the right is a graph of an antiderivative $G(x)$ of $g(x)$. It also has a vertical asymptote at $x = 0$.
Use the information in these graphs to determine whether the following three improper integrals **converge**, **diverge**, or whether there is **insufficient information to tell**. You may assume that f, g and h have no intersection points other than those shown in the graph. **Justify all your answers**.



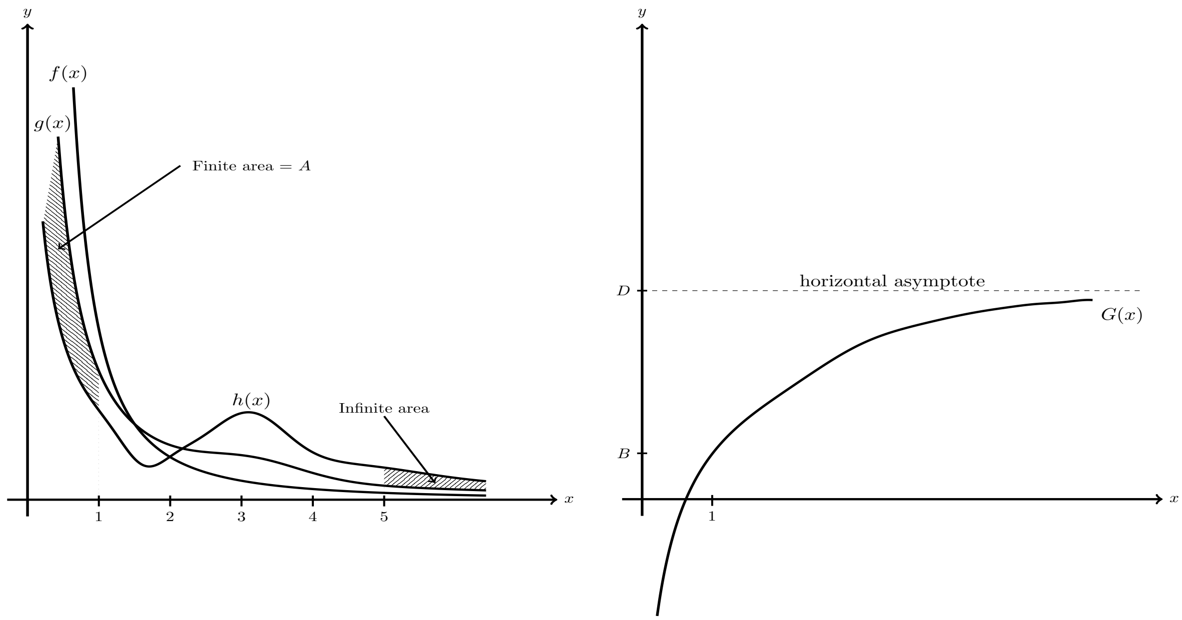


a. [3 points] $\displaystyle\int_1^\infty h(x)dx$

b. [4 points] $\displaystyle\int_0^1 g(x)dx$

(problem 8 continued)

These graphs are the same as those found on the previous page.



c. [3 points] $\int_0^1 h(x)dx$

d. [5 points] If $f(x) = 1/x^p$, what are all the possible values of p? **Justify your answer**.