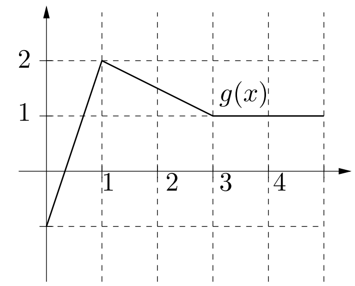
1. [12 points] For all of parts (a)–(d), let $f(x) = 2x - 4$ and let $g(x)$ be given in the graph to the right.



(a) [3 points of 12] Find $\int_1^5 g'(x)\,dx$.

Solution:
By the Fundamental Theorem of Calculus,

$$\int_0^5 g'(x)\,dx = g(5) - g(1) = 1 - (2) = -1.$$

Alternately, note that $g'(x) = 3$ for $0 < x < 1$, $g'(x) = -\frac{1}{2}$ for $1 < x < 3$, and $g'(x) = 0$ for $x > 3$. Thus $\int_1^5 g'(x)\,dx = \int_1^3 -\frac{1}{2}\,dx = -1$.

(b) [3 points of 12] Find $\int_0^5 g(x)\,dx$.

Solution:
This integral is just the area between the graph of $g(x)$ and the x-axis between $x = 0$ and $x = 5$, counting area below the axis as negative. This is

$$\int_0^5 g(x)\,dx = -\text{ (area between 0 and 2/3)} + \text{(area between 0 and 1)} + \text{(area between 1 and 3)} +$$
$$\text{(area between 3 and 5)} = -\left(\frac{1}{2}(\frac{1}{3})(1)\right) + \left(\frac{1}{2}(\frac{2}{3})(2)\right) + (3) + (2) = 5\frac{1}{2}.$$

(c) [3 points of 12] Find $\int_2^{4.5} g(f(x))\,dx$.

Solution:
Using substitution with $w = f(x) = 2x - 4$, we have $\frac{1}{2}dw = dx$, so that $\int_2^{4.5} g(f(x))\,dx = \frac{1}{2}\int_{w(2)}^{w(4.5)} g(w)\,dw = \frac{1}{2}\int_0^5 g(w)\,dw$. But the calculation above gives $\int_0^5 g(w)\,dw = 5\frac{1}{2} = \frac{11}{2}$, so $\int_2^{4.5} g(f(x))\,dx = \frac{1}{2} \cdot \frac{11}{2} = \frac{11}{4} = 2.75$.

(d) [3 points of 12] Find $\int_0^5 f(x) \cdot g'(x)\,dx$.

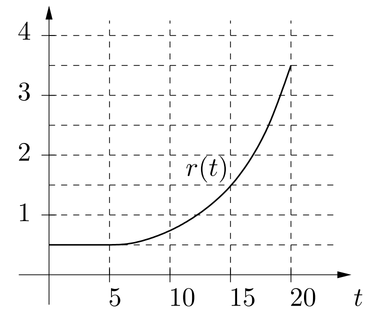
Solution:
Using integration by parts with $u = 2x - 4$ and $v' = g'$, we have $u' = 2$ and $v = g$, so that

$$\int_0^5 f(x) \cdot g'(x)\,dx = (2x-4) \cdot g(x)\Big|_0^5 - 2\int_0^5 g(x)\,dx$$
$$= (6)(1) - (-4)(-1) - 2(\frac{11}{2}) = 2 - 11 = -9.$$

Alternate solution: note that $g'(x) = 3$ for $0 < x < 1$, $g'(x) = -\frac{1}{2}$ for $1 < x < 3$, and $g'(x) = 0$ for $x > 3$. Thus

$$\int_0^5 f(x) \cdot g'(x)\,dx = \int_0^1 3(2x-4)\,dx + \int_1^3 -\frac{1}{2}(2x-4)\,dx$$
$$= 3(x^2 - 4x)\Big|_0^1 + (-\frac{x^2}{2} + 2x)\Big|_1^3 = -9 + (\frac{3}{2} - \frac{3}{2}) = -9.$$

3. [12 points] Having completed their team homework, Alex and Chris are making chocolate chip cookies to celebrate. The rate at which they make their cookies, $r(t)$, is given in cookies/minute in the figure to the right (in which t is given in minutes). After $t =$ 20 minutes they have completed their cookie making extravaganza.



(a) [3 of 12 points] Write an expression for the total number of cookies that they make in the 20 minutes they are baking. Why does your expression give the total number of cookies?

Solution:
We are given the rate at which the cookies are being produced, so we know by the Fundamental Theorem of Calculus that the total number of cookies produced is given by $\int_0^{20} r(t)\,dt$.

(b) [3 of 12 points] Using $\Delta t = 5$, find left- and right-Riemann sum and trapezoid estimates for the total number of cookies that they make.

Solution:
Using $\Delta t = 5$, the left- and right-hand Riemann sums are

$$\text{LEFT}(4) \approx 5(0.5 + 0.5 + 0.75 + 1.5) = 16.25$$
$$\text{RIGHT}(4) \approx 5(0.5 + 0.75 + 1.5 + 3.5) = 31.25.$$

Thus the trapezoid estimate is $\text{TRAP}(4) = \frac{1}{2}(16.25 + 31.25) = 23.75$, or about 24 cookies.

(c) [3 of 12 points] How large could the error in each of your estimates in (b) be?

Solution:
We know that the maximum error in the left- or right-hand sums is just $\Delta t(r(20) - r(0)) = 5(3.5 - 0.5) = 15$ cookies. The maximum error in the trapezoid estimate is half this, or 7.5 cookies.

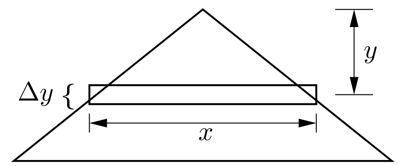
(d) [3 of 12 points] How would you have to change the way you found each of your estimates to reduce the possible errors noted in (c) to one quarter of their current values?

Solution:
The error in the left- or right-hand sums drops as n, so we would have to take four times as many steps, reducing Δt to $\frac{5}{4} = 1.25$ min to reduce the error in those estimates by a factor of four. The error in the trapezoid estimate drops as n^2, the square of the number of steps we take in the calculation, so if we recalculated our estimate with $\Delta t = 2.5$ minutes it would be four times as accurate.

3. [10 points] The Great Pyramid of Giza in Egypt was originally (approximately) 480 ft high. Its base was originally (again, approximately) a square with side lengths 760 ft.

(a) [6 points of 10] Sketch a slice that could be used to calculate the volume of the pyramid by integrating. In your sketch, indicate all variables you are using. Find an expression for the volume of the slice in terms of those variables.

Solution:
The pyramid is shown in cross-section to the right. A representative slice is given a distance y down from the top of the pyramid. It is a square box with side length x and height Δy, as shown. We can find x in terms of y by using similar triangles: the base is 760 ft and height 480 ft, so $\frac{x}{y} = \frac{760}{480}$, or $x = \frac{760}{480}y$. Thus the volume of the slice is $V_{sl} = (\text{length})^2\Delta y = (\frac{760}{480}y)^2\Delta y$ ft^3.



Alternately, we might take y to measure up from the base. In this case $\frac{x}{480-y} = \frac{760}{480}$ so that $V_{sl} = (760 - \frac{760}{480}y)^2\Delta y$ ft^3.

(b) [4 points of 10] Use your slice from (a) to find the volume of the pyramid.

Solution:
We let $\Delta y \to 0$, getting the integral $V = \int_0^{480} (\frac{760}{480}y)^2\,dy$. Expanding the square and integrating, we get

$$V = \left(\frac{760}{480}\right)^2 \int_0^{480} y^2\,dy = \frac{361}{144}\left(\frac{1}{3}\,y^3\right)\Big|_0^{480} = \frac{361}{432}(480^3 - 0) = 92,416,000 \text{ ft}^3.$$

Or, if we had $V_{sl} = (760 - \frac{760}{480}y)^2\Delta y$,

$$V = \int_0^{480} (760 - \frac{760}{480}\,y)^2\,dy = -(\frac{480}{760})(\frac{1}{3})(760 - \frac{760}{480})^3\Big|_0^{480}$$
$$= -(\frac{160}{760})(0 - 760^3) = 92,416,000 \text{ ft}^3.$$

6. [10 points] While eating cookies, Alex notes that the velocity of a student passing by is given, in meters/second, by the data shown below.

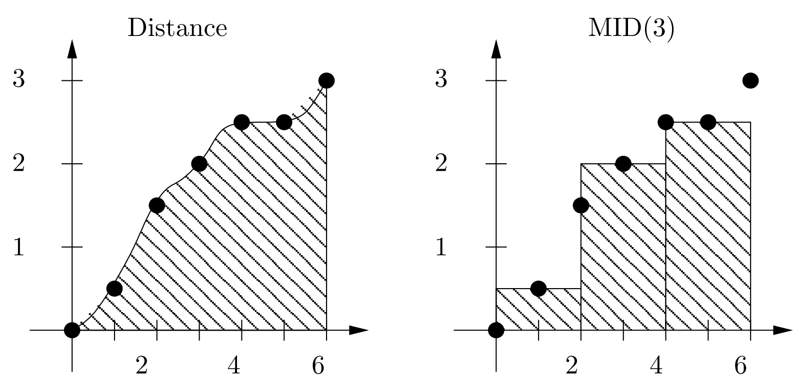
t (seconds)	0	1	2	3	4	5	6
$v(t)$ (m/s)	0	0.5	1.5	2	2.5	2.5	3

(a) [5 of 10 points] Using the midpoint rule, find as accurate an estimate as possible for the total distance the student travels in the six seconds shown in the table (use only the given data in your calculation).

Solution:
The smallest value of Δt we can use with the given data is $\Delta t = 2$ seconds; if we try to use $\Delta t = 1$, we need values of v at $t = \frac{1}{2}$, $\frac{3}{2}$, etc., which aren't available in the given data. Then, using $\Delta t = 2$, we have

$$\text{MID}(3) = 2(0.5 + 2 + 2.5) = 10 \text{ meters.}$$

(b) [5 of 10 points] Draw two figures on the axes below, one each to illustrate the total distance you are estimating and the estimate you found. Be sure it is clear how your figures illustrate the indicated quantities.



Solution:
We show the actual distance traveled and MID(3) in the figures above right. The actual distance traveled is the shaded area under the function $v(t)$ between $t = 0$ and $t = 6$; MID(3) is the shaded area in the indicated rectangles.

1. [12 points] While at home for Thanksgiving, Alex finds a forgotten can of corn that has been sitting on the shelf for a number of years. The contents have started to settle towards the bottom of the can, and the density of corn inside the can is therefore a function, $\delta(h)$, of the height h (measured in cm) from the bottom of the can. δ is measured in g/cm^3. The can has a radius of 4 cm, and a height of 12 cm.

(a) [3 points of 12] Write an expression that approximates the mass of corn in the cylindrical cross-section from height h to height $h + \Delta h$.

Solution:
The cylindrical cross-section is a disk with height Δh and radius 4 cm, so its volume is $\Delta V = \pi(4)^2 \cdot \Delta h$. The mass of the cross-section is then $\Delta M = \delta(h) \cdot \Delta V = 16\pi \cdot \delta(h) \cdot \Delta h$.

(b) [3 points of 12] Write a definite integral that gives the total mass of corn in the can.

Solution:
We let $\Delta h \to 0$ and add the contributions from each disk by integrating, to get

$$M = \int_0^{12} 16\pi \cdot \delta(h)\, dh.$$

(c) [3 points of 12] If $\delta(h) = 4e^{-0.03h}$, what is the total mass of corn inside the can?

Solution:
We have $M = \int_0^{12} 16\pi \cdot \delta(h)\, dh = \int_0^{12} 16\pi \cdot 4e^{-0.03h}\, dh$. Thus

$$M = 64\pi \int_0^{12} e^{-0.03h}\, dh = -\frac{64}{0.03}\pi\, e^{-0.03h}\Big|_0^{12} = -\frac{6400}{3}\pi \left(e^{-0.36} - 1\right) \approx 2026.2 \text{ g}.$$

(d) [3 points of 12] Write, but do not evaluate, an expression for the can's center of mass in the h direction. Would you expect the center of mass to be in the top or bottom half of the can? Do not solve for the center of mass, but in one sentence, justify your answer.

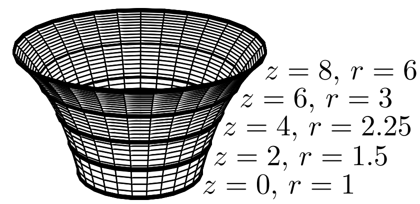
Solution:
The center of mass is

$$\overline{h} = \frac{\int_0^{12} 16\pi \cdot h \cdot \delta(h)\, dh}{M},$$

where M is the mass we found before. We expect this to be in the bottom half of the can, because the density decreases with increasing h. (Obviously, we could also write $\overline{h} = \frac{\int_0^{12} 16\pi \cdot h \cdot \delta(h)\, dh}{\int_0^{12} 16\pi \cdot \delta(h)}$; plugging in $\delta(h)$ from (c) is fine too.)

4. [16 points] An entrepreneurial University of Michigan Business Squirrel is marketing childrens' buckets with curved sides, as shown in the figure to the right, below. The figure gives the radius of the bucket, r, at different heights, z, from the bottom of the bucket. All lengths are given in inches. Suppose that a child fills one of these buckets with muddy water.

a. [4 points] If the density of the water in the bucket is $\delta(z)$ oz/in^3, write an integral that gives the mass of the water in the bucket.



Solution: Slicing the bucket horizontally, the mass is

$$M = \int_0^8 \pi\,\delta(z)\,(r(z))^2\,dz \quad \text{oz.}$$

b. [4 points] If $\delta(z) = (24 - z)$ oz/in^3, estimate the mass using your integral from (a).

Solution: The data given provide values of $r(z)$ at steps of size $\Delta z = 2$ in. We can find $\delta(z)$ at each of the points $z = 0$, 2, etc., and find a left or right Riemann sum for the mass. A better estimate would be the average of the two (the trapezoid estimate):

$$\begin{aligned} \text{LEFT} &= 2\pi\ \ 24(1)^2 + 22(3/2)^2 + 20(9/4)^2 + 18(3)^2 \ \approx 2116 \text{ oz} \\ \text{RIGHT} &= 2\pi\ \ 22(3/2)^2 + 20(9/4)^2 + 18(3)^2 + 16(6)^2 \ \approx 5584 \text{ oz} \\ \text{TRAP} &\approx 0.5\,(2116 + 5584) = 3850 \text{ oz} \end{aligned}$$

c. [8 points] Estimate the center of mass of the bucket.

Solution: By symmetry, the center of mass must be along the centerline of the bucket, so if $\overline{x}$ and $\overline{y}$ give the coordinates along the base of the bucket, we know $\overline{x} = 0$ and $\overline{y} = 0$. The z center of mass is

$$\overline{z} = \frac{\int_0^8 \pi\,z\,\delta(z)\,(r(z))^2\,dz}{M} \quad \text{in,}$$

where M is the mass given in (a). We have the mass from (b), and so need only estimate the integral $\int_0^8 \pi\,z\,\delta(z)\,(r(z))^2\,dz$. Again, we can find a left or right Riemann sum or a trapezoid estimate:

$$\begin{aligned} \text{LEFT} &= 2\pi\ \ 24(0)(1)^2 + 22(2)(3/2)^2 + 20(4)(9/4)^2 + 18(6)(3)^2 \ = 2952\pi \approx 9274 \\ \text{RIGHT} &= 2\pi\ \ 22(2)(3/2)^2 + 20(4)(9/4)^2 + 18(6)(3)^2 + 16(8)(6)^2 \ = 12,168\pi \approx 38,227 \\ \text{TRAP} &\approx 0.5\,(2952\pi + 12,168\pi) = 7560\pi \approx 23,750. \end{aligned}$$

Then the center of mass is $\overline{z} \approx 23,750/3850 = 6.17$ in. Using just a left-sum, we have $\overline{z} \approx 9274/2116 = 4.60$ in, and the right-sum gives $\overline{z} \approx 38,227/5584 = 6.85$ in.

6. [12 points] Recall that the Great Pyramid of Giza was (originally) approximately 480 ft high, with a square base approximately 760 ft to a side. The pyramid was made of close to 2.4 million limestone blocks, and has several chambers and halls that extended into its center. It is not too far from the truth to suppose that these open areas are located along the vertical centerline of the pyramid, and that we can therefore think of the density of the pyramid varying only along its vertical dimension. Suppose that the result is that the density of the pyramid is approximately $\delta(h) = (0.00011(h-240)^2 + 134.2)$ lb/ft^3, where h is the height measured up from the base of the pyramid.

(a) [6 points of 12] Set up an integral to find the weight W of the pyramid. You need not evaluate the integral to find the actual weight.

Solution:
Because the density of the pyramid varies along its vertical dimension, we have to slice it in that direction to find the weight of each slice and then sum them with an integral. If z is the distance down from the top of the pyramid and x is the length of the edge of the square slice, the volume of the slice is $\Delta V = x^2 \Delta z$. From similar triangles, we have that $\frac{x}{z} = \frac{760}{480}$, so that $x = \frac{760}{480}z$, so that $\Delta V = (\frac{760}{480})^2 z^2 \Delta z$. By symmetry we can see that $\delta(h) = \delta(z)$, or we can find this explicitly: $\delta(h) = \delta(480-z) = 0.00011(480 - z - 240)^2 + 134.2 = 0.00011(240-z)^2 + 134.2 = 0.0001(z-240)^2 + 134.2 = \delta(z)$. Thus the weight of the pyramid is

$$W = \int_0^{480} \left(\frac{760}{480}\right)^2 z^2 (0.0001(z-240)^2 + 134.2)\, dz.$$

Evaluating this numerically, we can find $W \approx 1.2615 \times 10^{10}$ lbs (about 12.6 billion pounds).

Alternately, in terms of h, the distance up from the ground, $\Delta V = (760 - \frac{760}{480}h)^2 \Delta h$, so that $W = \int_0^{480} (760 - \frac{760}{480}h)^2 (0.00011(h-240)^2 + 134.2)\, dh$.

(b) [6 points of 12] Give an expression, in terms of integral(s), that tells how far off the ground the center of mass of the pyramid is. Again, you need not evaluate the integral(s). *(Note that you may set up the expression in terms of the weight density without worrying about converting it to a mass density.)*

Solution:
Again working with z, the distance down from the top of the pyramid, we have

$$\overline{z} = \frac{\int_0^{480} \left(\frac{760}{480}\right)^2 z^2 \delta(z) \cdot z\, dz}{W} = \frac{\int_0^{480} \left(\frac{760}{480}\right)^2 z^3 (0.0001(z-240)^2 + 134.2)\, dz}{W},$$

where W is given in part (a). Then the height of the center of mass above the surface of the desert is $\overline{h} = 480 - \overline{z}$. Using $W = 1.2615 \times 10^{10}$ lbs, found in (a), we can find $\overline{z}$ and $\overline{h}$ by evaluating the integrals numerically: $\overline{z} \approx 361$ ft, so $\overline{h} \approx 480 - \overline{z} = 119$ feet above the ground.