5

PROBLEMS IN RECTANGULAR COÖRDINATES

The Cartesian coördinate system (x, y) is clearly particularly suited to the problem of determining the stresses in a rectangular body whose boundaries are defined by equations of the form $x = a, y = b$. A wide range of such problems can be treated using stress functions which are polynomials in x, y. In particular, polynomial solutions can be obtained for 'Mechanics of Materials' type beam problems in which a rectangular bar is bent by an end load or by a distributed load on one or both faces.

5.1 Biharmonic polynomial functions

In rectangular coördinates, the biharmonic equation takes the form

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \tag{5.1}$$

and it follows that any polynomial in x, y of degree less than four will be biharmonic and is therefore appropriate as a stress function. However, for higher order polynomial terms, equation (5.1) is not identically satisfied. Suppose, for example, that we consider just those terms in a general polynomial whose combined degree (the sum of the powers of x and y) is N. We can write these terms in the form

$$P_N(x,y) = A_0x^N + A_1x^{N-1}y + A_2x^{N-2}y^2 + \ldots + A_Ny^N \tag{5.2}$$

$$= \sum_{i=0}^{N} A_i x^{N-i} y^i \; , \tag{5.3}$$

where we note that there are $(N{+}1)$ independent coefficients, $A_i (i{=}0, N)$. If we now substitute $P_N(x, y)$ into equation (5.1), we shall obtain a new polynomial of degree $(N{-}4)$, since each term is differentiated four times. We can denote this new polynomial by $Q_{N-4}(x, y)$ where

$$Q_{N-4}(x,y) = \nabla^4 P_N(x,y) \tag{5.4}$$

$$= \sum_{i=0}^{N-4} B_i x^{(N-4-i)} y^i \ . \tag{5.5}$$

The $(N-3)$ coefficients $B_0, \ldots, B_{N-4}$ are easily obtained by expanding the right-hand side of equation (5.4) and equating coefficients. For example,

$$B_0 = N(N-1)(N-2)(N-3)A_0 + 4(N-2)(N-3)A_2 + 24A_4 \ . \tag{5.6}$$

Now the original function $P_N(x,y)$ will be biharmonic if and only if $Q_{N-4}(x,y)$ is zero for all x, y and this in turn is only possible if every term in the series (5.5) is identically zero, since the polynomial terms are all linearly independent of each other. In other words

$$B_i = 0 \ ; \ \ i = 0 \text{ to } (N-4) \ . \tag{5.7}$$

These conditions can be converted into a corresponding set of $(N-3)$ equations for the coefficients A_i. For example, the equation $B_0 = 0$ gives

$$N(N-1)(N-2)(N-3)A_0 + 4(N-2)(N-3)A_2 + 24A_4 = 0 \ , \tag{5.8}$$

from (5.6). We shall refer to the $(N-3)$ equations of this form as *constraints* on the coefficients A_i, since the coefficients are constrained to satisfy them if the original polynomial is to be biharmonic.

One approach would be to use the constraint equations to eliminate $(N–3)$ of the unknown coefficients in the original polynomial — for example, we could treat the first four coefficients, A_0, A_1, A_2, A_3, as unknown constants and use the constraint equations to define all the remaining coefficients in terms of these unknowns. Equation (5.8) would then be treated as an equation for A_4 and the subsequent constraint equations would each define one new constant in the series. It may help to consider a particular example at this stage. Suppose we consider the fifth degree polynomial

$$P_5(x,y) = A_0x^5 + A_1x^4y + A_2x^3y^2 + A_3x^2y^3 + A_4xy^4 + A_5y^5 \ , \tag{5.9}$$

which has six independent coefficients. Substituting into equation (5.4), we obtain the first degree polynomial

$$Q_1(x,y) = (120A_0 + 24A_2 + 24A_4)x + (24A_1 + 24A_3 + 120A_5)y \ . \tag{5.10}$$

The coefficients of x and y in Q_1 must both be zero if P_5 is to be biharmonic and we can write the resulting two constraint equations in the form

$$A_4 = -5A_0 - A_2 \tag{5.11}$$

$$A_5 = -A_1/5 - A_3/5 \ . \tag{5.12}$$

Finally, we use (5.11, 5.12) to eliminate A_4, A_5 in the original definition of P_5, obtaining the definition of the most general biharmonic fifth degree polynomial

$$P_5(x,y) = A_0(x^5 - 5xy^4) + A_1(x^4y - y^5/5) + A_2(x^3y^2 - xy^4) + A_3(x^2y^3 - y^5/5) \ . \tag{5.13}$$

This function will be biharmonic for any values of the four independent constants A_0, A_1, A_2, A_3. We can express this by stating that the biharmonic polynomial P_5 has four degrees of freedom.

In general, the polynomial Q is of degree 4 less than P because the biharmonic equation is of degree 4. It follows that there are always four fewer constraint equations than there are coefficients in the original polynomial P and hence that they can be satisfied leaving a polynomial with 4 degrees of freedom. However, the process degenerates if $N < 3$.

In view of the above discussion, it might seem appropriate to write an expression for the general polynomial of degree N in the form of equation (5.13) as a preliminary to the solution of polynomial problems in rectangular coördinates. However, as can be seen from equation (5.13), the resulting expressions are algebraically messy and this approach becomes unmanageable for problems of any complexity. Instead, it turns out to be more straightforward algebraically to define problems in terms of the simpler unconstrained polynomials like equation (5.2) and to impose the constraint equations at a later stage in the solution.

5.1.1 Second and third degree polynomials

We recall that the stress components are defined in terms of the stress function ϕ through the relations

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \tag{5.14}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \tag{5.15}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \ . \tag{5.16}$$

It follows that when the stress function is a polynomial of degree N in x, y, the stress components will be polynomials of degree $(N-2)$. In particular, constant and linear terms in ϕ correspond to null stress fields (zero stress everywhere) and can be disregarded.

The second degree polynomial

$$\phi = A_0x^2 + A_1xy + A_2y^2 \tag{5.17}$$

yields the stress components

$$\sigma_{xx} = 2A_2 \;\; ; \;\; \sigma_{xy} = -A_1 \;\; ; \;\; \sigma_{yy} = 2A_0 \tag{5.18}$$

and hence corresponds to the most general state of biaxial uniform stress.

The third degree polynomial

$$\phi = A_0x^3 + A_1x^2y + A_2xy^2 + A_3y^3 \tag{5.19}$$

yields the stress components

$$\sigma_{xx} = 2A_2x + 6A_3y \;\; ; \;\; \sigma_{xy} = -2A_1x - 2A_2y \;\; ; \;\; \sigma_{yy} = 6A_0x + 2A_1y \; . \tag{5.20}$$

If we arbitrarily set $A_0, A_1, A_2 = 0$, the only remaining non-zero stress will be

$$\sigma_{xx} = 6A_3y \; , \tag{5.21}$$

which corresponds to a state of pure bending, when applied to the rectangular beam $-a<x<a, -b<y<b$, as shown in Figure 5.1.

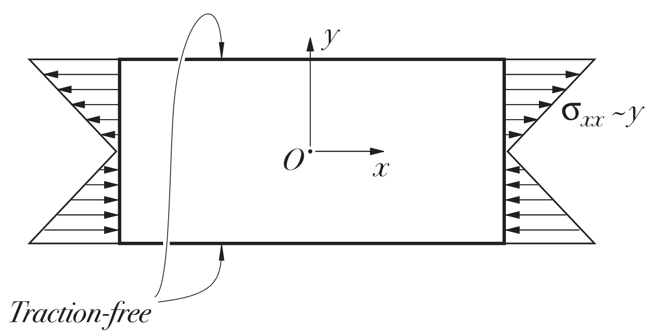

Figure 5.1: The rectangular beam in pure bending.

The other terms in equation (5.19) correspond to a more general state of bending. For example, the constant A_0 describes bending of the beam by tractions σ_{yy} applied to the boundaries $y = \pm b$, whilst the terms involving shear stresses σ_{xy} could be obtained by describing a general state of biaxial bending with reference to a Cartesian coördinate system which is not aligned with the axes of the beam.

The above solutions are of course very elementary, but we should remember that, in contrast to the Mechanics of Materials solutions for simple bending, they are obtained without making any simplifying assumptions about the stress fields. For example, we have not assumed that plane sections remain plane, nor have we demanded that the beam be long in comparison with its depth. Thus, the present section could be taken as verifying the *exactness* of the Mechanics of Materials solutions for uniform stress and simple bending, as applied to a rectangular beam.

5.2 Rectangular beam problems

5.2.1 Bending of a beam by an end load

Figure 5.2 shows a rectangular beam, $0 < x < a, -b < y < b$, subjected to a transverse force, F at the end $x = 0$, and built-in at the end $x = a$, the horizontal boundaries $y = \pm b$ being traction free. The boundary conditions for this problem are most naturally written in the form

$$\sigma_{xy} = 0 \; ; \; y = \pm b \tag{5.22}$$

$$\sigma_{yy} = 0 \; ; \; y = \pm b \tag{5.23}$$

$$\sigma_{xx} = 0 \; ; \; x = 0 \tag{5.24}$$

$$\int_{-b}^{b} \sigma_{xy} dy = F \; ; \; x = 0 \; . \tag{5.25}$$

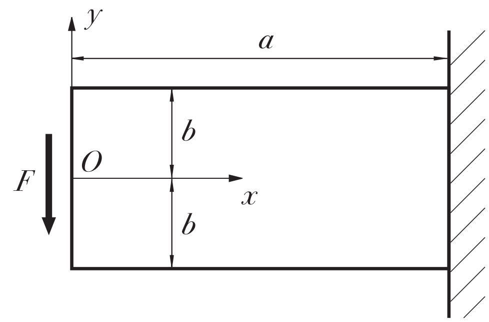

Figure 5.2: Cantilever with an end load.

The boundary condition (5.25) is imposed in the *weak* form, which means that the value of the traction is not specified at each point on the boundary — only the force resultant is specified. In general, we shall find that problems for the rectangular beam have finite polynomial solutions when the boundary conditions on the ends are stated in the weak form, but that the *strong* (i.e. pointwise) boundary condition can only be satisfied on all the boundaries by an infinite series or transform solution. This issue is further discussed in Chapter 6.

Mechanics of Materials considerations suggest that the bending moment in this problem will vary linearly with x and hence that the stress component σ_{xx} will have a leading term proportional to xy. This in turn suggests a fourth degree polynomial term xy^3 in the stress function ϕ. Our procedure is therefore to start with the trial stress function

$$\phi = C_1 xy^3 \; , \tag{5.26}$$

examine the corresponding tractions on the boundaries and then seek a *corrective* solution which, when superposed on equation (5.26), yields the solution to the problem. Substituting (5.26) into (5.14–5.16), we obtain

$$\sigma_{xx} = 6C_1xy \tag{5.27}$$
$$\sigma_{xy} = -3C_1y^2 \tag{5.28}$$
$$\sigma_{yy} = 0\ , \tag{5.29}$$

from which we note that the boundary conditions (5.23, 5.24) are satisfied identically, but that (5.22) is not satisfied, since (5.28) implies the existence of an unwanted uniform shear traction $-3C_1b^2$ on both of the edges $y=\pm b$. This unwanted traction can be removed by superposing an appropriate uniform shear stress, through the additional stress function term C_2xy. Thus, if we define

$$\phi = C_1xy^3 + C_2xy\ , \tag{5.30}$$

equations (5.27, 5.29) remain unchanged, whilst (5.28) is modified to

$$\sigma_{xy} = -3C_1y^2 - C_2\ . \tag{5.31}$$

The boundary condition (5.22) can now be satisfied if we choose C_2 to satisfy the equation

$$C_2 = -3C_1b^2\ , \tag{5.32}$$

so that

$$\sigma_{xy} = 3C_1(b^2 - y^2)\ . \tag{5.33}$$

The constant C_1 can then be determined by substituting (5.33) into the remaining boundary condition (5.25), with the result

$$C_1 = \frac{F}{4b^3}\ . \tag{5.34}$$

The final stress field is therefore defined through the stress function

$$\phi = \frac{F(xy^3 - 3b^2xy)}{4b^3}\ , \tag{5.35}$$

the corresponding stress components being

$$\sigma_{xx} = \frac{3Fxy}{2b^3} \tag{5.36}$$
$$\sigma_{xy} = \frac{3F(b^2 - y^2)}{4b^3} \tag{5.37}$$
$$\sigma_{yy} = 0\ . \tag{5.38}$$

The solution of this problem is given in the Mathematica and Maple files 'S521'.

We note that no boundary conditions have been specified on the built-in end, $x=a$. In the weak form, these would be

$$\int_{-b}^{b} \sigma_{xx} dy = 0 \;\; ; \;\; x = a \tag{5.39}$$

$$\int_{-b}^{b} \sigma_{xy} dy = F \;\; ; \;\; x = a \tag{5.40}$$

$$\int_{-b}^{b} \sigma_{xx} y dy = Fa \;\; ; \;\; x = a \, . \tag{5.41}$$

However, if conditions (5.22–5.25) are satisfied, (5.39–5.41) are merely equivalent to the condition that the whole beam be in equilibrium. Now the Airy stress function is so defined that whatever stress function is used, the corresponding stress field will satisfy equilibrium in the local sense of equations (2.5). Furthermore, if every particle of a body is separately in equilibrium, it follows that the whole body will also be in equilibrium. It is therefore not necessary to enforce equations (5.39–5.41), since if we were to check them, we should necessarily find that they are satisfied identically.

5.2.2 Higher order polynomials — a general strategy

In the previous section, we developed the solution by trial and error, starting from the leading term whose form was dictated by equilibrium considerations. A more general technique is to identify the highest order polynomial term from equilibrium considerations and then write down the most general polynomial of that degree and below. The constant multipliers on the various terms are then obtained by imposing boundary conditions and biharmonic constraint equations.

The only objection to this procedure is that it involves a lot of algebra. For example, in the problem of §5.2.1, we would have to write down the most general polynomial of degree 4 and below, which involves 12 separate terms even when we exclude the linear and constant terms as being null. However, this is not a serious difficulty if we are using Maple or Mathematica, so we shall first develop the steps needed for this general strategy. Shortcuts which would reduce the complexity of the algebra in a manual calculation will then be discussed in §5.2.3.

Order of the polynomial

Suppose we have a normal traction on the surface $y=b$ varying with x^n. In Mechanics of Materials terms, this corresponds to a distributed load proportional to x^n and elementary equilibrium considerations show that the bending moment can then be expected to contain a term proportional to x^{n+2}. This in turn implies a bending stress σ_{xx} proportional to $x^{n+2}y$ and a term in the stress function proportional to $x^{n+2}y^3$ — i.e. a term of polynomial order $(n+5)$. A corresponding argument for shear tractions proportional to x^m shows that we require a polynomial order of $(m+4)$.

We shall show in Chapter 28 that these arguments from equilibrium and elementary bending theory define the highest order of polynomial required to satisfy any polynomial boundary conditions on the lateral surfaces of a beam, even in three-dimensional problems. A sufficient polynomial order can therefore be selected by the following procedure:-

(i) Identify the highest order polynomial term n in the normal tractions σ_{yy} on the surfaces $y=\pm b$.
(ii) Identify the highest order polynomial term m in the shear tractions σ_{yx} on the surfaces $y=\pm b$.
(iii) Use a polynomial for ϕ including all polynomial terms of order $\max(m+4, n+5)$ and below, but excluding constant and linear terms.

In the special case where both surfaces $y=\pm b$ are traction-free, it is sufficient to use a polynomial of 4th degree and below (as in §5.2.1).

Solution procedure

Once an appropriate polynomial has been identified for ϕ, we proceed as follows:-

(i) Substitute ϕ into the biharmonic equation (5.1), leading to a set of constraint equations, as in §5.1.
(ii) Substitute ϕ into equations (5.14–5.16), to obtain the stress components as functions of x, y.
(iii) Substitute the equations defining the boundaries (e.g. $x=0, y=b, y=-b$ in the problem of §5.2.1) into appropriate[1] stress components, to obtain the tractions on each boundary.
(iv) For the longer boundaries (where strong boundary conditions will be enforced), sort the resulting expressions into powers of x or y and equate coefficients with the corresponding expression for the prescribed tractions.
(v) For the shorter boundaries, substitute the tractions into the appropriate weak boundary conditions, obtaining three further independent algebraic equations.

The equations so obtained will generally not all be linearly independent, but they will be sufficient to determine all the coefficients uniquely. The solvers in Maple and Mathematica can handle this redundancy.

[1] Recall from §1.1.1 that the only stress components that act on (e.g.) $y=b$ are those which contain y as one of the suffices.

Example

We illustrate this procedure with the example of Figure 5.3, in which a rectangular beam $-a < x < a$, $-b < y < b$ is loaded by a uniform compressive normal traction p on $y=b$ and simply supported at the ends.

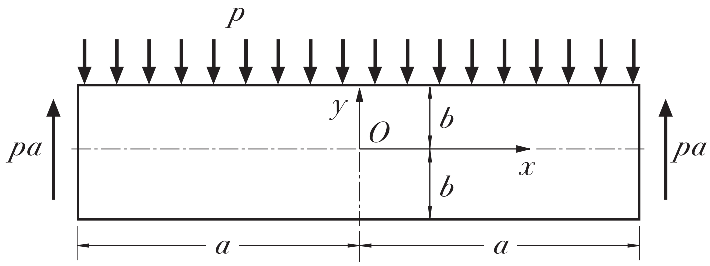

Figure 5.3: Simply supported beam with a uniform load.

The boundary conditions on the surfaces $y=\pm b$ can be written

$$\sigma_{yx} = 0 \;\; ; \;\; y = \pm b \tag{5.42}$$

$$\sigma_{yy} = -p \;\; ; \;\; y = b \tag{5.43}$$

$$\sigma_{yy} = 0 \;\; ; \;\; y = -b \; . \tag{5.44}$$

These boundary conditions are to be satisfied in the strong sense. To complete the problem definition, we shall require three linearly independent weak boundary conditions on one or both of the ends $x=\pm a$. We might use symmetry and equilibrium to argue that the load will be equally shared between the supports, leading to the conditions[2]

$$F_x(a) = \int_{-b}^{b} \sigma_{xx}(a, y)dy = 0 \tag{5.45}$$

$$F_y(a) = \int_{-b}^{b} \sigma_{xy}(a, y)dy = pa \tag{5.46}$$

$$M(a) = \int_{-b}^{b} \sigma_{xx}(a, y)ydy = 0 \tag{5.47}$$

[2] It is not necessary to use symmetry arguments to obtain three linearly independent weak conditions. Since the beam is simply supported, we know that

$$M(a) = \int_{-b}^{b} \sigma_{xx}(a, y)ydy = 0 \;\; ; \;\; M(-a) = \int_{-b}^{b} \sigma_{xx}(-a, y)ydy = 0$$

$$F_x(a) = \int_{-b}^{b} \sigma_{xx}(a, y)dy = 0 \; .$$

It is easy to verify that these conditions lead to the same solution as (5.45–5.47).

on the end $x = a$. As explained in §5.2.1, we do not need to enforce the additional three weak conditions on $x=-a$.

The normal traction is uniform — i.e. it varies with x^0 $(n = 0)$, so the above criterion demands a polynomial of order $(n+5)=5$. We therefore write

$$\begin{aligned}\phi = C_1x^2 + C_2xy + C_3y^2 + C_4x^3 + C_5x^2y + C_6xy^2 + C_7y^3 + C_8x^4 \\ +C_9x^3y + C_{10}x^2y^2 + C_{11}xy^3 + C_{12}y^4 + C_{13}x^5 + C_{14}x^4y \\ +C_{15}x^3y^2 + C_{16}x^2y^3 + C_{17}xy^4 + C_{18}y^5 \ .\end{aligned} \tag{5.48}$$

This is a long expression, but remember we only have to type it in once to the computer file. We can cut and paste the expression in the solution of subsequent problems (and the reader can indeed cut and paste from the web file 'polynomial'). Substituting (5.48) into the biharmonic equation (5.1), we obtain

$$\begin{aligned}(120C_{13} + 24C_{15} + 24C_{17})x + (24C_{14} + 24C_{16} + 120C_{18})y \\ +(24C_8 + 8C_{10} + 24C_{12}) = 0\end{aligned} \tag{5.49}$$

and this must be zero for all x, y leading to the three constraint equations

$$120C_{13} + 24C_{15} + 24C_{17} = 0 \tag{5.50}$$

$$24C_{14} + 24C_{16} + 120C_{18} = 0 \tag{5.51}$$

$$24C_8 + 8C_{10} + 24C_{12} = 0 \ . \tag{5.52}$$

The stresses are obtained by substituting (5.48) into (5.14–5.16) with the result

$$\begin{aligned}\sigma_{xx} &= 2C_3 + 2C_6x + 6C_7y + 2C_{10}x^2 + 6C_{11}xy + 12C_{12}y^2 + 2C_{15}x^3 \\ &\quad +6C_{16}x^2y + 12C_{17}xy^2 + 20C_{18}y^3\end{aligned} \tag{5.53}$$

$$\begin{aligned}\sigma_{xy} &= -C_2 - 2C_5x - 2C_6y - 3C_9x^2 - 4C_{10}xy - 3C_{11}y^2 - 4C_{14}x^3 \\ &\quad -6C_{15}x^2y - 6C_{16}xy^2 - 4C_{17}y^3\end{aligned} \tag{5.54}$$

$$\begin{aligned}\sigma_{yy} &= 2C_1 + 6C_4x + 2C_5y + 12C_8x^2 + 6C_9xy + 2C_{10}y^2 + 20C_{13}x^3 \\ &\quad +12C_{14}x^2y + 6C_{15}xy^2 + 2C_{16}y^3 \ .\end{aligned} \tag{5.55}$$

The tractions on $y=b$ are therefore

$$\begin{aligned}\sigma_{yx} &= -4C_{14}x^3 - (3C_9 + 6C_{15}b)x^2 - (2C_5 + 4C_{10}b + 6C_{16}b^2)x \\ &\quad -(C_2 + 2C_6b + 3C_{11}b^2 + 4C_{17}b^3)\end{aligned} \tag{5.56}$$

$$\begin{aligned}\sigma_{yy} &= 20C_{13}x^3 + (12C_8 + 12C_{14}b)x^2 + (6C_4 + 6C_9b + 6C_{15}b^2)x \\ &\quad +(2C_1 + 2C_5b + +2C_{10}b^2 + 2C_{16}b^3)\end{aligned} \tag{5.57}$$

and these must satisfy equations (5.42, 5.43) for all x, giving

$$4C_{14} = 0 \tag{5.58}$$
$$3C_9 + 6C_{15}b = 0 \tag{5.59}$$
$$2C_5 + 4C_{10}b + 6C_{16}b^2 = 0 \tag{5.60}$$
$$C_2 + 2C_6b + 3C_{11}b^2 + 4C_{17}b^3 = 0 \tag{5.61}$$
$$20C_{13} = 0 \tag{5.62}$$
$$12C_8 + 12C_{14}b = 0 \tag{5.63}$$
$$6C_4 + 6C_9b + 6C_{15}b^2 = 0 \tag{5.64}$$
$$2C_1 + 2C_5b + 2C_{10}b^2 + 2C_{16}b^3 = -p\ . \tag{5.65}$$

A similar procedure for the edge $y=-b$ yields the additional equations

$$3C_9 - 6C_{15}b = 0 \tag{5.66}$$
$$2C_5 - 4C_{10}b + 6C_{16}b^2 = 0 \tag{5.67}$$
$$C_2 - 2C_6b + 3C_{11}b^2 - 4C_{17}b^3 = 0 \tag{5.68}$$
$$12C_8 - 12C_{14}b = 0 \tag{5.69}$$
$$6C_4 - 6C_9b + 6C_{15}b^2 = 0 \tag{5.70}$$
$$2C_1 - 2C_5b + 2C_{10}b^2 - 2C_{16}b^3 = 0\ . \tag{5.71}$$

On $x=a$, we have

$$\sigma_{xx} = 2C_3 + 2C_6a + 6C_7y + 2C_{10}a^2 + 6C_{11}ay + 12C_{12}y^2 + 2C_{15}a^3 + 6C_{16}a^2y + 12C_{17}ay^2 + 20C_{18}y^3 \tag{5.72}$$
$$\sigma_{xy} = -C_2 - 2C_5a - 2C_6y - 3C_9a^2 - 4C_{10}ay - 3C_{11}y^2 - 4C_{14}a^3 - 6C_{15}a^2y - 6C_{16}ay^2 - 4C_{17}y^3\ . \tag{5.73}$$

Substituting into the weak conditions (5.45–5.47) and evaluating the integrals, we obtain the three additional equations

$$4C_3b + 4C_6ab + 4C_{10}a^2b + 8C_{12}b^3 + 4C_{15}a^3b + 8C_{17}ab^3 = 0 \tag{5.74}$$
$$-2C_2b - 4C_5ab - 6C_9a^2b - 2C_{11}b^3 - 8C_{14}a^3b - 4C_{16}ab^3 = pa \tag{5.75}$$
$$4C_7b^3 + 4C_{11}ab^3 + 4C_{16}a^2b^3 + 8C_{18}b^5 = 0\ . \tag{5.76}$$

Finally, we solve equations (5.50–5.52, 5.58–5.71, 5.74–5.76) for the unknown constants $C_1, ..., C_{18}$ and substitute back into (5.48), obtaining

$$\phi = \frac{p}{40b^3}(5x^2y^3 - y^5 - 15b^2x^2y - 5a^2y^3 + 2b^2y^3 - 10b^3x^2)\ . \tag{5.77}$$

The corresponding stress field is

$$\sigma_{xx} = \frac{p}{20b^3}(15x^2y - 10y^3 - 15a^2y + 6b^2y) \tag{5.78}$$
$$\sigma_{xy} = \frac{3px}{4b^3}(b^2 - y^2) \tag{5.79}$$
$$\sigma_{yy} = \frac{p}{4b^3}(y^3 - 3b^2y - 2b^3)\ . \tag{5.80}$$

The reader is encouraged to run the Maple or Mathematica files 'S522', which contain the above solution procedure. Notice that most of the algebraic operations are generated by quite simple and repetitive commands. These will be essentially similar for any polynomial problem involving rectangular coördinates, so it is a simple matter to modify the program to cover other cases.

5.2.3 Manual solutions — symmetry considerations

If the solution is to be obtained manually, the complexity of the algebra makes the process time consuming and increases the likelihood of errors. Fortunately, the complexity can be reduced by utilizing the natural symmetry of the rectangular beam. In many problems, the loading has some symmetry which can be exploited in limiting the number of independent polynomial terms and even when this is not the case, some saving of complexity can be achieved by representing the loading as the sum of symmetric and antisymmetric parts. We shall illustrate this procedure by repeating the solution of the problem of Figure 5.3.

The problem is symmetrical about the mid-point of the beam and hence, taking the origin there, we deduce that the resulting stress function will contain only *even* powers of x. This immediately reduces the number of terms in the general stress function to 10.

The beam is also symmetrical about the axis $y=0$, but the loading is not. We therefore decompose the problem into the two sub-problems illustrated in Figure 5.4(a,b).

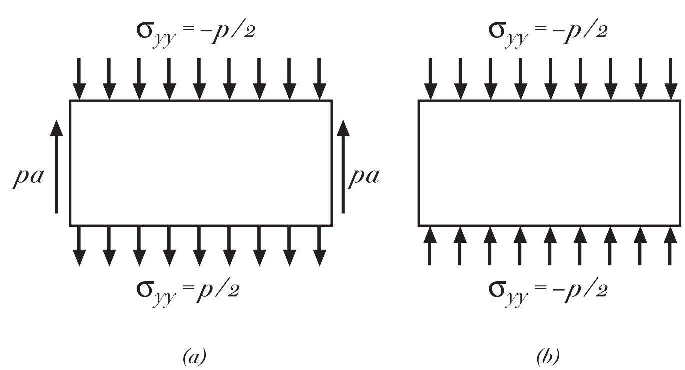

Figure 5.4: Decomposition of the problem into (a) antisymmetric and (b) symmetric parts.

The problem in Figure 5.4(a) is *antisymmetric* in y and hence requires a stress function with only *odd* powers of y, whereas that of Figure 5.4(b) is *symmetric* and requires only *even* powers. In fact, the problem of Figure 5.4(b)

clearly has the trivial solution corresponding to uniform uniaxial compression, $\sigma_{yy} = -p/2$, the appropriate stress function being $\phi = -px^2/4$.

For the problem of Figure 5.4(a), the most general fifth degree polynomial which is even in x and odd in y can be written

$$\phi = C_5x^2y + C_7y^3 + C_{14}x^4y + C_{16}x^2y^3 + C_{18}y^5 \; , \tag{5.81}$$

which has just five degrees of freedom. We have used the same notation for the remaining constants as in (5.48) to aid in comparing the two solutions. The appropriate boundary conditions for this sub-problem are

$$\sigma_{xy} = 0 \; ; \;\; y = \pm b \tag{5.82}$$

$$\sigma_{yy} = \mp\frac{p}{2} \; ; \;\; y = \pm b \tag{5.83}$$

$$\int_{-b}^{b} \sigma_{xx} dy = 0 \; ; \;\; x = \pm a \tag{5.84}$$

$$\int_{-b}^{b} \sigma_{xx} y dy = 0 \; ; \;\; x = \pm a \; . \tag{5.85}$$

Notice that, in view of the symmetry, it is only necessary to satisfy these conditions on one of each pair of edges (e.g. on $y = b, x = a$). For the same reason, we do not have to impose a condition on the vertical force at $x = \pm a$, since the symmetry demands that the forces be equal at the two ends and the *total* force must be $2pa$ to preserve global equilibrium, this being guaranteed by the use of the Airy stress function, as in the problem of §5.2.1.

It is usually better strategy to start a manual solution with the strong boundary conditions (equations (5.82, 5.83)), and in particular with those conditions that are homogeneous (in this case equation (5.82)), since these will often require that one or more of the constants be zero, reducing the complexity of subsequent steps. Substituting (5.81) into (5.15, 5.16), we find

$$\sigma_{xy} = -2C_5x - 4C_{14}x^3 - 6C_{16}xy^2 \tag{5.86}$$

$$\sigma_{yy} = 2C_5y + 12C_{14}x^2y + 2C_{16}y^3 \; . \tag{5.87}$$

Thus, condition (5.82) requires that

$$4C_{14}x^3 + (2C_5 + 6C_{16}b^2)x = 0 \; ; \;\; \text{for all } x \tag{5.88}$$

and this condition is satisfied if and only if

$$C_{14} = 0 \quad \text{and} \quad 2C_5 + 6C_{16}b^2 = 0 \; . \tag{5.89}$$

A similar procedure with equation (5.87) and boundary condition (5.83) gives the additional equation

$$2C_5b + 2C_{16}b^3 = -\frac{p}{2} \; . \tag{5.90}$$

Equations (5.89, 5.90) have the solution

$$C_5 = -\frac{3p}{8b} \ \ ; \ \ C_{16} = \frac{p}{8b^3} \ . \tag{5.91}$$

We next determine C_{18} from the condition that the function ϕ is biharmonic, obtaining

$$(24C_{14} + 24C_{16} + 120C_{18})y = 0 \tag{5.92}$$

and hence

$$C_{18} = -\frac{p}{40b^3} \ , \tag{5.93}$$

from (5.89, 5.91, 5.92).

It remains to satisfy the two weak boundary conditions (5.84, 5.85) on the ends $x = \pm a$. The first of these is satisfied identically in view of the antisymmetry of the stress field and the second gives the equation

$$4C_7b^3 + 4C_{16}a^2b^3 + 8C_{18}b^5 = 0 \ , \tag{5.94}$$

which, with equations (5.91, 5.93), serves to determine the remaining constant,

$$C_7 = \frac{p(2b^2 - 5a^2)}{40b^3} \ . \tag{5.95}$$

The final solution of the complete problem (the sum of that for Figures 5.4(a) and (b)) is therefore obtained from the stress function

$$\phi = \frac{p}{40b^3}(5x^2y^3 - y^5 - 15b^2x^2y - 5a^2y^3 + 2b^2y^3 - 10b^3x^2) \ , \tag{5.96}$$

as in the 'computer solution' (5.77), and the stresses are therefore given by (5.78–5.80) as before.

5.3 Fourier series and transform solutions

Polynomial solutions can, in principle, be extended to more general loading of the beam edges, as long as the tractions are capable of a power series expansion. However, the practical use of this method is limited by the algebraic complexity encountered for higher order polynomials and by the fact that many important traction distributions do not have convergent power series representations.

A more useful method in such cases is to build up a general solution by components of Fourier form. For example, if we write

$$\phi = f(y)\cos(\lambda x) \quad \text{or} \quad \phi = f(y)\sin(\lambda x) \ , \tag{5.97}$$

substitution in the biharmonic equation (5.1) shows that $f(y)$ must have the general form

$$f(y) = (A + By)e^{\lambda y} + (C + Dy)e^{-\lambda y} \ , \tag{5.98}$$

where A, B, C, D are arbitrary constants. Alternatively, by defining new arbitrary constants A', B', C', D' through the relations $A = (A'+C')/2,\ B = (B'+D')/2,\ C=(A'-C')/2,\ D=(B'-D')/2$, we can group the exponentials into hyperbolic functions, obtaining the equivalent form

$$f(y) = (A' + B'y)\cosh(\lambda y) + (C' + D'y)\sinh(\lambda y) \ . \tag{5.99}$$

The hyperbolic form enables us to take advantage of any symmetry about $y=0$, since $\cosh(\lambda y), y\sinh(\lambda y)$ are even functions of y and $\sinh(\lambda y), y\cosh(\lambda y)$ are odd functions.

More general biharmonic stress functions can be constructed by superposition of terms like (5.98, 5.99), leading to Fourier series expansions for the tractions on the surfaces $y=\pm b$. The theory of Fourier series can then be used to determine the coefficients in the series, using strong boundary conditions on $y=\pm b$. Quite general traction distributions can be expanded in this way, so Fourier series solutions provide a methodology applicable to any problem for the rectangular bar.

5.3.1 Choice of form

The stresses due to the stress function $\phi=f(y)\cos(\lambda x)$ are

$$\sigma_{xx} = f''(y)\cos(\lambda x) \ ; \ \sigma_{xy} = \lambda f'(y)\sin(\lambda x) \ ; \ \sigma_{yy} = -\lambda^2 f(y)\cos(\lambda x) \tag{5.100}$$

and the tractions on the edge $x=a$ are

$$\sigma_{xx}(a, y) = f''(y)\cos(\lambda a) \ ; \ \sigma_{xy}(a, y) = \lambda f'(y)\sin(\lambda a) \ . \tag{5.101}$$

It follows that we can satisfy homogeneous boundary conditions on one (but not both) of these tractions in the strong sense, by restricting the Fourier series to specific values of λ. In equation (5.101), the choice $\lambda=n\pi/a$ will give $\sigma_{xy}=0$ on $x=\pm a$, whilst $\lambda=(2n-1)\pi/2a$ will give $\sigma_{xx}=0$ on $x=\pm a$, where n is any integer.

Example

We illustrate this technique by considering the rectangular beam $-a<x<a$, $-b<y<b$, simply supported at $x=\pm a$ and loaded by compressive normal tractions $p_1(x)$ on the upper edge $y=b$ and $p_2(x)$ on $y=-b$ — i.e.

$$\sigma_{xy} = 0 \ ; \ \ y = \pm b \tag{5.102}$$

$$\sigma_{yy} = -p_1(x) \ ; \ \ y = b \tag{5.103}$$

$$= -p_2(x) \ ; \ \ y = -b \tag{5.104}$$

$$\sigma_{xx} = 0 \ ; \ \ x = \pm a \ . \tag{5.105}$$

Notice that we have replaced the weak conditions (5.84, 5.85) by the strong condition (5.105). As in §5.2.2, it is not necessary to enforce the remaining weak conditions (those involving the vertical forces on $x=\pm a$), since these will be identically satisfied by virtue of the equilibrium condition.

The algebraic complexity of the problem will be reduced if we use the geometric symmetry of the beam to decompose the problem into four sub-problems. For this purpose, we define

$$f_1(x) = f_1(-x) \equiv \frac{1}{4}\{p_1(x)+p_1(-x)+p_2(x)+p_2(-x)\} \tag{5.106}$$

$$f_2(x) = -f_2(-x) \equiv \frac{1}{4}\{p_1(x)-p_1(-x)+p_2(x)-p_2(-x)\} \tag{5.107}$$

$$f_3(x) = f_3(-x) \equiv \frac{1}{4}\{p_1(x)+p_1(-x)-p_2(x)-p_2(-x)\} \tag{5.108}$$

$$f_4(x) = -f_4(-x) \equiv \frac{1}{4}\{p_1(x)-p_1(-x)-p_2(x)+p_2(-x)\} \tag{5.109}$$

and hence

$$p_1(x) = f_1(x)+f_2(x)+f_3(x)+f_4(x) \;\;;\;\; p_2(x) = f_1(x)+f_2(x)-f_3(x)-f_4(x)\;. \tag{5.110}$$

The boundary conditions now take the form

$$\sigma_{xy} = 0 \;\;;\;\; y=\pm b \tag{5.111}$$

$$\sigma_{yy} = -f_1(x)-f_2(x)-f_3(x)-f_4(x) \;\;;\;\; y=b \tag{5.112}$$

$$= -f_1(x)-f_2(x)+f_3(x)+f_4(x) \;\;;\;\; y=-b \tag{5.113}$$

$$\sigma_{xx} = 0 \;\;;\;\; x=\pm a \tag{5.114}$$

and each of the functions f_1, f_2, f_3, f_4 defines a separate problem with either symmetry or antisymmetry about the x- and y-axes. We shall here restrict attention to the loading defined by the function $f_3(x)$, which is symmetric in x and antisymmetric in y. The boundary conditions of this sub-problem are

$$\sigma_{xy} = 0 \;\;;\;\; y=\pm b \tag{5.115}$$

$$\sigma_{yy} = \mp f_3(x) \;\;;\;\; y=\pm b \tag{5.116}$$

$$\sigma_{xx} = 0 \;\;;\;\; x=\pm a\;. \tag{5.117}$$

The problem of equations (5.115–5.117) is even in x and odd in y, so we use a cosine series in x with only the odd terms from the hyperbolic form (5.99) — i.e.

$$\phi = \sum_{n=1}^{\infty}\{A_n y\cosh(\lambda_n y)+B_n\sinh(\lambda_n y)\}\cos(\lambda_n x)\;, \tag{5.118}$$

where A_n, B_n are arbitrary constants. The strong condition (5.117) on $x=\pm a$ can then be satisfied in every term by choosing

$$\lambda_n = \frac{(2n-1)\pi}{2a} \; . \tag{5.119}$$

The corresponding stresses are

$$\sigma_{xx}=\sum_{n=1}^{\infty}\left\{2A_n\lambda_n\sinh(\lambda_n y)+A_n\lambda_n^2 y\cosh(\lambda_n y)+B_n\lambda_n^2\sinh(\lambda_n y)\right\}\cos(\lambda_n x)$$
$$\sigma_{xy}=\sum_{n=1}^{\infty}\left\{A_n\lambda_n\cosh(\lambda_n y)+A_n\lambda_n^2 y\sinh(\lambda_n y)+B_n\lambda_n^2\cosh(\lambda_n y)\right\}\sin(\lambda_n x)$$
$$\sigma_{yy}=-\sum_{n=1}^{\infty}\left\{A_n\lambda_n^2 y\cosh(\lambda_n y)+B_n\lambda_n^2\sinh(\lambda_n y)\right\}\cos(\lambda_n x) \tag{5.120}$$

and hence the boundary conditions (5.115, 5.116) on $y=\pm b$ require that

$$\sum_{n=1}^{\infty}\{A_n\lambda_n\cosh(\lambda_n b)+A_n\lambda_n^2 b\sinh(\lambda_n b)+B_n\lambda_n^2\cosh(\lambda_n b)\}\sin(\lambda_n x)=0 \tag{5.121}$$

$$\sum_{n=1}^{\infty}\left\{A_n\lambda_n^2 b\cosh(\lambda_n b)+B_n\lambda_n^2\sinh(\lambda_n b)\right\}\cos(\lambda_n x)=f_3(x) \; . \tag{5.122}$$

To invert the series, we multiply (5.122) by $\cos(\lambda_m x)$ and integrate from $-a$ to a, obtaining

$$\sum_{n=1}^{\infty}\int_{-a}^{a}\left\{A_n\lambda_n^2 b\cosh(\lambda_n b)+B_n\lambda_n^2\sinh(\lambda_n b)\right\}\cos(\lambda_n x)\cos(\lambda_m x)dx$$
$$=\int_{-a}^{a}f_3(x)\cos(\lambda_m x)dx \; . \tag{5.123}$$

The integrals on the left-hand side are all zero except for the case $m=n$ and hence, evaluating the integrals, we find

$$\left\{A_m\lambda_m^2 b\cosh(\lambda_m b)+B_m\lambda_m^2\sinh(\lambda_m b)\right\}a=\int_{-a}^{a}f_3(x)\cos(\lambda_m x)dx \; . \tag{5.124}$$

The homogeneous equation (5.121) is clearly satisfied if

$$A_m\lambda_m\cosh(\lambda_m b)+A_m\lambda_m^2 b\sinh(\lambda_m b)+B_m\lambda_m^2\cosh(\lambda_m b)=0 \; . \tag{5.125}$$

Solving (5.124, 5.125) for A_m, B_m, we have

$$A_m=\frac{\cosh(\lambda_m b)}{\lambda_m a\{\lambda_m b-\sinh(\lambda_m b)\cosh(\lambda_m b)\}}\int_{-a}^{a}f_3(x)\cos(\lambda_m x)dx$$
$$B_m=-\frac{(\cosh(\lambda_m b)+\lambda_m b\sinh(\lambda_m b))}{\lambda_m^2 a\{\lambda_m b-\sinh(\lambda_m b)\cosh(\lambda_m b)\}}\int_{-a}^{a}f_3(x)\cos(\lambda_m x)dx \; , \tag{5.126}$$

where λ_m is given by (5.119). The stresses are then recovered by substitution into equations (5.120).

The corresponding solutions for the functions f_1, f_2, f_4 are obtained in a similar way, but using a sine series for the odd functions f_2, f_4 and the even terms $y\sinh(\lambda y), \cosh(\lambda y)$ in ϕ for f_1, f_2. The complete solution is then obtained by superposing the solutions of the four sub-problems.

The Fourier series method is particularly useful in problems where the traction distribution on the long edges has no power series expansion, typically because of discontinuities in the loading. For example, suppose the beam is loaded only by a concentrated compressive force F on the upper edge at $x=0$, corresponding to the loading $p_1(x)=F\delta(x),\ p_2(x)=0$. For the symmetric/antisymmetric sub-problem considered above, we then have

$$f_3(x) = \frac{F\delta(x)}{4} \tag{5.127}$$

from (5.108) and the integral in equations (5.126) is therefore

$$\int_{-a}^{a} f_3(x)\cos(\lambda_m x)dx = \frac{F}{4}\ , \tag{5.128}$$

for all m.

This solution satisfies the end condition on σ_{xx} in the strong sense, but the condition on σ_{xy} only in the weak sense. In other words, the tractions σ_{xy} on the ends add up to the forces required to maintain equilibrium, but we have no control over the exact distribution of these tractions. This represents an improvement over the polynomial solution of §5.2.3, where weak conditions were used for both end tractions, so we might be tempted to use a Fourier series even for problems with continuous polynomial loading. However, this improvement is made at the cost of an infinite series solution. If the series were truncated at a finite value of n, errors would be obtained particularly near the ends or any discontinuities in the loading.

5.3.2 Fourier transforms

If the beam is infinite or semi-infinite ($a \to \infty$), the series (5.118) must be replaced by the integral representation

$$\phi(x, y) = \int_{0}^{\infty} f(\lambda, y)\cos(\lambda x)d\lambda\ , \tag{5.129}$$

where

$$f(\lambda, y) = A(\lambda)y\cosh(\lambda y) + B(\lambda)\sinh(\lambda y)\ . \tag{5.130}$$

Equation (5.129) is introduced here as a generalization of (5.97) by superposition, but $\phi(x, y)$ is in fact the Fourier cosine transform of $f(\lambda, y)$, the corresponding inversion being

$$f(\lambda, y) = \frac{2}{\pi}\int_0^\infty \phi(x, y)\cos(\lambda x)dx \; . \tag{5.131}$$

The boundary conditions on $y = \pm b$ will also lead to Fourier integrals, which can be inverted in the same way to determine the functions $A(\lambda), B(\lambda)$. For a definitive treatment of the Fourier transform method, the reader is referred to the treatise by Sneddon[3]. Extensive tables of Fourier transforms and their inversions are given by Erdelyi[4]. The cosine transform (5.129) will lead to a symmetric solution. For more general loading, the complex exponential transform can be used.

It is worth remarking on the way in which the series and transform solutions are natural generalizations of the elementary solution (5.97). One of the most powerful techniques in Elasticity — and indeed in any physical theory characterized by linear partial differential equations — is to seek a simple form of solution (often in separated-variable form) containing a parameter which can take a range of values. A more general solution can then be developed by superposing arbitrary multiples of the solution with different values of the parameter.

For example, if a particular solution can be written symbolically as $\phi = f(x, y, \lambda)$, where λ is a parameter, we can develop a general series form

$$\phi(x, y) = \sum_{i=0}^{\infty} A_i f(x, y, \lambda_i) \tag{5.132}$$

or an integral form

$$\phi(x, y) = \int_a^b A(\lambda) f(x, y, \lambda) d\lambda \; . \tag{5.133}$$

The series form will naturally arise if there is a discrete set of *eigenvalues*, λ_i for which $f(x, y, \lambda_i)$ satisfies some of the boundary conditions of the problem. Additional examples of this kind will be found in §§6.2, 11.2. In this case, the series (5.132) is most properly seen as an eigenfunction expansion. Integral forms arise most commonly (but not exclusively) in problems involving infinite or semi-infinite domains (see, for example, §§11.3, 30.2.2.).

Any particular solution containing a parameter can be used in this way and, since transforms are commonly named after their originators, the reader desirous of instant immortality might like to explore some of those which have not so far been used. Of course, the usefulness of the resulting solution depends upon its *completeness* — i.e. its capacity to represent all stress fields of a given class — and upon the ease with which the transform can be inverted.

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[3] I.N.Sneddon, *Fourier Transforms*, McGraw-Hill, New York, 1951.

[4] A.Erdelyi, **ed.**, *Tables of Integral Transforms*, Bateman Manuscript Project, California Institute of Technology, Vol.1, McGraw-Hill, New York, 1954.

PROBLEMS

1. The beam $-b<y<b,\ 0<x<L$, is built-in at the end $x=0$ and loaded by a uniform shear traction $\sigma_{xy}=S$ on the upper edge, $y=b$, the remaining edges, $x=L,\ y=-b$ being traction-free. Find a suitable stress function and the corresponding stress components for this problem, using the weak boundary conditions on $x=L$.

2. The beam $-b<y<b, -L<x<L$ is simply supported at the ends $x=\pm L$ and loaded by a shear traction $\sigma_{xy}=Sx/L$ on the lower edge, $y=-b$, the upper edge being traction-free. Find a suitable stress function and the corresponding stress components for this problem, using the weak boundary conditions on $x=\pm L$.

3. The beam $-b<y<b,\ 0<x<L$, is built-in at the end $x=L$ and loaded by a linearly-varying compressive normal traction $p(x)=Sx/L$ on the upper edge, $y=b$, the remaining edges, $x=0,\ y=-b$ being traction-free. Find a suitable stress function and the corresponding stress components for this problem, using the weak boundary conditions on $x=0$.

4. The beam $-b<y<b, -L<x<L$ is simply supported at the ends $x=\pm L$ and loaded by a compressive normal traction

$$p(x) = S\cos\left(\frac{\pi x}{2L}\right)$$

on the upper edge, $y=b$, the lower edge being traction-free. Find a suitable stress function and the corresponding stress components for this problem.

5. The beam $-b<y<b,\ 0<x<L$, is built-in at the end $x=L$ and loaded by a compressive normal traction

$$p(x) = S\sin\left(\frac{\pi x}{2L}\right)$$

on the upper edge, $y=b$, the remaining edges, $x=0,\ y=-b$ being traction-free. Use a combination of the stress function (5.97) and an appropriate polynomial to find the stress components for this problem, using the weak boundary conditions on $x=0$.

6. A large plate defined by $y>0$ is subjected to a sinusoidally varying load

$$\sigma_{yy} = S\sin\lambda x \ \ ; \ \ \sigma_{xy} = 0$$

at its plane edge $y=0$.

Find the complete stress field in the plate and hence estimate the depth y at which the amplitude of the variation in σ_{yy} has fallen to 10% of S.

Hint: You might find it easier initially to consider the case of the layer $0<y<h$, with $y=h$ traction-free, and then let $h \to \infty$.

7. The beam $-a<x<a$, $-b<y<b$ is loaded by a uniform compressive traction p in the central region $-a/2<x<a/2$ of both of the edges $y=\pm b$, as shown in Figure 5.5. The remaining edges are traction-free. Use a Fourier series with the appropriate symmetries to obtain a solution for the stress field, using the weak condition on σ_{xy} on the edges $x=\pm a$ and the strong form of all the remaining boundary conditions.

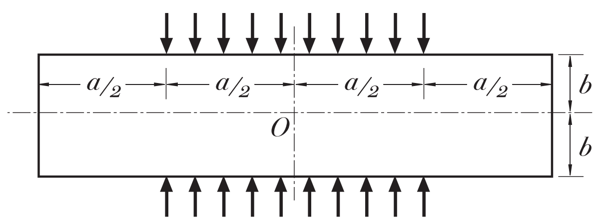

Figure 5.5

8. Use a Fourier series to solve the problem of Figure 5.4(a) in §5.2.3. Choose the terms in the series so as to satisfy the condition $\sigma_{xx}(\pm a, y) = 0$ in the strong sense.

If you are solving this problem in Maple or Mathematica, compare the solution with that of §5.2.3 by making a contour plot of the difference between the truncated Fourier series stress function and the polynomial stress function

$$\phi = \frac{p}{40b^3}(5x^2y^3 - y^5 - 15b^2x^2y - 5a^2y^3 + 2b^2y^3) .$$

Examine the effect of taking different numbers of terms in the series.

9. The large plate $y>0$ is loaded at its remote boundaries so as to produce a state of uniform tensile stress

$$\sigma_{xx} = S \ ; \ \sigma_{xy} = \sigma_{yy} = 0 ,$$

the boundary $y=0$ being traction-free. We now wish to determine the perturbation in this simple state of stress that will be produced if the traction-free boundary had a slight waviness, defined by the line

$$y = \epsilon \cos(\lambda x) ,$$

where $\lambda\epsilon \ll 1$. To solve this problem

(i) Start with the stress function

$$\phi = \frac{Sy^2}{2} + f(y)\cos(\lambda x)$$

and determine $f(y)$ if the function is to be biharmonic.

(ii) The perturbation will be localized near $y=0$, so select only those terms in $f(y)$ that decay as $y\to\infty$.

(iii) Find the stress components and use the stress transformation equations to determine the tractions on the wavy boundary. Notice that the inclination of the wavy surface to the plane $y=0$ will be everywhere small if $\lambda\epsilon \ll 1$ and hence the trigonometric functions involving this angle can be approximated using $\sin(x)\approx x,\ \cos(x)\approx 1, x\ll 1$.

(iv) Choose the free constants in $f(y)$ to satisfy the traction-free boundary condition on the wavy surface.

(v) Determine the maximum tensile stress and hence the stress concentration factor as a function of $\lambda\epsilon$.