THE THERMAL DISTORTION DUE TO A UNIFORM CIRCULAR HEAT SOURCE ON THE SURFACE OF A SEMI-INFINITE SOLID

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Summary—An expression is derived for the surface displacement of a semi-infinite solid which is heated uniformly over a circular area after time \( t = 0 \). The displacement is shown to be a function of the group \( t/a^2 \), where \( a \) is the radius of the heated area. Numerical results for the displacement at the centre of the heated area are tabulated. A description of the numerical method involved in evaluating the integrals derived is included in the Appendix.

NOTATION

- \( a \): radius of heated area
- \( A \): \( a/(4kt) \)
- \( \alpha \): coefficient of linear expansion
- \( c \): specific heat
- \( E \): Young's modulus
- \( k \): thermal diffusivity
- \( \lambda \): surface displacement
- \( q \): heat input rate
- \( Q \): \( \pi a^2 q \)
- \( r \): radius
- \( R \): \( r/(4kt) \)
- \( \rho \): density
- \( \sigma_\theta \): tangential stress
- \( t \): time
- \( T \): temperature
- \( \nu \): Poisson's ratio
- \( x, y \): radial distances from an axis
- \( X \): \( x/(4kt) \)
- \( Y \): \( y/(4kt) \)
- \( \psi(R) \): \( \frac{\text{erf} R}{2R^2} - \text{erfc} R - \frac{\exp(-R^2)}{R \sqrt{\pi}} \)

INTRODUCTION

The problem considered in this paper arose in connexion with some experiments on instability in sliding friction, see Barber1. When two flat surfaces are in sliding contact, they only touch at a few points because of inevitable surface irregularity. Since heat is generated at these points of contact only, the resultant temperature field causes them to expand outwards...
from the surface, exaggerating the surface irregularity. In certain circumstances an instability can occur causing the largest area of contact to take all the load and leading to extremes of temperature.

In order to analyse this problem it is necessary to calculate the thermal distortion due to an axially symmetric heat source on the surface of a semi-infinite solid. The solution for a point heat source is obtained by considering the equivalent infinite body problem and superposing surface forces equal and opposite to the stresses transmitted across the surface plane. The circular heat source solution is then obtained by integration. Numerical values of the central displacement are presented in a Table.

THE CONTINUOUS POINT SOURCE

Consider a semi-infinite solid, heated at a point on the surface by a continuous source of strength \( q \), starting at time \( t = 0 \).

The temperature at a radius \( r \), time \( t \) is

\[
T = \frac{q}{2\pi k\rho c} \text{erfc} \frac{r}{\sqrt{4kt}}
\]

(see Carslaw and Jaeger\(^2\)), where \( k \) denotes thermal diffusivity, \( \rho \) the density and \( c \) is the specific heat (all assumed constant).

If the surface is restrained to be flat, the thermal stresses will be spherically symmetric and the resultant tangential* stress at a radius \( r \) is

\[
\sigma_t = \frac{\alpha E}{1-\nu} \left( \frac{1}{r^2} \int_0^r Tr^2 dr - T \right)
\]

(see Timoshenko and Goodier\(^3\)), where \( \alpha \) is the coefficient of linear expansion, \( E \) is Young's modulus and \( \nu \) is Poisson's ratio.

Substituting the value of \( T \) obtained above,

\[
\int_0^r Tr^2 dr = \frac{q}{2\pi k\rho c} \int_0^r \text{erfc} \frac{r}{\sqrt{4kt}} dr
\]

\[
= \frac{q}{2\pi k\rho c} \left[ \frac{r^2}{2} \text{erfc} \frac{r}{\sqrt{4kt}} - \frac{k}{\sqrt{4kt}} \exp \left( -\frac{r^2}{4kt} \right) + \frac{1}{\sqrt{4kt}} \text{erf} \frac{r}{\sqrt{4kt}} \right]
\]

whence

\[
\sigma_t = \frac{q\alpha E}{4\pi k\rho c(1-\nu)} \left[ \frac{2kt}{r^2} \text{erf} \frac{r}{\sqrt{4kt}} - \left( \frac{4kt}{\pi} \right)^\frac{1}{2} \text{erf} \left( -\frac{r^2}{4kt} \right) - \frac{1}{r} \text{erfc} \frac{r}{\sqrt{4kt}} \right]
\]

(1)

The surface displacements corresponding to a stress free surface may be found by superposing surface stresses equal and opposite to the stresses \( \sigma_t \).

The displacement on the surface of a semi-infinite solid due to a point load \( P \) at a distance \( r \) is equal to \( [P(1-\nu^2)/\pi Er^2] \) (Ref. 3).

The displacement due to an annulus, radius \( r \), thickness \( dr \), stress \( \sigma_t \) at a distance \( r_0 \) from its centre is therefore

\[
\int_0^\pi \frac{r \sigma_t (1-\nu^2) d\theta dr}{\pi E (r_0^2 + r^2 - 2r_0r \cos \theta)^\frac{3}{2}}
\]

* The tangential stress here referred to is the direct stress tangential to the spheres of symmetry of the infinite body problem. At the surface plane of the semi-infinite solid it is, therefore, normal to the surface. Since the infinite body problem is spherically symmetric, no shear stress can be transmitted across the surface plane.
Thermal distortion on the surface of a semi-infinite solid

Summing for \( r = 0 \) to \( r = \infty \) we find

\[
\lambda = \int_0^\infty \int_0^{2\pi} \frac{\rho \alpha (1 - \nu^2)}{E(r^2 + r^2 - 2rr_0 \cos \theta)} \, d\theta \, dr
\]

(2)

**THE AREA HEAT SOURCE**

The displacement due to a uniform circular heat source at a distance \( y \) from its centre is obtained by integrating equation (2), substituting for \( \alpha \) from equation (1) and is

\[
\psi(R) = \frac{Q \alpha (1 + \nu)}{4\pi^2 k \rho c a^2} \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \frac{2kt}{r^2} \text{erf} \left( \frac{r}{\sqrt{4kt}} \right) - \text{erfo} \left( \frac{r}{\sqrt{4kt}} \right) - \left( \frac{4kt}{r} \right) \exp \left( \frac{r^2}{4kt} \right) \frac{x \, dr \, d\theta \, d\phi \, dx}{y^2 + x^2 + r^2 - 2yx \cos \phi - 2(y^2 + x^2 - 2yx \cos \phi) r \cos \theta}
\]

The number of independent variables may conveniently be reduced by expressing all lengths in the non-dimensional form \( r/\sqrt{4kt} \), i.e. \( Y = y/(4kt) \), \( X = x/\sqrt{4kt} \), \( R = r/\sqrt{4kt} \) and \( A = a/\sqrt{4kt} \), whence

\[
\lambda = \frac{Q \alpha (1 + \nu)}{4\pi^2 k \rho c a^2} \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \frac{\psi(R) X \, dr \, d\theta \, d\phi \, dx}{(Y^2 + X^2 + R^2 - 2XY \cos \phi - 2(X^2 + Y^2 - 2XY \cos \phi) R \cos \theta)}
\]

(3)

where

\[
\psi(R) = \frac{\text{erf} \, R}{2R^2} - \frac{\text{erfo} \, R - \exp \left( \frac{-R^2}{4\pi} \right)}{R \sqrt{\pi}}
\]

**RESULTS**

Numerical values have been obtained for the point \( Y = 0 \) and are presented in the Table below.

<table>
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<tr>
<th>( A )</th>
<th>( \frac{4\pi^2 k\lambda}{Q \alpha (1 + \nu)} )</th>
<th>( A )</th>
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<td>0.50</td>
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<td>34.793</td>
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**CONCLUSIONS**

These results may be extended to cover any unsteady, non-uniform, axi-symmetric heat input condition by suitable integration.

An interesting consequence of this thermal distortion applies to the transfer of heat between static surfaces in contact. Any disturbance in the heat flow, due to irregularities in the surfaces, will cause an expansion of one solid and a contraction of the other. If the normal load is constant, the surfaces will either move apart or together depending on the relative size of these displacements.
In general, though the total area of actual contact is a function of the normal load, the contact areas will be more uniformly distributed if the surfaces move together. The thermal contact resistance is therefore dependent on thermal distortion. From equation (3) we see that the parameter determining this behaviour is \[ \alpha(1 + \nu)/kpc. \]

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REFERENCES

1. J. R. Barber, Wear, 10, 155 (1967).

APPENDIX

The numerical evaluation of the integrals

The expression (3) for \( \lambda \) reduces for \( Y = 0 \) to

\[
\lambda = \frac{Q\alpha(1 + \nu)}{2\pi^2 kA^2} \int_0^\infty \int_0^{2\pi} \psi(R) X dR d\theta dX
\]

since the integrand is independent of \( \phi \). Further, the integrand is symmetric in \( \theta \) about \( \theta = \pi \) so that only the range (0 to \( \pi \)) need be considered. There remains the problem of evaluating the triple integral

\[
I = \int_0^\infty \int_0^{2\pi} \psi(R) X dR d\theta dX
\]

There are three main problems to be taken care of before numerical integration can be used

(i) The singularities in the integrand for \( X = R, \theta = 0 \) and \( X = R = 0 \).
(ii) The infinite range in \( R \).
(iii) The evaluation of \( \psi(R) \).

The last of these is perhaps the easiest. The main difficulty is that the expression for \( \psi(R) \) given in (4) has terms which tend to infinity as \( R \) tends to 0, but cancellation occurs and \( \psi(0) \) is finite. This cancellation implies that the form given for \( \psi(R) \) is unsuitable for numerical evaluation near \( R = 0 \), so a power series expansion was developed thus,

\[
\psi(R) = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r(4r^2 - 1)} - \frac{1}{2} \exp \left( -\frac{1}{2r^2} \right) \left( 2 - \frac{1}{2r^2} - \frac{1}{2r^4} \cdots \right)
\]

This expansion is clearly not suitable for large \( R \) but for this range an asymptotic expansion was found as

\[
\psi(R) \approx \frac{1}{2R^2} - \frac{1}{\pi^4} \left( 2 - \frac{1}{2R^2} - \frac{1}{2R^4} \cdots \right)
\]

These two forms are sufficient to evaluate \( \psi(R) \) for any \( R \) as their ranges of usefulness just about overlap. A third direct evaluation would be slightly more efficient in the middle of the range (particularly if high accuracy were required), but it was not used here.

Since the integrand has no singularity for \( R > A \) it is convenient to divide the \( R \) integral into three parts

\[
I = I_1 + I_2 + I_3
\]

where the ranges in \( R \) are (0, \( A \)), (\( A \), \( B \)) and (\( B \), \( \infty \)) respectively.
Thermal distortion on the surface of a semi-infinite solid

If \( B \) is chosen so that only the first term in (A4) is significant (and \( B > A \)) the integral \( I_3 \) can be evaluated analytically. For

\[
I_3 = \int_0^A \int_0^\pi \int_B^\infty \frac{X}{2R^2} \left(1 + \frac{X^2 - 2XR \cos \theta}{R^2}\right)^{-1} dR\ d\theta\ dX
\]

(A5)

it becomes on expanding and integrating

\[
I_3 = \frac{\pi}{4} \left(\frac{A^4}{2B^4} + \frac{A^4}{32B^4}\right) + O\left(\frac{A}{B}\right)^5
\]

It is interesting to note that this expansion for \( I_3 \) is still dependent only on the ratio \( A/B \) if the whole series is kept, so that if \( A \) is large enough \( \int A \) is independent of \( A \) and consequently needs to be calculated only once.

Returning to the evaluation of \( I_1 \) and \( I_2 \) it is apparent that the integration with respect to \( X \) can be performed analytically,

\[
\int_0^A X(X^2 + R^2 - 2XR \cos \theta)^{-1} dX = \left(A^2 + R^2 - 2AR \cos \theta\right)^{\frac{1}{2}} - R + R \cos \theta \left[\log \left(\frac{A - R \cos \theta + (A^2 + R^2 - 2AR \cos \theta)^{\frac{1}{2}}}{R - R \cos \theta}\right)\right]
\]

For small \( \theta \) the last term is approximately \( R \log \left(\frac{A - R + \frac{1}{2}R\theta^2 + [(A - R)^2 + AR\theta^2]^{\frac{1}{2}}}{\frac{1}{4}R\theta^2}\right) \) which is finite for \( R > A \) but for \( R < A \) is infinite—of the form \(-2 \log \theta\). At \( R = A \) there is a complicated form of singularity,

\[
\log \left(\frac{1}{2}A\theta^2 - S + (S^2 + A^2 \theta^2)^{\frac{1}{2}}\right)
\]

where \( S = R - A \).

For \( R < A \), the \( \log \theta \) singularity can be subtracted and integrated analytically, but it does not appear possible to remove the whole infinite singularity for \( R = A \), \( \theta = 0 \), as the local form is not analytically integrable. In spite of this, the numerical integration produced satisfactory results without a prohibitive amount of calculation.

Since the integrand is modified for \( R < A \), the ranges in \( R \) of \( (0, A) \) and \( (A, B) \) must be treated separately. For simplicity, \( B \) was taken as a multiple of \( A \) and the range \( (A, B) \) divided into subranges of length \( A \). This is a little inefficient for small values of \( A \), but it was not considered worthwhile changing the method for these cases.

The method of numerical integration employed for both the \( R \) and \( \theta \) integration was Gauss quadrature, which avoids requiring values of the integrand on the edges of the domain of integration and seems reasonably efficient in practice. For small values of \( A \) about 5 points in each direction were needed (making 25 in all) to give 3 significant figures. For larger values of \( A \) up to 20 points were used. The values were checked by altering the number of points and splitting the domain of integration further.